SCHEMES: SPACES FOR ALGEBRAIC GEOMETRY

By

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ABSTRACT

We give an exposition of basic scheme theory.

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CONVENTIONS

We shall use the following conventions throughout this paper.

The set $\mathbf{N}$ contains 0.

All rings are unital and commutative.

Contents

ABSTRACT ii
ACKNOWLEDGEMENT iii
CONVENTIONS iv
Introduction 1
1. Sheaves of Rings 2
1.1. Presheaves and Sheaves 2
1.2. Sheafification and Stalks 4
1.3. Sheaves on a Base 8
1.4. Direct and Inverse Images 11
1.5. Ringed and Local Ringed Spaces 17
2. The Spectrum of a Ring 22
2.1. The Underlying Space 23
2.2. The Sheaf of Rings 28
2.3. Maps into the Spectrum 34
2.4. Affine Space over a Ring 37
3. Schemes as Local Ringed Spaces 40
3.1. Maps Between Schemes 41
3.2. Gluing Schemes 52
3.3. Projective Space over a Ring 59
4. Schemes as Functors 68
4.1. The Topology of Functors 71
4.2. Pullbacks of Schemes 80
4.3. Maps into Projective Space 83
Appendix A. Category Theory 92

Appendix B. Commutative Algebra 104

References 110

Introduction

In mathematics, there are many notions of space serving various purposes. The most obvious notion for algebraic geometry is that of an *algebraic variety*. However the theory of algebraic varieties has the limitation that it only makes sense over a fixed field K (and only really works if K is algebraically closed). In 1960, Alexander Grothendieck proposed an alternative notion of space: that of a *scheme*. The theory of schemes contains within it the theory of varieties, but works over an arbitrary ring. In this paper, we develop the basic theory of schemes, with the aim of justifying that schemes are good spaces for algebraic geometry.

To define a scheme, we need the concept of a *sheaf of rings*, which we study in Section 1. In Section 2, we see our first example of a scheme: the *spectrum* of a ring. We define schemes in general in Section 3. In Section 4 we give an equivalent definition of scheme (also due to Grothendieck) that simplifies many constructions.

It will be necessary that the reader have some knowledge of category theory and commutative algebra; we review the relevant definitions in the appendix. A knowledge with classical algebraic geometry is not required, but would be helpful for motivation.

The author learned the material presented here through the book [1] of Eisenbud and Harris and the online textbook [2]. Whereas this paper focuses on the foundations of scheme theory, [1] shows what doing algebraic geometry with schemes actually looks like. The author used [2] to fill in some of the details that [1] glosses over.

1. Sheaves of Rings

Classically, affine varieties over a field K are dual to certain algebras over K: an affine variety X corresponds to its coordinate ring $\Gamma(X)$ of rational functions from X to the affine line $\mathbf{A}^1_K$. An arbitrary variety is not determined by its coordinate ring, however: for example there is an isomorphism $\Gamma(\mathbf{P}^1_{\mathbf{C}}) \cong \mathbf{C}$ as every rational function on the Riemann Sphere is constant. This can be solved by keeping track of not just the globally defined rational functions, but those on all open subsets U of the variety. The structure that encodes this data is a sheaf of rings.

Taking inspiration from this, in Section 3 we *define* a scheme to be a topological space X together with a sheaf of rings $\mathcal{F}$ on X (satisfying some requirements) and interpret the values of $\mathcal{F}$ on an open subset U of X as the ring of rational functions $U \to \mathbf{A}^1$. In this section, we outline the sheaf theory we will need.

1.1. **Presheaves and Sheaves.**

Definition 1.1. Let X be a topological space. We write $\Omega(X)$ for the poset of open subsets of X under inclusion as in Example A.5. We write $\Omega(X)^{\mathrm{op}}$ for its opposite category (Example A.6), or equivalently the category obtained from the opposite order on $\Omega(X)$.

Definition 1.2. Let X be a topological space. A *presheaf of rings* on X is a functor $\mathcal{F} : \Omega(X)^{\mathrm{op}} \to \mathbf{Ring}$ (Definition A.15).

Remark 1.3. Let X be a topological space. Explicitly, a presheaf of rings on X assigns to each open subset U of X a ring $\mathcal{F}(U)$ and to each pair of open

subsets U, V of X with $V \subset U$ homomorphism of rings $\mathcal{F}(U) \to \mathcal{F}(V)$. We call this homomorphism a restriction map and write its value on $x \in \mathcal{F}(U)$ as $x|_V$. The restriction maps must satisfy the following requirements:

- For all open subsets U of X and $x \in \mathcal{F}(U)$, $x|_U = x$.
- For all open subsets U, V, W of X such that $W \subset V$ and $V \subset U$ and $x \in \mathcal{F}(U)$, $(x|_V)|_W = x|_W$.

Example 1.4. For X be a topological space, we define a presheaf of rings $C(X, \mathbf{R})$ on X as follows:

- For an open subset U of X, $C(X, \mathbf{R})(U)$ is the ring of continuous functions from U to $\mathbf{R}$.
- For open subsets U, V of X such that $V \subset U$, the restriction map $C(X, \mathbf{R})(U) \to C(X, \mathbf{R})(V)$ is given by restriction of functions.

