A Conjectural Characterization for $\mathbb{F}_q(t)$-Linear Relations between Multizeta Values

George Todd

Department of Mathematics, Bailey Hall 202, Union College Schenectady, NY 12309

Abstract

We provide a conjectural characterization for $\mathbb{F}_q(t)$-linear relations between multizeta values and for the dimensions of their fixed-weight $\mathbb{F}_q(t)$ span, as well as many new parameterized families of relations. These two conjectures provide the function field analog of the conjectures provide by Zagier and others. In contrast to the classical case which uses regularized stuffle and shuffle products to produce relations, we will posit that all relations in the function field setting can be generated from a single relation in weight q from Thakur's stuffle product. We also prove many of the currently existing relations in the literature in this setting.

---

1. Introduction

There exist two products on the $\mathbb{Q}$-span of classical multizeta values, the shuffle product and the stuffle product, from which $\mathbb{Q}$-linear relations can be derived. It has been conjectured that the regularization of this process accounts for all such linear relations (see [1], [2]).

Recent research in the function field case has investigated multizeta values in positive characteristic (See [3] and [4] for a survey of recent results.). Although the function field case does not have two products, we will define two parameterized families of maps between the space of power sum relations. We will posit that these two families of maps, along with one relation, describes all relations. As in the classical case, we restrict our attention to the $\mathbb{F}_q(t)$-span of multizeta values of fixed-weight since it is not expected that there are relations between multizeta values of differing weights [5].

We also provide the following conjecture for the dimension of the span of fixed-weight multizeta values in positive characteristic. Let $g_k = \dim\left(\mathrm{Span}_{\mathbb{F}_q(t)}(\text{MZV's of weight } k)\right)$, then

$$g_k = \begin{cases} 2^{k-1} & \text{if } 1 \leq k < q \\ 2^{k-1} - 1 & \text{if } k = q \\ \displaystyle\sum_{i=1}^{q} g_{k-i} & \text{if } k > q. \end{cases}$$

Email address: `toddg@union.edu` (George Todd)

Compare this with Zagier's conjecture for the classical multizeta values: $d_k = d_{k-2} + d_{k-3}$ (see [2]).

We also provide a theorem which shows that the two conjectures in the function field setting are equivalent for all q and weights k with $1 \leq k \leq \max(q+3, 11)$.

2. Notation

We fix some notation.

$$\begin{aligned}
q &= \text{a power of a prime } p\\
A &= \mathbb{F}_q[t]\\
A_+ &= \text{monics in } A\\
A_{\leq d} &= \text{elements of } A \text{ of degree at most } d\\
A_{d+} &= \text{monics in } A \text{ of degree exactly } d\\
K &= \mathbb{F}_q(t)\\
K_\infty &= \mathbb{F}_q((1/t)) = \text{completion of } K \text{ at } \infty\\
[n] &= t^{q^n} - t\\
l_n &= \prod_{i=1}^{n} (t - t^{q^i})\\
L_n &= (-1)^n l_n\\
D_n &= \prod_{i=0}^{n-1} (t^{q^n} - t^{q^i})
\end{aligned}$$

3. Multizeta Values

For integers $d \geq 0$ and $s > 0$, we define the power sum $S_d(s)$ by

$$S_d(s) = \sum_{\substack{a \in A_+ \\ \deg(a) = d}} \frac{1}{a^s} \in K.$$

Fix a positive integer w and let $s_1, \ldots, s_r$ be positive integers such that $\sum s_i = w$. Then we call $(s_1, \ldots, s_r)$ a composition of w. Let $V = (s_1, \ldots, s_r)$ be a composition of w, then we define the multizeta value $\zeta(V) = \zeta(s_1, \ldots, s_r)$ as

$$\zeta(V) = \zeta(s_1, \ldots, s_r) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \in K_\infty.$$

We will frequently say that $\zeta(V)$ has depth r and weight $\sum s_i$. Strictly speaking, depth and weight are really properties of V rather than $\zeta(V)$. The weight and depth, then, are possibly not unique to a multizeta value. As in the classical case, it is conjectured that there are no relations between multizeta values of differing weights. We also define

$$S_d(V) = S_d(s_1, \dots, s_r) = \sum_{\substack{d=d_1>\cdots>d_r\geq 0\\ \deg(a_i)=d_i}} \frac{1}{a_1^{s_1}\cdots a_r^{s_r}} \in K.$$

Then we have

$$\zeta(V) = \sum_{d=0}^{\infty} S_d(V) \in K_\infty.$$

We define $S_{<d}(V)$ by

$$S_{<d}(V) = \sum_{i=0}^{d-1} S_i(V)$$

so that

$$S_d(s_1, \dots, s_r) = S_d(s_1)S_{<d}(s_2, \dots, s_r).$$

Finally, we extend the definition of $S_d(V)$ to any integer d by defining empty sums to be zero and empty products to be unity. Thus

$$S_d(s_1, \dots, s_r) = 0, d \in \mathbb{Z}, d < r-1.$$

We now introduce some results of Carlitz and Thakur concerning power sums which we will use frequently (see [6], [5], and [7]). With $\binom{x}{q^d}$ defined as

$$\binom{x}{q^d} = \prod_{\substack{a\in A\\ \deg(a)<d}} (x+a)/D_d,$$

Carlitz derived the first generating function and Thakur derived the second:

$$\frac{x}{l_d(1-\binom{x}{q^d})} = \sum_{k=1}^{\infty} S_d(k)x^k, d \geq 1 \tag{3.1}$$

$$\frac{x}{l_d\binom{x}{q^d}} = 1 + \sum_{\substack{k>0\\ q-1|k}} S_{<d}(k)x^k, d \geq 1. \tag{3.2}$$

Thus, we get formulas (although complicated) for the power sums

$$S_d(k+1) = \frac{1}{l_d^{k+1}} \sum_{\substack{\sum_{i=0}^{d} k_i q^i = k\\ k_i\geq 0}} \binom{\sum k_i}{k_i} \prod_{i=1}^{d} \frac{(l_d/l_{d-i})^{q^i k_i}}{D_i^{k_i}}, k \geq 0.$$

The following special cases will be used frequently and are derived in [5]:

$$S_d(a) = \frac{1}{l_d^a}, \text{if } a \le q, d \ge 0 \tag{3.3}$$

$$S_d(q+b) = \frac{1}{l_d^{q+b}} \left(1 - b\frac{[d]^q}{[1]}\right), \text{if } 1 \le b < q, d \ge 0 \tag{3.4}$$

$$S_{<d}(m(q-1)) = \left(\frac{[d]}{[1]l_{d-1}^{q-1}}\right)^m, \text{if } m \le q, d \ge 1 \tag{3.5}$$

$$S_{<d}(q^i - 1) = \left(\frac{l_{d+i-1}}{l_i l_{d-1}^{q^i}}\right), d \ge 1. \tag{3.6}$$

The following product formula for the power sums is due to Chen (See [3], [4]):

Theorem 3.7.

$$S_d(a)S_d(b) = S_d(a+b) + \sum_j \left((-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1}\right) S_d(a+b-j, j)$$

where the sum is over all j such that $q-1 \mid j$ and $0 < j < a+b$.

Beginning with this product for the product of two power sums of depth 1, we can define the product of two power sums of arbitrary depth. The following is proved in [8].

Theorem 3.8 (Thakur)**.** *Let $S_d(a_1, \dots, a_r)$ and $S_d(b_1, \dots, b_k)$ be power sums of weight $\sum a_i$ and $\sum b_j$ and depth r and k, respectively, then*

$$S_d(a_1, \dots, a_r)S_d(b_1, \dots, b_k) = \sum_i f_i S_d(c_{i1}, \dots, c_{im_i})$$

with $f_i \in \mathbb{F}_p$, c_{ij} and m_i independent of d, $\sum a_i + \sum b_j = \sum_j c_{ij}$ for each i, and $m_i \le r+k$ for each i.

Thus the product of two multizeta values is an $\mathbb{F}_p$-linear combination of multizeta values, turning the $\mathbb{F}_p$-span of all multizeta values into an algebra. A simple corollary of Theorem 3.8 is:

Corollary 3.9. *Let $S_{<d}(a_1, \dots, a_r)$ and $S_{<d}(b_1, \dots, b_k)$ be sums of power sums of weight $\sum a_i$ and $\sum b_j$ and depth r and k, respectively, then*

$$S_{<d}(a_1, \dots, a_r)S_{<d}(b_1, \dots, b_k) = \sum_i g_i S_{<d}(e_{i1}, \dots, e_{im_i})$$

with $g_i \in \mathbb{F}_p$, e_{ij} and m_i independent of d, $\sum a_i + \sum b_j = \sum_j e_{ij}$ for each i, and $m_i \le r+k$ for each i.