Definition 1.5. Let X be a topological space and $\mathcal{F}$ be a presheaf of rings on X. For U an open subset of X and $(U_i)_{i \in I}$ an open covering of U, we say the *sheaf condition* holds for the covering $(U_i)_{i \in I}$ if for all families $(f_i \in \mathcal{F}(U_i))_{i \in I}$ such that for all $i, j \in I$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$ for all $i \in I$. We say $\mathcal{F}$ is a *sheaf of rings* or simply a sheaf if the sheaf condition holds for all open coverings.

Remark 1.6. Let X be a topological space, $\mathcal{F}$ a presheaf of rings on X, U an open subset of X and $(U_i)_{i \in I}$ an open covering of U. For $f \in \mathcal{F}(U)$, and $i, j \in I$, we have $f|_{U_i}|_{U_i \cap U_j} = f|_{U_i \cap U_j} = f|_{U_j}|_{U_i \cap U_j}$. It follows that the sheaf condition for $(U_i)_{i \in I}$ amounts to two separate assertions:

(existence): For all families $(f_i \in \mathcal{F}(U_i))_{i \in I}$ such that for all $i, j \in I$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$ for all $i \in I$.

(uniqueness): For all $f, f' \in \mathcal{F}(U)$, if $f|_{U_i} = f'|_{U_i}$ for all $i \in I$ then $f = f'$.

Example 1.7. For X a topological space, the presheaf of rings $C(X, \mathbf{R})$ of Example 1.4 is a sheaf of rings: this follows from the gluing lemma of topology.

Definition 1.8. Let X be a topological space. Presheaves on X form a category $\mathbf{PSh}(X) = [\Omega(X)^{\mathrm{op}}, \mathbf{Ring}]$ (Definition A.24). Sheaves on X form a full subcategory of $\mathbf{Sh}(X)$ of $\mathbf{PSh}(X)$ (Definition A.7).

Remark 1.9. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ presheaves of rings on X. Explicitly, a map of presheaves of rings $\alpha : \mathcal{F} \to \mathcal{G}$ assigns to each open subset U of X a homomorphism of rings $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for all open subsets U, V with $V \subset U$ and $f \in \mathcal{F}(U)$, we have $\alpha_U(f)|_V = \alpha_V(f|_V)$.

Example 1.10. Write 1 for the topological space with one point. Let F be a presheaf on 1. The sheaf condition for an open covering $(U_i)_{i \in I}$ says $F(\varnothing)$ is the zero ring if I is empty and is automatically satisfied otherwise. It follows that the functor $\mathcal{F} \mapsto \mathcal{F}(1) : \mathbf{Sh}(1) \to \mathbf{Ring}$ is an equivalence of categories (Definition A.27).

1.2. **Sheafification and Stalks.** Let X be a topological space and $\mathcal{F}$ a presheaf of rings on X. In this section we give a construction for freely making $\mathcal{F}$ into a sheaf of rings $\mathcal{F}^\sharp$ on X. In the language of adjoint functors (Lemma A.51), this means the following.

Proposition 1.11. *Let X be a topological space. The inclusion $\mathbf{Sh}(X) \to \mathbf{PSh}(X)$ admits a left adjoint $\mathbf{PSh}(X) \to \mathbf{Sh}(X)$ called sheafification. We write the $\mathcal{F}^\sharp$ for the image of a presheaf of rings $\mathcal{F}$ on X.*

The proof of Proposition 1.11 will use the notion of the stalk of a presheaf of rings, which will be important in its own right.

Definition 1.12. Let X be a topological space, $\mathcal{F}$ a presheaf of rings on X and $x \in X$. The *stalk* of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$, is the ring $\operatorname{colim}_{x \in U} \mathcal{F}(U)$, where the colimit is over the opposite of the poset of open neighborhoods U of x (see Example A.36).

Remark 1.13. Let X be a topological space, $\mathcal{F}$ a presheaf of rings on X and $x \in X$. Explictly, a point of $\mathcal{F}_x$ is an equivalence class $[(U, f)]$ for U an open neighborhood of x and $f \in \mathcal{F}(U)$, which we write as f_x. For open neighborhoods U and U' of x, $f \in \mathcal{F}(U)$ and $f' \in \mathcal{F}(U')$, f_x and f'_x are equal if and only if theree an open neighborhood U'' of x with $U'' \subset U \cap U'$ such that $f|_{U''} = f'|_{U''}$.

Example 1.14. Let X be a topological space and $x \in X$. The stalk of $C(X, \mathbf{R})$ at x is commonly referred to as the *germ* of continuous functions on X at x. It consists of real valued functions defined on a neighborhood of X, with two functions regarded as equal if they agree on a sufficiently small neighborhood of x.

Definition 1.15. Let X be a topological space and $\mathcal{F}$ a presheaf of rings on X. For an open subset U of X, we define $\mathcal{F}^\sharp(U)$ to be the subring of $\prod_{x \in U} \mathcal{F}_x$ consisting of $f \in \prod_{x \in U} \mathcal{F}_x$ such that for all $x \in U$, there exists an open neighborhood U' of x contained in U and $f' \in \mathcal{F}(U')$ such that $f(x') = f'_{x'}$ for all $x' \in U'$.