In light of the classical case of both a shuffle and stuff product, the product resulting from Theorem 3.7 might be emphasized with suggestive notation such as $S_d(a) * S_d(b)$. However, as there is no known analogue of the shuffle product for Thakur multizeta values, it will be understood that when we wish to write $S_d(a)S_d(b)$ as an $\mathbb{F}_p$-linear combination of power sums, this product will be used.

We seek a characterization of $\mathbb{F}_q(t)$-linear relations between multizeta values. As examples, the following are proved in [5]:

Theorem 3.10 (Thakur). *For any q we have*

$$\zeta(m, m(q-1)) = \zeta(mq)/l_1^m, m \leq q \tag{3.11}$$

$$\zeta(1, q^2-1) = \zeta(q^2)(1/l_2 + 1/l_1). \tag{3.12}$$

Fix a positive integer k. We introduce the vector space of relations between multizeta values of weight k and we attempt to identify a spanning set for these spaces. Let $\mathcal{Z}_k$ be the K-span of the multizeta values of weight k and let $\{V_1, \ldots, V_{2^{k-1}}\}$ be a fixed ordering of the 2^{k-1} compositions of k. Then we define a subset of $K^{2^{k-1}}$, the space of relations between multizeta values of weight k, by:

$$\mathcal{R}_k = \left\{ (r_1, \ldots, r_{2^{k-1}}) \in K^{2^{k-1}} : \sum_{i=1}^{2^{k-1}} r_i \zeta(V_i) = 0 \right\}.$$

Note that $\dim(\mathcal{Z}_k) + \dim(\mathcal{R}_k) = 2^{k-1}$. We define two subsets of $\mathcal{R}_k$ as follows: the space of *fixed relations*, $\mathcal{F}_k$:

$$\mathcal{F}_k = \left\{ (r_1, \ldots, r_{2^{k-1}}) \in \mathcal{R}_k : \sum_{i=1}^{2^{k-1}} r_i S_d(V_i) = 0, \text{for all } d \in \mathbb{Z} \right\}$$

and the space of *binary relations*, $\mathcal{N}_k$:

$$\mathcal{N}_k = \big\{ (r_1, \ldots, r_{2^{k-1}}) \in \mathcal{R}_k : \text{there exist } a_i, b_i \text{ with } a_i + b_i = r_i, \\ \sum_{i=1}^{2^{k-1}} a_i S_d(V_i) + b_i S_{d+1}(V_i) = 0, \text{for all } d \in \mathbb{Z} \big\}.$$

Note that these are both K-vector spaces with $\mathcal{F}_k \subset \mathcal{N}_k \subseteq \mathcal{R}_k$. We will motivate the necessity of the complexity in the definitions of $\mathcal{N}_k$ which allows terms in the relation between the power sums that do not appear in the relation between the multizeta values. Essentially, the extension of a relation between multizeta values to power sums is not always obvious and, in general, not unique. We give an example.

Example 3.13. For $q = 2$, we have the relation, proved in [5]:

$$(l_1 + l_2)\zeta(4) + l_1 l_2 \zeta(1, 3) = 0. \tag{3.14}$$

It is easily verified using (3.3) - (3.6) that:

$$l_1 S_d(4) + l_2 S_{d+1}(4) + l_1 l_2 S_{d+2}(1,3) = 0, \text{for all } d \in \mathbb{Z} \tag{3.15}$$

and

$$l_1 S_d(4) + l_2 S_d(4) + l_1^2 S_{d+1}(2,2) + l_1^2 S_d(2,2) + l_1 l_2 S_{d+1}(1,3) = 0, \text{for all } d \in \mathbb{Z}. \tag{3.16}$$

Both (3.15) and (3.16) give the relation (3.14) when summed over all d, but we cannot conclude from (3.15) that the relation associated to (3.14) is fixed, but it does follow from (3.16).

Because relations between multizeta values do not always extend uniquely to relations between power sums, it makes defining maps on these spaces problematic. We introduce the vector space of relations between power sums, which will somewhat remedy this situation. Define the space of relations between power sums of weight k by:

$$\mathcal{P}_k = \left\{ (a_1, \ldots, a_{2^{k-1}}, b_1, \ldots, b_{2^{k-1}}) \in K^{2^k} : \sum_{i=1}^{2^{k-1}} a_i S_d(V_i) + \sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i) = 0, \text{for all } d \in \mathbb{Z} \right\}.$$

There is an obvious K-linear map from $\mathcal{P}_k$ to $\mathcal{R}_k$, $\mathcal{I}_k : \mathcal{P}_k \to \mathcal{R}_k$ given by

$$\mathcal{I}_k((a_1, \ldots, a_{2^{k-1}}, b_1, \ldots, b_{2^{k-1}})) = (a_1 + b_1, \ldots, a_{2^{k-1}} + b_{2^{k-1}}).$$

Note that the image of $\mathcal{I}_k$ is actually in $\mathcal{N}_k$. If for $(a_1, \ldots, a_{2^{k-1}}, b_1, \ldots, b_{2^{k-1}}) \in \mathcal{P}_k$ either $a_i = 0$ for all i or $b_i = 0$ for all i, then the image of this element under $\mathcal{I}_k$ is in $\mathcal{F}_k$. As a notational convenience, if $R \in \mathcal{P}_k$, we define $\overline{R} = \mathcal{I}_k(R) \in \mathcal{R}_k$.

We now define several families of maps between the K vector spaces just discussed. These maps will be parameterized by compositions. First, we define a map from $\mathcal{P}_k$ to $\mathcal{P}_{k+w}$ whose image under $\mathcal{I}_{k+w}$ is in $\mathcal{F}_{k+w}$ (that is, a map that always results in a fixed relation).

Definition 3.17. Let $W = (w_1, \ldots, w_r)$ be a composition of w. Define a map

$$\mathcal{B}_W : \mathcal{P}_k \to \mathcal{P}_{k+w}$$

as follows. Let $R = (a_1, \ldots, a_{2^{k-1}}, b_1, \ldots, b_{2^{k-1}}) \in \mathcal{P}_k$. Then

$$\left(\sum_{i=1}^{2^{k-1}} a_i S_d(V_i) \right) + \left(\sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i) \right) = 0, \text{for all } d \in \mathbb{Z}.$$

Fix a $D \in \mathbb{Z}$ and let $\{U_1, \ldots, U_{2^{k+w-1}}\}$ be a fixed ordering of the 2^{k+w-1} compositions of $k + w$. By Theorem 3.8 and Corollary 3.9, we have

$$0 = S_D(W) \sum_{d<D} \left(\sum_{i=1}^{2^{k-1}} a_i S_d(V_i) + \sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i) \right) = \sum_{i=1}^{2^{k+w-1}} c_i S_D(U_i),$$

for some $c_i \in K$. We define $\mathcal{B}_W(R) = (c_1, \ldots, c_{2^{k+w-1}}, 0, \ldots, 0)$. Note that for all $R \in \mathcal{P}_k$, $\overline{\mathcal{B}_W(R)} \in \mathcal{F}_{k+w}$ and $\mathcal{B}_W$ is a K-linear map.