Remark 1.16. Let X be a topological space, $\mathcal{F}$ a presheaf of rings on X, U an open subset of X and $f \in \prod_{x \in U} \mathcal{F}_x$. For each $x \in U$, $f(x)$ may be written as

f'_x for U' a neighborhood of x and $f' \in \mathcal{F}(U')$. To say f belongs to $\mathcal{F}^\sharp(U)$ is to say we can cover U by opens $(U_i)_{i\in I}$ such that for each $i \in I$, we can choose a single $f_i \in \mathcal{F}(U_i)$ which satisfies $f(x) = (f_i)_x$ for all $x \in U_i$.

Lemma 1.17. *Let X be a topological space, $\mathcal{F}$ a presheaf of rings on X and U, V open subsets of X such that $V \subset U$. The restriction map $\prod_{x\in U} \mathcal{F}_x \to \prod_{x\in V} \mathcal{F}_x$ carries $\mathcal{F}^\sharp(U)$ to $\mathcal{F}^\sharp(V)$.*

Proof. Let $f \in \mathcal{F}^\sharp(U)$ and fix $x \in V$. By assumption, we may choose an open neighborhood U' of x contained in U and $f' \in \mathcal{F}(U')$ so that $f(x) = f'_{x'}$ for all $x' \in U$. Then $U' \cap V$ is a neighborhood of x contained in V, $f'|_{U'\cap V} \in \mathcal{F}(U' \cap V)$ and $f|_V(x') = f(x') = f'_{x'} = (f'|_{U\cap V})_{x'}$ for all $x' \in U' \cap V$. Thus $f|_V \in \mathcal{F}^\sharp(V)$. □

Definition 1.18. Let X be a topological space and $\mathcal{F}$ a presheaf of rings on X. It follows from Lemma 1.17 that $U \mapsto \mathcal{F}^\sharp(U)$ determines a presheaf of rings $\mathcal{F}^\sharp$ on X.

Lemma 1.19. *Let X be a topological space and $\mathcal{F}$ a presheaf of rings on X. The presheaf of rings $\mathcal{F}^\sharp$ of Definition 1.18 is a sheaf.*

Proof. Let U be an open subset of X and $(U_i)_{i\in I}$ an open covering of U; we show the sheaf condition holds for the covering $(U_i)_{i\in I}$. Let $(f_i \in \mathcal{F}^\sharp(U_i))_{i\in I}$ be a family such that for all $i, j \in I$, $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. It is immediate that there is a unique $f \in \prod_{x\in U} \mathcal{F}_x$ with $f|_{U_i} = f_i$ for all $i \in I$. To conclude, we show $f \in \mathcal{F}^\sharp(U)$. Fix $x \in U$. Since $(U_i)_{i\in I}$ covers U, we may choose $i \in I$ with $x \in U_i$. Since $f|_{U_i} = f_i$ is in $\mathcal{F}^\sharp(U_i)$, there is an open neighborhood U' of x contained in U_i and $f \in \mathcal{F}(U')$ such that $f(x') = f|_{U_i}(x') = f'_{x'}$ for all $x' \in U'$. These are the desiderata. □

Definition 1.20. Let X be a topological space and $\mathcal{F}$ a presheaf of rings on X. For an open subset U of X, we define $\eta_U : \mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ by $\eta_U(f)(x) = f_x$ for $f \in \mathcal{F}(U)$ and $x \in U$. Letting U vary, we obtain a map of presheaves of rings $\eta : \mathcal{F} \to \mathcal{F}^\sharp$.

Lemma 1.21. *Let X be a topological space and $\mathcal{F}$ a sheaf of rings on X. The map $\eta : \mathcal{F} \to \mathcal{F}^\sharp$ is an isomorphism of sheaves of rings.*

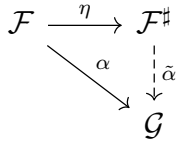
Proof. Let U be an open subset of X; we must show $\eta_U : \mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ is a bijection.

(injectivity): Let $f, f' \in \mathcal{F}(U)$ and suppose $\eta_U(f) = \eta_U(f')$, that is $f_x = f'_x$ for all $x \in U$. For a fixed $x \in U$, this means there is a neighborhood $U' \subset U$ such that $f|_{U'} = f'|_{U'}$ (see Remark 1.13). Thus $\{U' \in \Omega(X) \mid U' \subset U \text{ and } f|_{U'} = f'|_{U'}\}$ is an open cover of U. By the sheaf condition, $f = f'$.

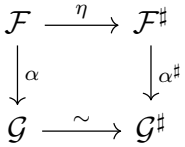
(surjectivity): Let $f \in \mathcal{F}^\sharp(U)$. By Remark 1.16, there is a covering $(U_i)_{i\in I}$ of U by opens and family $(f_i \in \mathcal{F}(U_i))_{i\in I}$ so that $f(x) = (f_i)_x$ for all $i \in I$. Fix $i, j \in I$; we claim $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. This follows from injectivity of $\mathcal{F}(U_i \cap U_j) \to \mathcal{F}^\sharp(U_i \cap U_j)$ proved above. Thus the sheaf condition provides $\bar{f} \in \mathcal{F}(U)$ with $\bar{f}|_{U_i} = f_i$ for all $i \in I$. To conclude, we show $\eta_U(\bar{f}) = f$. Fix $x \in X$. Since $(U_i)_{i\in I}$ covers U, we may choose $i \in I$ with $x \in U_i$. Then $\eta_U(\bar{f})(x) = \bar{f}_x = (\bar{f}|_{U_i})_x = (f_i)_x = f(x)$. □

Proof of Proposition 1.11. For $\mathcal{F}$ be a presheaf of rings on X, let $\mathcal{F}^\sharp$ be the presheaf of rings on X of Definition 1.18 which is a sheaf by Lemma 1.19. It is clear that the assignment of $\mathcal{F}$ to $\mathcal{F}^\sharp$ is functorial and that the maps $\eta : \mathcal{F} \to \mathcal{F}^\sharp$ for $\mathcal{F}$ a presheaf of rings are natural in $\mathcal{F}$. Now fix a presheaf of rings $\mathcal{F}$ on X, a sheaf of rings $\mathcal{G}$ on X and $\alpha : \mathcal{F} \to \mathcal{G}$. We wish to show

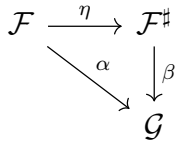
there is a unique map of sheaves $\tilde{\alpha} : \mathcal{F}^\sharp \to \mathcal{G}$ making the following diagram commute.



Existence follows from the commutative diagram



where the bottom map is an isomorphism by Lemma 1.21. For uniqueness, suppose β makes



commute. To show $\beta = \tilde{\alpha}$, fix an open subset U of X and $f \in \mathcal{F}^\sharp(U)$. We shall prove $\beta_U(f) = \tilde{\alpha}_U(f)$. By Remark 1.16, there is a covering $(U_i)_{i\in I}$ of U by opens and family $(f_i \in \mathcal{F}(U_i))_{i\in I}$ such that $f|_{U_i} = \eta_{U_i}(f_i)$ for all $i \in I$. Thus $(\beta_U(f))|_{U_i} = \beta_{U_i}(f|_{U_i}) = \beta_{U_i}(\eta_{U_i}(f_i)) = \alpha_{U_i}(f_i)$. Similarly, $(\tilde{\alpha}_U(f))|_{U_i} = \alpha_{U_i}(f_i)$. The sheaf condition finishes the proof. □

1.3. **Sheaves on a Base.** Let X be a topological space and $\mathcal{B}$ be a base for X. For our purposes, it will suffice to assume $\mathcal{B}$ is closed under binary intersections. Fix an open subset U of X. Since $\mathcal{B}$ is a base, we may cover U by open sets $(U_i)_{i\in I}$ in $\mathcal{B}$. The sheaf condition ensures that $\mathcal{F}(U)$ is determined by the $\mathcal{F}(U_i)$, $\mathcal{F}(U_i \cap U_j)$, and the restriction maps between them: it consists

of families $(f_i \in \mathcal{F}(U_i))_{i \in I}$ such that for all $i, j \in I$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. In this sense, $\mathcal{F}$ is determined by its values on $\mathcal{B}$. With some definitions, we can say something stronger.

Definition 1.22. Let X be a topological space and $\mathcal{B}$ be a base for X closed under binary intersections. A *presheaf of rings* on $\mathcal{B}$ is a functor $\mathcal{B}^{\mathrm{op}} \to \mathbf{Ring}$, where $\mathcal{B}$ is viewed as a full subcategory of $\Omega(X)$. A *sheaf of rings* on $\mathcal{B}$ is a presheaf of rings on $\mathcal{B}$ satisfying the analog of the sheaf condition for coverings of $U \in \mathcal{B}$ by $U_i \in \mathcal{B}$ (here we use that $\mathcal{B}$ is closed under binary intersections). We denote the category of sheaves of rings on $\mathcal{B}$ by $\mathbf{Sh}(\mathcal{B})$.

Definition 1.23. Let X be a topological space and $\mathcal{B}$ be a base for X closed under binary intersections. It is immediate that we obtain a functor $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$ by restriction to $\mathcal{B}$. We denote the value of this functor on a sheaf of rings $\mathcal{F}$ on X by $\mathcal{F}|_{\mathcal{B}}$.

Proposition 1.24. *Let X be a topological space and $\mathcal{B}$ be a base for X closed under binary intersections. The functor $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$ is an equivalence of categories.*

Proof. Let $\mathcal{F}$ be a sheaf of rings on $\mathcal{B}$ and $x \in X$. We define $\mathcal{F}_x$ to be the colimit $\mathrm{colim}_{x \in U \in \mathcal{B}} \mathcal{F}(U)$ where the colimit is over the opposite of the poset of open subsets U of X containing x and in $\mathcal{B}$. Note that, for $\mathcal{G}$ is a sheaf of rings on X we have an isomorphism $(\mathcal{G}|_{\mathcal{B}})_x \cong \mathcal{G}_x$, where the right term is the usual stalk of $\mathcal{G}$ at x (Definition 1.12). This follows from the fact that the poset of open subsets of X containing x and in $\mathcal{B}$ is cofinal in the poset of open subsets of X containing x.