Example 3.18. Fix $q = 2$. From [5] we have the relation $\zeta(2) + l_1\zeta(1,1) = 0$, which is a binary relation since

$$S_d(2) + l_1 S_{d+1}(1,1) = 0.$$

Denoting the associated vector in $\mathcal{P}^2$ by R_1 we calculate $\mathcal{B}_{(2,1)}(R_1)$:

$$\begin{aligned}
&S_d(2,1)\sum_{i<d}\left(S_i(2) + l_1 S_{i+1}(1,1)\right)\\
&= S_d(2,1)\left(S_{<d}(2) + l_1 S_{<d}(1,1) + l_1 S_d(1,1)\right)\\
&= \left(S_d(2)S_{<d}(1)S_{<d}(2)\right) + l_1\left(S_d(2)S_{<d}(1)S_{<d}(1,1)\right) + l_1\left(S_d(2)S_d(1)S_{<d}(1)S_{<d}(1)\right)\\
&= S_d(2)\left(S_{<d}(1,2) + S_{<d}(3)\right) + l_1 S_d(2)\left(S_{<d}(1,1,1) + S_{<d}(2,1) + S_{<d}(1,2)\right)\\
&+ l_1(S_d(2)S_d(1))(S_{<d}(1)S_{<d}(1))\\
&= S_d(2,1,2) + S_d(2,3) + l_1(S_d(2,1,1,1) + S_d(2,2,1) + S_d(2,1,2))\\
&+ l_1(S_d(3) + S_d(2,1))S_{<d}(2)\\
&= S_d(2,1,2) + S_d(2,3) + l_1 S_d(2,1,1,1) + l_1 S_d(2,2,1) + l_1 S_d(2,1,2) + l_1 S_d(3,2)\\
&+ l_1 S_d(2,1)S_{<d}(2)\\
&= S_d(2,1,2) + S_d(2,3) + l_1 S_d(2,1,1,1) + l_1 S_d(2,2,1) + l_1 S_d(2,1,2) + l_1 S_d(3,2)\\
&+ l_1 S_d(2)S_{<d}(1)S_{<d}(2)\\
&= S_d(2,1,2) + S_d(2,3) + l_1 S_d(2,1,1,1) + l_1 S_d(2,2,1) + l_1 S_d(2,1,2) + l_1 S_d(3,2)\\
&+ l_1 S_d(2)(S_{<d}(1,2) + S_{<d}(3))\\
&= S_d(2,1,2) + S_d(2,3) + l_1 S_d(2,1,1,1) + l_1 S_d(2,2,1) + l_1 S_d(2,1,2) + l_1 S_d(3,2)\\
&+ l_1 S_d(2,1,2) + l_1 S_d(2,3).
\end{aligned}$$

So we see that $\overline{\mathcal{B}_{(2,1)}(R_1)}$ is the relation associated to

$$\zeta(2,1,2) + (l_1+1)\zeta(2,3) + l_1\zeta(2,1,1,1) + l_1\zeta(2,2,1) + l_1\zeta(3,2) + l_1\zeta(2,1,2) = 0.$$

3.1. $\mathcal{C}$ *Map*

Definition 3.19. Let $W = (w_1, \ldots, w_r)$ be a composition of w, then we define a map $\mathcal{C}_W$

$$\mathcal{C}_W : \mathcal{P}_k \to \mathcal{P}_{k+w}$$

as follows. Let $R = (a_1, \ldots, a_{2^{k-1}}, b_1, \ldots, b_{2^{k-1}}) \in \mathcal{P}_k$. Then

$$\left(\sum_{i=1}^{2^{k-1}} a_i S_d(V_i)\right) + \left(\sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i)\right) = 0, \text{for all } d \in \mathbb{Z}.$$

Let $\{U_1, \ldots, U_{2^{k+w-1}}\}$ be a fixed ordering of the 2^{k+w-1} compositions of $k+w$. By Theorem 3.8 and Corollary 3.9, we have

$$0 = S_{<d+1}(W)\left(\sum_{i=1}^{2^{k-1}} a_i S_d(V_i) + \sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i)\right) = \left(\sum_{i=1}^{2^{k+w-1}} c_i S_d(U_i)\right) + \left(\sum_{i=1}^{2^{k+w-1}} d_i S_{d+1}(U_i)\right),$$

for some $c_i, d_i \in K$. We define $\mathcal{C}_W(R) = (c_1, \ldots, c_{2^{k+w-1}}, d_1, \ldots, d_{2^{k+w-1}})$. Note that $\mathcal{C}_W$ is a K-linear map.

3.2. Maps $\mathcal{B}^*$ and $\mathcal{C}^*$

We define iterated versions of the maps $\mathcal{B}$ and $\mathcal{C}$. These maps are particularly useful in describing fixed relations that are short in length.

Definition 3.20. Let $W = (w_1, \ldots, w_r)$ be a composition of w. Define $\mathcal{B}^*_W$ by

$$\mathcal{B}^*_W = \mathcal{B}_{w_1} \circ \mathcal{B}_{w_2} \circ \cdots \circ \mathcal{B}_{w_r}.$$

Definition 3.21. Let $W = (w_1, \ldots, w_r)$ be a composition of w. Define $\mathcal{C}^*_W$ by

$$\mathcal{C}^*_W = \mathcal{C}_{w_1} \circ \mathcal{C}_{w_2} \circ \cdots \circ \mathcal{C}_{w_r}.$$

Note that when $W = (w)$ is a composition of w of length 1, then $\mathcal{B}^*_W = \mathcal{B}_W$ and $\mathcal{C}^*_W = \mathcal{C}_W$. The map $\mathcal{B}^*$ restricted to fixed relations is quite simple.

Theorem 3.22. *If $\sum_i a_i\zeta(V_i) = 0$ is a fixed relation with associated power sum relation $\sum_i a_i S_d(V_i) = 0$ (for all $d \in \mathbb{Z}$), and W is a composition of w, then the relation associated to $\mathcal{B}^*_W(R)$ is*

$$\sum_i a_i Sd(W, V_i) = 0$$

so that

$$\sum_i a_i\zeta(W, V_i) = 0.$$

Proof. For any length 1 composition (w) and fixed relation, we have

$$\begin{aligned} &S_d(w) \sum_{j<d} \left(\sum_i a_i S_j(V_i) \right) \\ &= S_d(w) \sum_i a_i S_{<d}(V_i) \\ &= \sum_i a_i S_d(w, V_i). \end{aligned}$$

By definition $\mathcal{B}^*_W = \mathcal{B}_{w_1} \circ \ldots \circ \mathcal{B}_{w_r}$, and the claim follows. □

Example 3.23. We prove in Theorem 6.1 that for $q = 2$, a fixed relation is

$$\zeta(1, 2) + L_1\zeta(2, 1) + L_1\zeta(1, 1, 1) = 0.$$

By Theorem 3.22 we also have

$$\begin{aligned} &\zeta(3, 1, 2) + L_1\zeta(3, 2, 1) + L_1\zeta(3, 1, 1, 1) = 0 \\ &\zeta(1, 4, 5, 1, 2) + L_1\zeta(1, 4, 5, 2, 1) + L_1\zeta(1, 4, 5, 1, 1, 1) = 0 \\ &\text{etc.} \end{aligned}$$

Something similar is true for binary relations:

Theorem 3.24. *Let $R = (r_1, \ldots, r_{2^{k-1}}) \in \mathcal{P}$ be the relation associated to*

$$\left(\sum_{i=1}^{2^{k-1}} a_i S_d(V_i)\right) + \left(\sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i)\right) = 0$$

for all d. Write $V_i = (v_i, V_i')$ and let $W = (w)$ be a composition of w of length 1 such that $w + v_i \leq q$ for all i such that $b_i \neq 0$, then we have

$$\left(\sum_{i=1}^{2^{k-1}} (a_i + b_i)\zeta(w, V_i)\right) + \left(\sum_{i=1}^{2^{k-1}} b_i \zeta(w + v_i, V_i')\right) = 0.$$

Proof. By hypothesis we have

$$\sum_{i=1}^{2^{k-1}} a_i S_d(V_i) + \sum_{i=1}^{2^{k-1}} b_i S_{d+1}(V_i) = 0, \text{for all } d \in \mathbb{Z}.$$

First note that taking $d = -1$ gives

$$\sum_{i=1}^{2^{k-1}} b_i S_0(V_i) = 0.$$

Summing both sides from $d = 0$ to $d = D - 1$ we have

$$\begin{aligned}
0 &= \sum_{i=1}^{2^{k-1}} (a_i S_{<D}(V_i)) + \sum_{i=1}^{2^{k-1}} (b_i S_{<D}(V_i) + b_i S_D(v_i, V_i')) \\
&= \sum_{i=1}^{2^{k-1}} (a_i S_{<D}(V_i)) + \sum_{i=1}^{2^{k-1}} (b_i S_{<D}(V_i) + b_i S_D(v_i) S_{<D}(V_i')) \\
&= \sum_{i=1}^{2^{k-1}} (a_i S_{<D}(V_i)) + \sum_{i=1}^{2^{k-1}} \left(b_i S_{<D}(V_i) + \frac{b_i}{l_D^{v_i}} S_{<D}(V_i')\right).
\end{aligned}$$