For $\mathcal{F}$ is a sheaf of rings on $\mathcal{B}$ and U an open subset of X (not necessarily in $\mathcal{B}$), we define $\mathcal{F}^\sharp(U)$ to be the subring of $\prod_{x\in U}\mathcal{F}_x$ consisting of $f \in \prod_{x\in U}\mathcal{F}_x$ such that for all $x \in U$, there is an open neighborhood U' of x contained in U and in $\mathcal{B}$ and $f' \in \mathcal{F}(U')$ such that $f(x') = f'_{x'}$ for all $x' \in U'$. By a similar proof to Lemma 1.17, the restriction map $\prod_{x\in U}\mathcal{F}_x \to \prod_{x\in V}\mathcal{F}_x$ carries $\mathcal{F}^\sharp(U)$ to $\mathcal{F}^\sharp(V)$ for U, V open subsets of X such that $V \subset U$. Thus we obtain a presheaf of rings $\mathcal{F}^\sharp$ on X. By a similar proof to Lemma 1.19, $\mathcal{F}^\sharp$ determines is a sheaf. We take the inverse functor $\mathbf{Sh}(\mathcal{B}) \to \mathbf{Sh}(X)$ to be the evident functor assigning a sheaf of rings $\mathcal{F}$ on $\mathcal{B}$ to $\mathcal{F}^\sharp$.

For $\mathcal{G}$ a sheaf of rings on X, there is an isomorphism $(\mathcal{G}|_\mathcal{B})^\sharp \cong \mathcal{G}^\sharp$, where the right term is the sheafification of $\mathcal{G}$ (Definition 1.18). This follows from the observation above and the fact that $\mathcal{B}$ is a base. Lemma 1.21 provides an isomorphism $\mathcal{G}^\sharp \cong \mathcal{G}$, yielding $(\mathcal{G}|_\mathcal{B})^\sharp \cong \mathcal{G}$. Naturality in $\mathcal{G}$ is clear.

For $\mathcal{F}$ a sheaf of rings on $\mathcal{B}$, we may define a map $\mathcal{F} \to (\mathcal{F}^\sharp)|_\mathcal{B}$ as in Definition 1.20. A similar argument to Lemma 1.21 shows this map is an isomorphism. Naturality in $\mathcal{F}$ is clear. □

Remark 1.25. Let X be a topological space and $\mathcal{B}$ be a base for X closed under binary intersections. By Lemma A.28, the assertion that $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$ is an equivalence of categories is tantamount to the assertions that $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$ is essentially surjective, full and faithful. Essential surjectivity says we may define a sheaf on X by specifying its values on $\mathcal{B}$ and checking the sheaf axiom for these values. Fullness says we may define a map of sheaves on X by defining its values on $\mathcal{B}$ and checking naturality for these values. Faithfulness says we may show two maps are equal by showing they are equal on $\mathcal{B}$.

1.4. **Direct and Inverse Images.** So far, we have fixed a topological space X and studied its category of sheaves $\mathbf{Sh}(X)$. In this section, we investigate the relationship between the categories $\mathbf{Sh}(X)$ as X varies.

Definition 1.26. Let X and Y be topological spaces and $f : X \to Y$ a continuous map, so f induces an order-preserving map $\Omega(f) : \Omega(Y) \to \Omega(X)$ by $\Omega(f)(U) = f^{-1}(U)$. Viewing $\Omega(f)$ as a functor $\Omega(Y)^{\mathrm{op}} \to \Omega(X)^{\mathrm{op}}$, we obtain a restriction functor $\mathbf{PSh}(X) \to \mathbf{PSh}(Y)$ by precomposition. It is easy to see that this carries sheaves on Y to sheaves on X and so it gives a functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$, which we call the *direct image* functor associated to f.

Remark 1.27. Let X and Y be topological spaces, $f : X \to Y$ a continuous map and $\mathcal{F}$ a sheaf of rings on X. Explicitly, the sheaf $f_*\mathcal{F}$ on Y is given by the following prescription:

- For an open subset V of Y, $(f_*\mathcal{F})(V)$ is the ring $\mathcal{F}(f^{-1}(V))$
- For open subsets V, W of Y such that $V \subset W$, the restriction map $(f_*\mathcal{F})(V) \to (f_*\mathcal{F})(W)$ is the restriction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(W))$.

Example 1.28. Let X be a topological space. The direct image functor associated to the unique continuous map $X \to 1$ corresponds under the equivalence $\mathbf{Sh}(1) \xrightarrow{\sim} \mathbf{Ring}$ of Example 1.10 to the *global sections* functor $\Gamma : \mathbf{Sh}(X) \to \mathbf{Ring}$ given by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$ for $\mathcal{F} \in \mathbf{Sh}(X)$.

Remark 1.29. Let X be a topological space. The direct image functor associated to 1_X is the identity functor $\mathbf{Sh}(X) = \mathbf{Sh}(X)$. In symbols, $(1_X)_* = 1_{\mathbf{Sh}(X)}$.

Remark 1.30. Let X, Y and Z be topological spaces, $f : X \to Y$ and $g : Y \to Z$ be continuous maps. The direct image functor asociated to the composite $g \circ f : X \to Z$ is the composite fo the direct image functors associated to g and f. In symbols, $(g \circ f)_* = g_* \circ f_*$.