Multiplying both sides of the equation by $S_D(w) = 1/l_D^w$ (since $w < q$) gives

$$0 = \sum_{i=1}^{2^{k-1}} (a_i S_D(w, V_i)) + \sum_{i=1}^{2^{k-1}} \left(b_i S_D(w, V_i) + \frac{b_i}{l_D^{v_i+w}} S_{<D}(V_i')\right)$$

$$0 = \sum_{i=1}^{2^{k-1}} (a_i S_D(w, V_i)) + \sum_{i=1}^{2^{k-1}} (b_i S_D(w, V_i) + b_i S_D(w + v_i, V_i'))$$

$$0 = \sum_{i=1}^{2^{k-1}} ((a_i + b_i) S_D(w, V_i)) + \sum_{i=1}^{2^{k-1}} (b_i S_D(w + v_i, V_i')).$$

Summing over all D gives the result. □

4. Describing Previous Work

Previous work on $\mathbb{F}_q(t)$-linear relations has been done in [5], [9], [10], [11], [12], [13], and [14]. In this section we describe much of that work with the maps just defined. The following relation, proved in [5], is of special importance and we denote it by R_m (for fixed $m \leq q$):

$$S_d(mq) - l_1^m S_{d+1}(m, m(q-1)) = 0, \text{for all } d \in \mathbb{Z}$$

and associated multizeta relation

$$\zeta(mq) - l_1^m \zeta(m, m(q-1)) = 0,\ m \leq q.$$

Notice that this relation is binary.

Theorem 4.1. *Let a, b be positive integers with $a + b = m \leq q$, then*

$$R_m = \mathcal{C}_{(bq)}(R_a) - l_1^b \mathcal{C}_{(b,b(q-1))}(R_a) + l_1^a \mathcal{B}_{(a,a(q-1))}(R_b) - \mathcal{B}_{(aq)}(R_b).$$

Proof. $\mathcal{C}_{(bq)}(R_a)$ is the relation associated to

$$\begin{aligned}
&S_{<d+1}(bq)\left(S_d(aq) - l_1^a S_{d+1}(a, a(q-1))\right), \text{for all } d \in \mathbb{Z} \\
&= S_{<d+1}(bq)S_d(aq) - l_1^a S_{<d+1}(bq)S_{d+1}(a, a(q-1)) \\
&= \left(S_{<d}(bq) + S_d(bq)\right) S_d(aq) - l_1^a S_{<d+1}(bq)S_{d+1}(a, a(q-1)) \\
&= \underbrace{S_d(aq, bq)}_{A_1} + S_d(aq + bq) \underbrace{- l_1^a S_{d+1}(a, a(q-1))S_{<d+1}(bq)}_{B_1}.
\end{aligned}$$

Here we have used that $S_d(aq)S_d(bq) = S_d((a+b)q)$ since $a + b \leq q$.

$(-1)l_1^b \mathcal{C}_{(b,b(q-1))}(R_a)$ is the relation associated to

$$\begin{aligned}
&(-1)l_1^b S_{<d+1}(b, b(q-1))\left(S_d(aq) - l_1^a S_{d+1}(a, a(q-1))\right) \\
&= (-1)l_1^b S_{<d+1}(b, b(q-1))S_d(aq) + l_1^{a+b} S_{d+1}(a, a(q-1))S_{<d+1}(b, b(q-1)) \\
&= \underbrace{(-1)l_1^b S_d(aq, b, b(q-1))}_{C_1} \underbrace{- l_1^b S_d(aq)S_d(b, b(q-1))}_{D_1} \\
&\quad + \underbrace{l_1^{a+b} S_{d+1}(a, a(q-1))S_{<d+1}(b, b(q-1))}_{E_1}.
\end{aligned}$$

$l_1^a \mathcal{B}_{(a,a(q-1))}(R_b)$ is the relation associated to

$$\begin{aligned}
&l_1^a S_{d+1}(a, a(q-1)) \sum_{i<d+1} \left(S_i(bq) - l_1^b S_{i+1}(b, b(q-1))\right) \\
&= l_1^a S_{d+1}(a, a(q-1)) \left(S_{<d+1}(bq) - l_1^b S_{<d+1}(b, b(q-1)) - l_1^b S_{d+1}(b, b(q-1))\right) \\
&= \underbrace{l_1^a S_{d+1}(a, a(q-1))S_{<d+1}(bq)}_{B_2} \underbrace{- l_1^{a+b} S_{d+1}(a, a(q-1))S_{<d+1}(b, b(q-1))}_{E_2} \\
&\quad - l_1^{a+b} S_{d+1}(a, a(q-1))S_{d+1}(b, b(q-1)).
\end{aligned}$$

Finally, $-\mathcal{B}_{(aq)}(R_b)$ is the relation associated to

$$\begin{aligned}
&- S_d(aq) \sum_{i<d} \big(S_i(bq) - l_1^b S_{i+1}(b, b(q-1))\big) \\
&= -S_d(aq) \big(S_{<d}(bq) - l_1^b S_{<d}(b, b(q-1)) - l_1^b S_d(b, b(q-1))\big) \\
&= \underbrace{-S_d(aq, bq)}_{A_2} + \underbrace{l_1^b S_d(aq, b, b(q-1))}_{C_2} + \underbrace{l_1^b S_d(aq) S_d(b, b(q-1))}_{D_2}.
\end{aligned}$$

Note that $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = D_1 + D_2 = E_1 + E_2 = 0$ for all d. This gives that

$$\mathcal{C}_{(bq)}(R_a) - l_1^b \mathcal{C}_{(b,b(q-1))}(R_a) + l_1^a \mathcal{B}_{(a,a(q-1))}(R_b) - \mathcal{B}_{(aq)}(R_b)$$

is the relation associated to

$$S_d(aq + bq) - l_1^{a+b} S_{d+1}(a, a(q-1)) S_{d+1}(b, b(q-1)).$$

This second term we can write as

$$\begin{aligned}
S_{d+1}(a, a(q-1)) S_{d+1}(b, b(q-1)) &= S_{d+1}(a) S_{d+1}(b) S_{<d+1}(a(q-1)) S_{<d+1}(b(q-1)) \\
&= S_{d+1}(a+b) S_{<d+1}((a+b)(q-1)) \\
&= S_{d+1}(a+b, (a+b)(q-1))
\end{aligned}$$

and the result follows. □

As a particular case, by putting $a = 1$ and $b = n$, we have:

Corollary 4.2.

$$R_{n+1} = \mathcal{C}_{(nq)}(R_1) - l_1^n \mathcal{C}_{(n,n(q-1))}(R_1) + l_1 \mathcal{B}_{(1,q-1)}(R_n) - \mathcal{B}_{(q)}(R_n)$$

for $n + 1 \leq q$.

A similar calculation gives this relation in terms of the $\mathcal{B}^*$ map:

Theorem 4.3. *With* $m \leq q$ *we have*

$$R_{m+1} = \\ \mathcal{C}_{(mq)}(R_1) - l_1^m \mathcal{C}_{(m,m(q-1))}(R_1) - \mathcal{B}^*_{(q)}(R_m) + l_1 \mathcal{B}^*_{(1,q-1)}(R_m) + l_1 \mathcal{B}^*_{(1)} \circ \mathcal{C}_{(q-1)}(R_m).$$

Lemma 4.4. *We have*

$$S_d(q^2 - q) S_d(q-1) = S_d(q^2 - 1) - S_d(q^2 - q, q-1)$$

and

$$S_{<d}(q^2 - q) S_{<d}(q-1) = S_{<d}(q-1, q^2 - q) + S_{<d}(q^2 - 1).$$

Proof. The left-hand side of the first relation is

$$S_d(q(q-1))S_d(q-1) = \left(\frac{1}{l_d^{q(q-1)}}\right)\left(\frac{1}{l_d^{q-1}}\right) = \frac{1}{l_d^{q^2-1}}.$$