Let X and Y be topological spaces and $f : X \to Y$ a continuous map. Definition 1.26 provides a functor $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. Less obviously, there is a natural functor going in the other direction.

Proposition 1.31. *Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The direct image functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ has a right adjoint $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ called the inverse image functor associated to $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a sheaf of rings on Y. Define a presheaf of rings $f_!\mathcal{F}$ on X as follows:

- For an open subset U of X, $(f_!\mathcal{F})(U)$ is the colimit $\operatorname{colim}_{f(U)\subset V}\mathcal{F}(V)$ where the colimit is over the opposite of the poset of open subsets V of Y containing $f(U)$ (or equivalently such that $f^{-1}(V)$ contains U).
- For U, U' open subsets of X such that $U \subset U'$, the restriction map $(f_!\mathcal{F})(U') \to (f_!\mathcal{F})(U)$ is the unique homomorphism making the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \operatorname{colim}_{f(U')\subset V}\mathcal{F}(V) \\ & \searrow & \downarrow \\ & & \operatorname{colim}_{f(U)\subset V}\mathcal{F}(V) \end{array}$$

  commute for all open subsets V of Y containing $f(U')$.

The presheaf $\mathcal{F}$ is not necessarily a sheaf, so we first sheafify (Proposition 1.11), defining $f^*\mathcal{F}$ to be $(f_!\mathcal{F})^\sharp$. It is not hard to see that the assignment

of $\mathcal{F}$ to $f^*\mathcal{F}$ is functorial. Let $\mathcal{G}$ be a sheaf on X. We show that maps of sheaves on X from $f^*\mathcal{F}$ to $\mathcal{G}$ correspond naturally to maps of sheaves on Y from $\mathcal{F}$ to $f_*\mathcal{G}$. By Proposition 1.11, the former correspond to maps of presheaves on X from $f_!\mathcal{F}$ to $\mathcal{G}$. Unwinding the definitions, such a map corresponds to, for each open subset U of X and open subset V of Y containing $f(U)$, a homomorphism of rings $\mathcal{F}(V) \to \mathcal{G}(U)$ satisfying the following two conditions:

- For all open subsets U of X, open subsets V of Y containing $f(U)$ and open subsets V' of Y containing V, the diagram

$$\begin{array}{ccc} \mathcal{F}(V') & & \\ \downarrow & \searrow & \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(U) \end{array}$$

  commutes.
- For all open subsets U' of X, open subsets V of Y containing $f(U')$ and open subsets U of X contained in U', the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{G}(U') \\ & \searrow & \downarrow \\ & & \mathcal{G}(U) \end{array}$$

  commutes.

If we have this data, we get for each open subset V of Y a homomorphism of rings $\mathcal{F}(V) \to \mathcal{G}(f^{-1}(V))$. For open subsets V, V' of Y such that $V \subset Y'$, the two triangles in

$$\begin{array}{ccc} \mathcal{F}(V') & \longrightarrow & \mathcal{G}(f^{-1}(V')) \\ \downarrow & \searrow & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(f^{-1}(V)) \end{array}$$

commute, showing this is a map of sheaves $\mathcal{F} \to f_*\mathcal{G}$.

Conversely, if we have a map of sheaves $\mathcal{F} \to f_*\mathcal{G}$, U is an open subset of X and V an open subset of Y containing $f(U)$, the composite $\mathcal{F}(V) \to \mathcal{G}(f^{-1}(V)) \to \mathcal{G}(U)$ is a homomorphism of rings $\mathcal{F}(V) \to \mathcal{G}(U)$. For U is an open subset of X, V an open subset of Y containing $f(U)$ and V' an open subset of Y containing V, the square and the triangle in the diagram

$$\begin{array}{ccccc} \mathcal{F}(V') & \longrightarrow & \mathcal{G}(f^{-1}(V')) & & \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(f^{-1}(V)) & \longrightarrow & \mathcal{G}(U) \end{array}$$

commute by naturality and functoriality of $\mathcal{G}$, respectively, giving the first condition. For U' is an open subset of X, V an open subset of Y and U an open subset of X contained in U',

$$\begin{array}{ccccc} \mathcal{F}(V') & \longrightarrow & \mathcal{G}(f^{-1}(V')) & & \\ & & \downarrow & \searrow & \\ & & \mathcal{G}(U) & \longrightarrow & \mathcal{G}(U') \end{array}$$

commutes by functoriality of $\mathcal{G}$, for the second condition. Thus we obtain a map $f_!\mathcal{F} \to \mathcal{G}$. If we begin with a map $f_!\mathcal{F} \to \mathcal{G}$, make it into a map $\mathcal{F} \to f_*\mathcal{G}$ and make that into a map $f_!\mathcal{F} \to \mathcal{G}$, the following commutative diagram for U an open subset of X and V and open subset of Y contained in V shows we

recover the original map.