For the right-hand side, we have

$$\begin{aligned} S_d(q^2-1) - S_d(q(q-1), q-1) &= \left(\frac{l_{d+1}}{l_1 l_d^{q^2}}\right) - \left(\frac{1}{l_d^{q(q-1)}}\right)\left(\frac{[d]}{[1]l_{d-1}^{q-1}}\right) \\ &= \left(\frac{l_{d+1}/l_d}{l_1 l_d^{q^2-1}}\right) - \left(\frac{-(-1)^{q-1}[d]^q}{l_1 l_d^{q^2-1}}\right) \\ &= \frac{1}{l_d^{q^2-1}} \end{aligned}$$

where we have used (3.6) and that

$$S_d(q^2-1) = S_{<d+1}(q^2-1) - S_{<d}(q^2-1).$$

This proves the first relation. The second relation follows from

$$\begin{aligned} &S_{<d}(q^2-q)S_{<d}(q-1) \\ &= \left(\sum_{i<d} S_i(q^2-q)\right)\left(\sum_{j<d} S_j(q-1)\right) \\ &= \sum_{i<j<d} S_i(q^2-q)S_j(q-1) + \sum_{j<i<d} S_j(q-1)S_i(q^2-q) + \sum_{i=j<d} S_i(q^2-q)S_i(q-1) \\ &= S_{<d}(q^2-q, q-1) + S_{<d}(q-1, q^2-q) + \sum_{i<d} S_i(q^2-q)S_i(q-1) \\ &= S_{<d}(q^2-q, q-1) + S_{<d}(q-1, q^2-q) + \sum_{i<d} \big(S_i(q^2-1) - S_i(q^2-q, q-1)\big) \\ &= S_{<d}(q^2-q, q-1) + S_{<d}(q-1, q^2-q) + \big(S_{<d}(q^2-1) - S_{<d}(q^2-q, q-1)\big) \\ &= S_{<d}(q-1, q^2-q) + S_{<d}(q^2-1). \end{aligned}$$

□

Theorem 4.5. *The relation associated to* $\mathcal{C}_{(q-1)}(R_{q-1})$ *is*

$$S_d(q^2-1) - l_1^{q-1} S_{d+1}(q-1, q(q-1)) = 0, \textit{for all } d \in \mathbb{Z}.$$

Moreover, we have

$$\zeta(q^2-1) - l_1^{q-1} S_{d+1}(q-1, q(q-1)) = 0.$$

Proof. We check

$$\begin{aligned}0 &= S_{<d+1}(q-1)\left(S_d(q(q-1)) - l_1^{q-1}S_{d+1}(q-1,(q-1)^2)\right)\\ &= S_d(q-1)S_d(q(q-1)) + S_d(q(q-1),q-1) - l_1^{q-1}S_{d+1}(q-1,q(q-1))\\ &= S_d(q^2-1) - S_d(q(q-1),q-1) + S_d(q(q-1),q-1) - l_1^{q-1}S_{d+1}(q-1,q(q-1))\\ &= S_d(q^2-1) - l_1^{q-1}S_{d+1}(q-1,q(q-1))\end{aligned}$$

where we have used Lemma 4.4 and 3.5. Summing over all d gives the Theorem. □

Theorem 4.6. *With Q_1 the relation associated to*

$$\begin{aligned}&\big((l_1+l_2)S_d(q^2) - l_1l_2S_{d+1}(1,q^2-1)\\ &\quad + l_1^{q+1}S_d(q,q^2-q) - l_1^{q+1}S_{d+1}(q,q^2-q)\big) = 0.\end{aligned}$$

We have

$$Q_1 = (l_1+l_1^2)R_q + l_1^{q+1}\mathcal{C}_{(q(q-1))}(R_1) - l_1^3\mathcal{B}_{(1)}\circ\mathcal{C}_{(q-1)}(R_{q-1}).$$

Moreover $\overline{Q_1}$ is

$$(l_1+l_2)\zeta(q^2) - l_1l_2\zeta(1,q^2-1) = 0.$$

Proof. We compute term by term. For the first term, $(l_1+l_1^2)R_q$ is the relation associated to

$$(l_1+l_1^2)S_d(q^2) - (l_1^{q+1}+l_1^{q+2})S_{d+1}(q,q(q-1)).$$

For the second term, $l_1^{q+1}\mathcal{C}_{(q^2-q)}(R_1)$ is the relation associated to

$$\begin{aligned}0 &= l_1^{q+1}S_{<d+1}(q^2-q)\left(S_d(q) - l_1S_{d+1}(1,q-1)\right)\\ &= l_1^{q+1}\left(S_d(q)S_d(q^2-q) + S_d(q,q^2-q) - l_1S_{d+1}(1)S_{<d+1}(q-1)S_{<d+1}(q^2-q)\right)\\ &= l_1^{q+1}\left(S_d(q^2) + S_d(q,q^2-q) - l_1S_{d+1}(1)S_{<d+1}(q-1)S_{<d+1}(q^2-q)\right)\\ &= l_1^{q+1}\left(S_d(q^2) + S_d(q,q^2-q) - l_1S_{d+1}(1)\left(S_{<d+1}(q-1,q^2-q) + S_{<d+1}(q^2-1)\right)\right)\\ &= l_1^{q+1}\left(S_d(q^2) + S_d(q,q^2-q) - l_1S_{d+1}(1,q-1,q^2-q) - l_1S_{d+1}(1,q^2-1)\right)\\ &= l_1^{q+1}S_d(q^2) + l_1^{q+1}S_d(q,q^2-q) - l_1^{q+2}S_{d+1}(1,q-1,q^2-q) - l_1^{q+2}S_{d+1}(1,q^2-1)\end{aligned}$$

where we have used Lemma 4.4 from the third equality to the fourth.

Finally, compute the relation associated to $-l_1^3\mathcal{B}_{(1)}\circ\mathcal{C}_{(q-1)}(R_{q-1})$. From Theorem 4.5, the relation associated to $\mathcal{C}_{(q-1)}(R_{q-1})$ is

$$S_d(q^2-1) - l_1^{q-1}S_{d+1}(q-1,q(q-1)) = 0.$$

To compute $-l_1^3\mathcal{B}_{(1)}\circ\mathcal{C}_{(q-1)}(R_{q-1})$, we use Theorem 3.24 to get

$$-l_1^3S_d(1,q^2-1) + l_1^{q+2}S_d(1,q-1,q(q-1)) + l_1^{q+2}S_d(q,q(q-1)) = 0.$$

Adding together and noting that $-(l_1^{q+2}+l_1^3) = -l_1l_2$ and $l_1+l_1^2+l_1^{q+1} = l_1+l_2$ gives the result. □

Theorem 4.7. *With R as*

$$R = -l_1^{q-2}\overline{Q_1} + (l_1^{q-1}+1)\overline{R_q} + (l_1^{2q-1}+l_1^q)\overline{\mathcal{C}_{(q(q-1))}(R_1)},$$

the relation between multizeta avlues associated to $\overline{R}$ is

$$\zeta(q^2) - [1]^q[2]\zeta(1, q-1, q^2-q) = 0.$$

Proof. For $-l_1^{q-2}Q_1$ we have

$$\begin{aligned} &- l_1^{q-2}\big((l_1+l_2)S_d(q^2) - l_1 l_2 S_{d+1}(1, q^2-1) \\ &\quad + l_1^{q+1}S_d(q, q^2-q) - l_1^{q+1}S_{d+1}(q, q^2-q)\big) = 0 \end{aligned}$$

for all d, while $(l_1^{q-1}+1)R_q$ gives

$$(l_1^{q-1}+1)\left(S_d(q^2) - l_1^q S_{d+1}(q, q^2-q)\right) = 0$$

for all $d \in \mathbb{Z}$. Finally, $(l_1^{2q-1}+l_1^q)\mathcal{C}_{(q(q-1))}(R_1)$ gives

$$\begin{aligned} &(l_1^{2q-1}+l_1^q)S_{<d+1}(q^2-q)\left(S_d(q) - l_1 S_{d+1}(1, q-1)\right) \\ &= (l_1^{2q-1}+l_1^q)S_d(q^2-q)S_d(q) + S_{<d}(q^2-q)S_d(q) - l_1 S_{d+1}(1, q-1)S_{<d+1}(q^2-q) \\ &= (l_1^{2q-1}+l_1^q)S_d(q^2) + S_d(q, q^2-q) - l_1 S_{d+1}(1)S_{<d+1}(q-1)S_{<d+1}(q(q-1)) \\ &= (l_1^{2q-1}+l_1^q)S_d(q^2) + S_d(q, q^2-q) - l_1 S_{d+1}(1, q-1, q^2-q) - l_1 S_{d+1}(1, q^2-1). \end{aligned}$$

Adding these terms, noting that $l_1^{2q-1}+l_1^q = l_1^{q-1}(l_1^q+l_1) = l_1^{q-2}l_2$, and summing over all d gives the result. □

5. Relation Conjecture

We now provide a conjecture for the nature of all $\mathbb{F}_q(t)$-linear relations between multizeta values in terms of these maps.

Conjecture 5.1. *A spanning set for $\mathcal{R}_{q+w}$ with $w > 0$ is given by*

$$\mathcal{S} = \left(\bigcup\left\{\overline{\mathcal{B}_V(R_1)}, \overline{\mathcal{C}_V(R_1)}\right\}\right) \cup \left(\bigcup \overline{\mathcal{B}_U \circ \mathcal{C}_V(R_1)}\right)$$

where the first union is over all compositions of w and the second union is over all compositions U and V such that (U, V) is a composition of w. Moreover, this set is also a spanning set if $\mathcal{B}$ is replaced by $\mathcal{B}^*$ in both unions.

Note that the conjecture requires only one "generating" relation, namely R_1. We now compute $\mathcal{S}$ explicitly for $q+1, q+2$, and $q+3$ for all q (using $\mathcal{B}^*$; similar, but much longer, relations result from $\mathcal{B}$). We list each power sum vector with its associated relation between multizeta values.

5.1. Weight q

$$R_1 \quad \zeta(q) + L_1\zeta(1, q-1) = 0$$

This relation is binary and is proved in [5].

5.2. Weight q + 1

$$\begin{aligned}
&\mathcal{B}_{(1)}(R_1) \quad \zeta(1,q) + L_1\zeta(2,q-1) + L_1\zeta(1,1,q-1) = 0 && (5.2)\\
&\mathcal{C}_{(1)}(R_1) \quad \zeta(q+1) + L_1\zeta(1,q) + L_1\zeta(1,1,q-1) + L_1\zeta(1,q-1,1) && (5.3)\\
&\qquad\qquad + \zeta(q,1) + \zeta(2,q-1) = 0
\end{aligned}$$

5.2 follows from Theorem 3.24 with $W = (1)$.
For 5.3, we have

$$\begin{aligned}
0 &= S_{<d+1}(1)\left(S_d(q) + L_1 S_{d+1}(1, q-1)\right)\\
&= S_{<d+1}(1)S_d(q) + L_1 S_{<d+1}(1)S_{d+1}(1, q-1)\\
&= (S_{<d}(1) + S_d(1))S_d(q) + L_1 S_{d+1}(1)S_{<d+1}(1)S_{<d+1}(q-1)\\
&= S_d(1)S_d(q) + S_d(q,1) + L_1 S_{d+1}(1,1,q-1) + L_1 S_{d+1}(1,q-1,1) + L_1 S_{d+1}(1,q)\\
&\quad + L_1 A S_{d+1}(1,1,q-1).
\end{aligned}$$

Here A is given by

$$A = (-1)^0\binom{q-2}{0} + (-1)^{q-2}\binom{q-2}{q-2}.$$

Since the characteristic is 2 if $q \equiv 0 \bmod 2$, we have that $A = 0$ for all q, so

$$\begin{aligned}
0 &= S_{<d+1}(1)\left(S_d(q) + L_1 S_{d+1}(1, q-1)\right)\\
&= S_d(1)S_d(q) + S_d(q,1) + L_1 S_{d+1}(1,1,q-1) + L_1 S_{d+1}(1,q-1,1) + L_1 S_{d+1}(1,q).
\end{aligned}$$

For $S_d(1)S_d(q)$ we have

$$S_d(1)S_d(q) = S_d(q+1) + \sum_j f_j S_d(q+1-j, j)$$

where the sum is over all j such that $q-1|j$ and $0 < j < a+b$. If $q > 2$, then note that

$$\begin{aligned}
2(q-1) &= (q+1) + q - 3\\
&\geq q+1
\end{aligned}$$

so that, in this case

$$\begin{aligned} S_d(1)S_d(q) &= S_d(q+1) + \left((-1)^0\binom{q-2}{0} + (-1)^{q-1}\binom{q-2}{q-1}\right) S_d(2,q-1) \\ &= S_d(q+1) + S_d(2,q-1). \end{aligned}$$

In the case that $q = 2$, we have

$$\begin{aligned} S_d(1)S_d(2) &= S_d(3) + S_d(2,1) + \left((-1)\binom{1}{0} + (-1)\binom{1}{1}\right) S_d(1,2) \\ &= S_d(3) + S_d(2,1). \end{aligned}$$

This proves 5.3. Note that the set $\{\overline{B_{(1)}(R_1)}, \overline{\mathcal{C}_{(1)}(R_1)}\}$ is linearly independent in $\mathcal{R}_{q+1}$.

5.3. Weight $q+2$

For $q > 2$ we have

$$\begin{aligned} &\mathcal{B}^*_{(2)}(R_1) && \zeta(2,q) + L_1\zeta(3,q-1) + L_1\zeta(2,1,q-1) = 0 && (5.4) \\ &\mathcal{B}^*_{(1,1)}(R_1) && \zeta(1,1,q) + L_1\zeta(1,2,q-1) + L_1\zeta(1,1,1,q-1) = 0 && (5.5) \\ &\mathcal{B}^*_{(1)}(\mathcal{C}_{(1)}(R_1)) && \zeta(1,q+1) + L_1\zeta(2,q) + L_1\zeta(1,1,q) + \zeta(1,2,q-1) && (5.6) \\ & && \zeta(1,q,1) + L_1\zeta(2,1,q-1) + L_1\zeta(2,q-1,1) && \\ & && + L_1\zeta(1,1,q-1,1) + L_1\zeta(1,1,1,q-1) = 0 && \\ &\mathcal{C}_{(2)}(R_1) && \zeta(q+2) + L_1\zeta(1,q+1) + \zeta(q,2) + L_1\zeta(1,q-1,2) && (5.7) \\ & && + 2\zeta(3,q-1) + 2L_1\zeta(1,2,q-1) = 0 && \\ &\mathcal{C}_{(1,1)}(R_1) && \zeta(2,q) + \zeta(q+1,1) + L_1\zeta(1,1,q) + L_1\zeta(1,1,1,q-1) && (5.8) \\ & && + L_1\zeta(1,q,1) + \zeta(2,1,q-1) + \zeta(2,q-1,1) + \zeta(q,1,1) && \\ & && + L_1\zeta(1,1,q-1,1) + L_1\zeta(1,q-1,1,1) = 0. && \end{aligned}$$

The only exception for $q = 2$ is that we have $(L_1+1)\zeta(2,2) + L_1\zeta(3,1) + L_1\zeta(2,1,1) = 0$ in place of relation 5.4. Relations 5.4, 5.5, and 5.6 follow from 3.22 applied to R_1, R_1, and 5.3, respectively. 5.7 follows from the same method as in Theorem 4.5.

For 5.8 we compute

$$\begin{aligned} &S_{<d+1}(1,1)\left(S_d(q) - l_1 S_{d+1}(1,q-1)\right) \\ &= S_d(1,1)S_d(q) + S_d(q,1,1) - l_1 S_{d+1}(1)S_{<d+1}(q-1)S_{<d+1}(1,1). \end{aligned}$$

We expand the first term:

$$\begin{aligned} S_d(1)S_d(q)S_{<d}(1) &= \left(S_d(q+1) + S_d(2,q-1)\right) S_{<d}(1) \\ &= S_d(q+1,1) + S_d(2,q-1)S_{<d}(1) \\ &= S_d(q+1,1) + S_d(2)S_{<d}(q-1)S_{<d}(1) \\ &= S_d(q+1,1) + S_d(2)\left(S_{<d}(q-1,1) + S_{<d}(1,q-1) + S_{<d}(q)\right) \\ &= S_d(q+1,1) + S_d(2,q-1,1) + S_d(2,1,q-1) + S_d(2,q). \end{aligned}$$

Similarly the third term is

$$\begin{aligned}
&- l_1 S_{d+1}(1) S_{<d+1}(q-1) S_{<d+1}(1,1) \\
&= -l_1 S_{d+1}(1)\big(S_{<d+1}(q-1,1,1) + \sum_{i<d+1} S_i(q-1) S_i(1,1) \\
&\quad + \sum_{j<d+1} S_j(1) S_{<j}(1) S_{<j}(q-1)\big) \\
&= -l_1 S_{d+1}(1)\big(S_{<d+1}(q-1,1,1) + S_{<d+1}(1,1,q-1) + S_{<d+1}(1,q-1,1) \\
&\quad + S_{<d+1}(1,q) + S_{<d+1}(q,1)\big) \\
&= -l_1 S_{d+1}(1,q-1,1,1) - l_1 S_{d+1}(1,1,1,q-1) - l_1 S_{d+1}(1,1,q-1,1) \\
&\quad - l_1 S_{d+1}(1,1,q) - l_1 S_{d+1}(1,q,1).
\end{aligned}$$

Adding all three terms, summing over all d and noting that $-l_1 = L_1$ gives the result. We now show that these relations are linearly independent in $\mathcal{R}_{q+2}$. For $q > 3$ (strict), we fix an ordering of the compositions of $q+2$ as $\{(2,q),(1,1,q),(1,q,1),(q,2),$ $(q,1,1),\ldots\}$. Then writing these five relations as row vectors, we have

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
L_1 & L_1 & 1 & 0 & 0 & \cdots \\
0 & 0 & L_1 & 1 & 0 & \cdots \\
1 & L_1 & 0 & 0 & 1 & \cdots
\end{bmatrix}$$
$$(2,q) \quad (1,1,q) \quad (1,q,1) \quad (q,2) \quad (q,1,1)$$

from which it is clear that they are linearly independent. Independence of relations for $q = 3$ is similarly easily verified.