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{G}(f^{-1}(V)) \\ & \searrow & \downarrow \\ & & \mathcal{G}(U) \end{array}$$

If we begin with a map $\mathcal{F} \to f_*\mathcal{G}$, make it into a map $f_!\mathcal{F} \to \mathcal{G}$ and make that into a map $\mathcal{F} \to f_*\mathcal{G}$, the following diagram for V an open subset of Y

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{G}(f^{-1}(V)) \\ & \searrow & \downarrow \\ & & \mathcal{G}(f^{-1}(V)) \end{array}$$

commutes by functoriality of $\mathcal{G}$, showing we recover the original map. The correspondence defined above is natural in $\mathcal{F}$ and $\mathcal{G}$. □

Warning 1.32. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The functor f_* (resp. f^*) is called a direct image functor (resp. an inverse image functor) because it goes in the same direction (resp. the opposite direction) of f, even though it is defined using inverse images of open sets (resp. direct images of open sets).

We analyze two special cases of Proposition 1.31.

Example 1.33. Let X be a topological space and $x \in X$, corresponding to a continuous map $x : 1 \to X$. The inverse image functor associated to the map x corresponds under the equivalence $\mathbf{Sh}(1) \xrightarrow{\sim} \mathbf{Ring}$ of Example 1.10 to the stalk functor $\mathbf{Sh}(X) \to \mathbf{Ring}$ assigning $\mathcal{F}$ to $\mathcal{F}_x$. This follows from the formula for $x_!$ in the proof of Proposition 1.31.

Example 1.34. Let X be a topological space and U an open subset of X, regarded as a subspace of X. The inclusion $i : U \hookrightarrow X$ induces an order-preserving map $\Omega(U) \to \Omega(X)$. Viewing this as a functor $\Omega(U)^{\mathrm{op}} \to \Omega(X)^{\mathrm{op}}$, we obtain a restriction functor $\mathbf{PSh}(X) \to \mathbf{PSh}(U)$. It is immediate that this carries sheaves on X to sheaves on U and so it gives a functor $\mathbf{Sh}(X) \to \mathbf{Sh}(U)$. We call this functor *restriction* to U and write the image of $\mathcal{F}|_U$ for the image of a sheaf on X under it.

Let $\mathcal{F}$ be a sheaf of rings on X and V an open subset of U. Then V is the least element of the poset of open subsets of X containing $i(V) = V$, hence we obtain an isomorphism $(i_!\mathcal{F})(V) \cong \mathcal{F}(V)$. Letting V vary we obtain an isomorphism between $i_!\mathcal{F}$ and the restriction $\mathcal{F}|_U$. Since $\mathcal{F}|_U$ is a sheaf, this yields an isomorphism $i^*\mathcal{F} \cong i_!\mathcal{F} \cong \mathcal{F}|_U$. Letting $\mathcal{F}$ vary, we have an isomorphism between i^* and the restriction to U functor.

Let $\mathcal{F}$ be a sheaf of rings on X. We note for later use that the isomorphism $i^*\mathcal{F} \cong \mathcal{F}|_U$ defined above corresponds as in Proposition 1.31 to the map $\mathcal{F} \to i_*(\mathcal{F}|_U)$ with value $\mathcal{F}(V) \to \mathcal{F}(V \cap U)$ for V an open subset of X.

Remark 1.35. Let X be a topological space. Combining Remark 1.29, the fact that $1_{\mathbf{Sh}(X)}$ is left adjoint to itself and Example A.52, we obtain an isomorphism $(1_X)^* \cong 1_{\mathbf{Sh}(X)}$.

Remark 1.36. Let X, Y and Z be topological spaces, $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Combining Remark 1.30, the fact that $f^* \circ g^* : \mathbf{Sh}(\mathbf{Z}) \to \mathbf{Sh}(\mathbf{X})$ is left adjoint to $g_* \circ f_* : \mathbf{Sh}(\mathbf{X}) \to \mathbf{Sh}(\mathbf{Z})$ and Example A.52, we obtain an isomorphism $(g \circ f)^* \cong f^* \circ g^*$.

Example 1.37. Let X be a topological space, $V \subset U$ an inclusion of open subsets of X and $\mathcal{F}$ a sheaf of rings on X. Then the diagram

$$\begin{array}{ccc} (V \hookrightarrow U)^*(U \hookrightarrow X)^*\mathcal{F} & \xrightarrow{\sim} & (V \hookrightarrow X)^*\mathcal{F} \\ \downarrow\sim & & \downarrow\sim \\ \mathcal{F}|_U|_V & = & \mathcal{F}|_V \end{array}$$

commutes, where the vertical isomorphisms are from Example 1.34 and the top horizontal isomorphism is from Remark 1.36. By adjointness, this reduces to commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & (V \hookrightarrow X)_*(\mathcal{F}|_V) \\ \downarrow & & \| \\ (U \hookrightarrow X)_*(\mathcal{F}|_U) & \longrightarrow & (U \hookrightarrow X)_*(V \hookrightarrow U)_*((\mathcal{F}|_U)|_V) \end{array}$$

This is immediate.

1.5. **Ringed and Local Ringed Spaces.** Recall that our goal is to define a scheme as a topological space X equipped with a sheaf of rings $\mathcal{F}$ on X (satisfying some requirements). We first assemble the collection of all such objects into a category.

Definition 1.38. A *ringed space* is a topological space X equipped with a sheaf of rings $\mathcal{F}$ on X. We abuse notation and write X for both the underlying topological space and the ringed space itself and write $\mathcal{O}_X$ for the sheaf of rings $\mathcal{F}$ on X.