5.4. Weight $q+3$

For $q > 3$, we have

$$\mathcal{B}_{(3)}(R_1) \quad \zeta(3,q) + L_1\zeta(4,q-1) + L_1\zeta(3,1,q-1) = 0 \tag{5.9}$$

$$\mathcal{B}^*_{(1,2)}(R_1) \quad \zeta(1,2,q) + L_1\zeta(1,3,q-1) + L_1\zeta(1,2,1,q-1) = 0 \tag{5.10}$$

$$\mathcal{B}^*_{(2,1)}(R_1) \quad \zeta(2,1,q) + L_1\zeta(2,2,q-1) + L_1\zeta(2,1,1,q-1) = 0 \tag{5.11}$$

$$\mathcal{B}^*_{(1,1,1)}(R_1) \quad \zeta(1,1,1,q) + L_1\zeta(1,1,2,q-1) + L_1\zeta(1,1,1,1,q-1) = 0 \tag{5.12}$$

$$\begin{aligned}
\mathcal{B}^*_{(1,1)}(\mathcal{C}_{(1)}(R_1)) \quad & \zeta(1,1,q+1) + L_1\zeta(1,1,1,q) + L_1\zeta(1,2,q) \\
& + L_1\zeta(1,1,1,1,q-1) + L_1\zeta(1,2,1,q-1) + L_1\zeta(1,1,1,q-1,1) \\
& + L_1\zeta(1,2,q-1,1) + \zeta(1,1,q,1) + \zeta(1,1,2,q-1) = 0
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\mathcal{B}_{(2)}(\mathcal{C}_{(1)}(R_1)) \quad & \zeta(2,q+1) + L_1\zeta(2,1,q) + L_1\zeta(3,q) + L_1\zeta(2,1,1,q-1) \\
& + L_1\zeta(3,1,q-1) + L_1\zeta(2,1,q-1,1) + L_1\zeta(3,q-1,1) + \zeta(2,q,1)
\end{aligned} \tag{5.14}$$

$$
\begin{aligned}
& + \zeta(2,2,q-1) = 0 \\
\mathcal{B}_{(1)}(\mathcal{C}_{(2)}(R_1)) \quad & \zeta(1,q+2) + \zeta(1,q,2) + 2\zeta(1,3,q-1) + L_1\zeta(1,1,q+1) \\
& + L_1\zeta(1,1,q-1,2) + 2L_1\zeta(1,1,2,q-1) + L_1\zeta(2,q+1) \\
& + L_1\zeta(2,q-1,2) + 2L_1\zeta(2,2,q-1) = 0 \qquad (5.15) \\
\mathcal{B}_{(1)}(\mathcal{C}_{(1,1)}(R_1)) \quad & \zeta(1,2,q) + \zeta(1,q+1,1) + \zeta(1,2,1,q-1) \\
& + \zeta(1,2,q-1,1) + \zeta(1,q,1,1) + L_1\zeta(1,1,1,q) + \\
& L_1\zeta(1,1,1,1,q-1) + L_1\zeta(1,1,q,1) + \\
& L_1\zeta(1,1,1,q-1,1) + L_1\zeta(1,1,q-1,1,1) \\
& + L_1\zeta(2,1,q) + L_1\zeta(2,1,1,q-1) + L_1\zeta(2,q,1) \\
& + L_1\zeta(2,1,q-1,1) + L_1\zeta(2,q-1,1,1) = 0 \qquad (5.16) \\
\mathcal{C}_{(3)}(R_1) \quad & \zeta(q+3) + L_1\zeta(1,q+2) + 3L_1\zeta(1,3,q-1) \\
& + L_1\zeta(1,q-1,3) + \zeta(q,3) + 3\zeta(4,q-1) = 0 \qquad (5.17) \\
\mathcal{C}_{(1,2)}(R_1) \quad & \zeta(2,q+1) + \zeta(q+1,2) + L_1\zeta(1,1,q+1) + \zeta(q,1,2) \\
& + L_1\zeta(1,q,2) + \zeta(2,q-1,2) + L_1\zeta(1,1,q-1,2) \\
& + L_1\zeta(1,q-1,1,2) + 2\zeta(2,2,q-1) + 2L_1\zeta(1,1,2,q-1) = 0 \qquad (5.18) \\
\mathcal{C}_{(2,1)}(R_1) \quad & 2\zeta(3,q) + \zeta(q+2,1) + 2L_1\zeta(1,2,q) + L_1\zeta(1,q+1,1) \\
& + 2\zeta(3,1,q-1) + \zeta(q,2,1) + L_1\zeta(1,q-1,2,1) + 2\zeta(3,q-1,1) \\
& + 2L_1\zeta(1,2,q-1,1) + 2L_1\zeta(1,2,1,q-1) = 0 \qquad (5.19) \\
\mathcal{C}_{(1,1,1)}(R_1) \quad & \zeta(2,1,q) + \zeta(2,q,1) + \zeta(q+1,1,1) + \zeta(2,1,1,q-1) \\
& + \zeta(2,1,q-1,1) + \zeta(2,q-1,1,1) + \zeta(q,1,1,1) \\
& + L_1\zeta(1,1,1,q) + L_1\zeta(1,1,q,1) + L_1\zeta(1,q,1,1) \\
& + L_1\zeta(1,1,1,1,q-1) + L_1\zeta(1,1,1,q-1,1) + L_1\zeta(1,1,q-1,1,1) \\
& + L_1\zeta(1,q-1,1,1,1) = 0. \qquad (5.20)
\end{aligned}
$$

Relations 5.9-5.16 follow from Theorem 3.22 applied to either R_1 or previously proved relations. Relations 5.17-5.20 follow from the same methods as in 5.8. For $q > 4$, we order the 2^{q+2} compositions of $q+3$ as

$$
\{(3,q),(1,2,q),(2,1,q),(1,1,1,q),(1,1,q,1),(2,q,1),(1,q,2),(1,q,1,1), (q,3),(q,1,2),(q,2,1),(q,1,1,1),\ldots\}.
$$

Writing the 12 relations in 5.9-5.20 as row vectors, we see the same lower-triangular pattern. $q = 2,3,4$ can easily be computed by hand.

6. General Relations

We give some examples of parameterized families of relations.

Theorem 6.1. *The following are true:*

$$\begin{aligned}
&(1)\quad L_1\zeta(k,1,q-1)+L_1\zeta(1+k,q-1)+\zeta(k,q)=0, \qquad k\le q-1\\
&(2)\quad L_1\zeta(1,2q-2)+\zeta(2q-1)=0\\
&(3)\quad \zeta(m,2q-1)+L_1\zeta(m+1,2q-2)+L_1\zeta(m,1,2q-2)=0, \qquad m+1\le q\\
&(4)\quad l_1^m\zeta(n,m,m(q-1))+l_1^m\zeta(n+m,m(q-1))-\zeta(n,m,q)=0, \qquad m+n\le q\\
&(5)\quad \zeta(2q)+\zeta(q,q)+L_1\zeta(1,2q-1)+L_1\zeta(1,q-1,q)=0\\
&(6)\quad \zeta(m,2q)+\zeta(m,q,q)+L_1\zeta(m+1,2q-1)+L_1\zeta(m+1,q-1,q)\\
&\qquad +L_1\zeta(m,1,2q-1)+L_1\zeta(m,1,q-1,q)=0, \qquad m+1\le q.
\end{aligned}$$

Proof of (1). For the first relation, we provide two proofs to show how the maps can sometimes simplify proofs.

We have that

$$\begin{aligned}
L_1S_d(2+k,q-1) &= L_1S_d(2+k)S_{<d}(q-1)\\
&= L_1\frac{1}{l_d^{k+2}}\frac{l_d}{l_1l_{d-1}^q}\\
&= -\frac{1}{l_d^{k+1}}\frac{1}{l_{d-1}^q}
\end{aligned}$$

and

$$\begin{aligned}
L_1S_d(1+k,1,q-1) &= L_1S_d(1+k)S_{<d}(1,q-1)\\
&= L_1S_d(1+k)\sum_{i=1}^{d-1}S_i(1)S_{<i}(q-1)\\
&= L_1\frac{1}{l_d^{k+1}}\sum_{i=1}^{d-1}\frac{1}{l_i}\frac{l_i}{l_1l_{i-1}^q}\\
&= \frac{1}{l_d^{k+1}}\left(\frac{-1}{l_0^q}+\frac{-1}{l_1^q}+\frac{-1}{l_2^q}+\cdots+\frac{-1}{l_{d-2}^q}\right)
\end{aligned}$$

and finally

$$\begin{aligned}
S_d(1+k,q) &= S_d(1+k)S_{<d}(q)\\
&= \frac{1}{l_d^{k+1}}\sum_{i=0}^{d-1}S_i(q)\\
&= \frac{1}{l_d^{k+1}}\left(\frac{1}{l_0^q}+\frac{1}{l_1^q}+\frac{1}{l_2^q}+\cdots+\frac{1}{l_{d-1}^q}\right).
\end{aligned}$$

Adding these together gives 0 and summing over $d\ge 0$ gives the result.

We now show that the relation in (1) is the relation associated to $\mathcal{B}_{(k)}(R_1)$:

$$\begin{aligned}
0 &= S_d(k) \sum_{i=0}^{d-1} \left(S_i(q) - l_1 S_{i+1}(1, q-1)\right) \\
&= S_d(k) \left(S_{<d}(q) - l_1 S_{<d}(1, q-1) - l_1 S_d(1, q-1)\right) \\
&= S_d(k, q) - l_1 S_d(k, 1, q-1) - l_1 S_d(k+1, q-1).
\end{aligned}$$

Summing over all d gives the relation in (1), as desired. □

Proof of (2). We compute

$$\begin{aligned}
S_d(2q-1) &= S_d(q + (q-1)) \\
&= \frac{1}{l_d^{2q-1}} \left(1 - (q-1)\frac{[d]^q}{[1]}\right) \\
&= \frac{1}{l_d^{2q-1}} \left(\frac{[d+1]}{[1]}\right)
\end{aligned}$$

and

$$\begin{aligned}
L_1 S_{d+1}(1, 2q-2) &= \frac{L_1}{l_{d+1}} \left(\frac{[d+1]^2}{[1]^2}\right) \left(\frac{1}{l_d^{2q-2}}\right) \\
&= \left(\frac{[d+1]}{[1]}\right) \left(\frac{-1}{l_d}\right) \left(\frac{1}{l_d^{2q-2}}\right).
\end{aligned}$$

The two are equal. Summing over $d \geq 0$ gives the result. □

Proof of (3). This is the image of (2) under the map $\mathcal{B}_{(m)}$. From (2), we have

$$S_d(2q-1) + L_1 S_{d+1}(1, 2q-2) = 0.$$

Applying $\mathcal{B}_{(m)}$ to this relation gives

$$\begin{aligned}
0 &= S_d(m) \sum_{i=0}^{d-1} \left(S_i(2q-1) - l_1 S_{i+1}(1, 2q-2)\right) \\
&= S_d(m) \left(S_{<d}(2q-1) - l_1 S_{<d}(1, 2q-2) - l_1 S_d(1, 2q-2)\right) \\
&= S_d(m, 2q-1) - l_1 S_d(m, 1, 2q-2) - l_1 S_d(m+1, 2q-2).
\end{aligned}$$

Summing over all d gives the relation in (3), as desired.

□

Proof of (4). This is the relation $\mathcal{B}_{(n)}(R_m)$ and follows from Theorem 3.24 applied to R_m.
□

Proof of (5). This is the relation $\mathcal{C}_{(q)}(R_1)$ and follows from a calculation similar to Theorem 4.5. □

Proof of (6). This is the image of (5) under $\mathcal{B}_{(m)}$ and follows from Theorem 3.24 applied to (5). □

7. Dimension Conjecture

We give the following conjecture for the dimensions of the $\mathbb{F}_q(t)$-span of multizeta values of fixed weight.

Conjecture 7.1. *Let* $g_k = dim\left(Span_{\mathbb{F}_q(t)}(MZV\text{'s of weight } k)\right)$. *Then*

$$g_k = \begin{cases} 2^{k-1} & \text{if } 1 \leq k < q \\ 2^{k-1} - 1 & \text{if } k = q \\ \sum_{i=1}^{q} g_{k-i} & \text{if } k > q. \end{cases}$$

Note that the generating function for g_k is

$$\sum_{n=0}^{\infty} g_n x^n = \frac{1 - x^q}{1 - x - \cdots - x^q}$$

where we define $g_0 = 1$.

By computing the spanning set in Conjecture 5.1, we can get an upper-bound for g_k. This has been done in Section 5 for $q \leq k \leq q + 3$ and with the help of sage for $q = 2, 3, 4, 5, 7, 8, 9, 11$ with $k \leq 11$. Noting that $\dim(\mathcal{Z}^k) + \dim(\mathcal{R}_k) = 2^{k-1}$, we have the following theorem:

Theorem 7.2. *For all q and all weights $k = q + w$ with $1 \leq q + w \leq \max(q + 3, 11)$, Conjecture 5.1 is true if and only if Conjecture 7.1 is true.*

8. Acknowledgements

The author wishes to thank his thesis advisor Dinesh Thakur for suggesting this topic and for helpful encouragement, as well as the University of Arizona, where most of this work took place. We are also greatly indebted to the anonymous referee for providing numerous corrections and suggestions which have greatly improved the original draft of this article.

[1] K. Ihara, M. Kaneko, D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compos. Math. 142 (2) (2006) 307–338.

[2] D. Zagier, Values of zeta functions and their applications, in: First European Congress of Mathematics, Vol. II (Paris, 1992), Vol. 120 of Progr. Math., Birkhäuser, Basel, 1994, pp. 497–512.

[3] H.-J. Chen, On shuffle of double zeta values over $\mathbb{F}_q[t]$, J. Number Theory 148 (2015) 153–163.

[4] D. S. Thakur, Power sums of polynomials over finite fields and applications: a survey, Finite Fields Appl. 32 (2015) 171–191.
[5] D. S. Thakur, Relations between multizeta values for $\mathbb{F}_q[t]$, Int. Math. Res. Not. IMRN (12) (2009) 2318–2346.
[6] L. Carlitz, On certain functions connected with polynomials in a Galois field, Duke Math. J. 1 (2) (1935) 137–168.
[7] D. S. Thakur, Function field arithmetic, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
[8] D. S. Thakur, Shuffle relations for function field multizeta values, Int. Math. Res. Not. IMRN (11) (2010) 1973–1980.
[9] J. A. Lara Rodríguez, D. S. Thakur, Zeta-like multizeta values for $\mathbb{F}_q[t]$, Indian J. Pure Appl. Math. 45 (5) (2014) 787–801.
[10] Y.-L. Kuan, Y.-H. Lin, Criterion for deciding zeta-like multizeta values in positive characteristic, Exp. Math. 25 (3) (2016) 246–256. doi:10.1080/10586458.2015.1069228.
URL `http://dx.doi.org/10.1080/10586458.2015.1069228`
[11] C.-Y. Chang, Linear relations among double zeta values in positive characteristic, Camb. J. Math. 4 (3) (2016) 289–331. doi:10.4310/CJM.2016.v4.n3.a1.
URL `http://dx.doi.org/10.4310/CJM.2016.v4.n3.a1`
[12] C.-Y. Chang, M. A. Papanikolas, J. Yu, An effective criterion for eulerian multizeta values in positive characteristic, To appear in Journal of the European Mathematical Society.
[13] R. Harada, On lara rodríguez' full conjecture for double zeta values in function fields, To appear in International Journal of Number Theory.
[14] H.-J. Chen, Anderson-thakur polynomials and multizeta values in positive characteristic, To appear in Asian Journal of Mathematics.