Example 1.39. Let X is a topological space. We may promote X to ringed space by defining $\mathcal{O}_X$ to be $C(X, \mathbf{R})$.

Example 1.40. Let X be a ringed space and U an open subset of X. We may promote the open subspace U of X to be a ringed space by defining $\mathcal{O}_U$ to be $\mathcal{O}_X|_U$ (see Example 1.34). We call U an open ringed subspace of X.

Definition 1.41. Let X and Y be ringed spaces. A *map* of ringed spaces from X to Y consists of a continuous map $f : X \to Y$ together with a map of sheaves on Y from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$. By Proposition 1.31, the latter is equivalent to a map of sheaves on X from $f^*\mathcal{O}_Y$ to $\mathcal{O}_X$. As in Definition 1.38, we abuse notation by writing f for both the underlying map of topological spaces and map of ringed spaces, while leaving the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ (and the map $f^*\mathcal{O}_Y \to \mathcal{O}_X$) unnamed.

Example 1.42. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. We may promote f to a map of ringed spaces between the ringed spaces of Example 1.39 by defining $C(Y, \mathbf{R}) \to f_*C(X, \mathbf{R})$ as follows:

- For an open subset V of Y, then $C(Y, \mathbf{R})(V) \to C(X, \mathbf{R})(f^{-1}(V))$ is given by precomposition by $f^{-1}(V) \to V$.

Example 1.43. Let X be a ringed space and U an open subset of X, viewed as a ringed space as in Example 1.40. The inclusion may be promoted to a map of ringed spaces $i : U \hookrightarrow X$ using the isomorphism $i^*\mathcal{O}_X \cong \mathcal{O}_U$ of Example 1.34.

Definition 1.44. We define the category **RS** of ringed spaces having objects the ringed spaces (Definition 1.38) and maps the maps of ringed spaces (Definition 1.41) with the following identities and composition laws:

- For X a ringed space, the identity map $1_X : X \to X$ is the identity map $1_X : X \to X$ of underlying spaces together with the map of sheaves $\mathcal{O}_X = (1_X)_*\mathcal{O}_X$ (see Remark 1.29)

- For X, Y and Z ringed spaces and $f : X \to Y$, $g : Y \to Z$ are maps of ringed spaces, the composite $g \circ f : X \to Z$ is the composite $g \circ f : X \to Z$ of the underlying maps together with the map of sheaves $\mathcal{O}_Z \to g_* \mathcal{O}_Y \to g_* f_* \mathcal{O}_X = (g \circ f)_* \mathcal{O}_X$ (see Remark 1.30).

If we instead view maps of ringed spaces using inverse images, the identities and composition laws are the following:

- For X a ringed space, the identity map $1_X : X \to X$ is the identity map $1_X : X \to X$ of underlying spaces together with the isomorphism $(1_X)^* \mathcal{O}_X \cong \mathcal{O}_X$ (see Remark 1.35)
- For X, Y and Z ringed spaces and $f : X \to Y$, $g : Y \to Z$ are maps of ringed spaces, the composite $g \circ f : X \to Z$ is the composite $g \circ f : X \to Z$ of the underlying maps together with the map of sheaves $(g \circ f)^* \mathcal{O}_Z \cong f^* g^* \mathcal{O}_Z \to f^* \mathcal{O}_Y \to \mathcal{O}_X$ (see Remark 1.36).

We will restrict our attention to a special class of ringed spaces.

Definition 1.45. A *local ringed space* is a ringed space X such that for all $x \in X$, the stalk $(\mathcal{O}_X)_x$ is a local ring (see Lemma B.10). We write $\mathfrak{m}_x$ for the unique maximal ideal of $(\mathcal{O}_X)_x$.

A motivating example of local ringed spaces is the following.

Example 1.46. Let X be a topological space, viewed as a ringed space by way of Example 1.39. Then X is a local ringed space.

To see this, fix $x \in X$ and define a homomorphism of rings $\mathrm{ev}_x : (\mathcal{O}_X)_x \to \mathbf{R}$ by $\mathrm{ev}_x(f_x) = f(x)$ for U an open neighborhood of x and $f \in \mathbf{C}(X, \mathbf{R})(U)$. Let us show an element of $(\mathcal{O}_X)_x$ is invertible if and only if its image under ev_x is

non-zero. The forward direction is clear. For the reverse direction, suppose U is an open neighborhood of x and $f \in \mathbf{C}(X, \mathbf{R})(U)$ with $\mathrm{ev}_x(f_x) = f(x) \neq 0$. Then $U' = U \cap f^{-1}(\mathbf{R} - \{0\})$ is an open neighborhood of x and $f|_{U'}$ is invertible, so $f_x = (f|_{U'})_x$ is as well. We conclude that $(\mathcal{O}_X)_x$ is a local ring with maximal ideal the kernel of ev_x.

Let X be a local ringed space. Taking inspiration from Example 1.46, we view $\mathcal{O}_X$ as a sheaf of functions on X. For $x \in X$, we view $(\mathcal{O}_X)_x$ as the germ of functions on X at x and $\mathfrak{m}_x$ as the ideal of functions vanishing at x. The fact that $\mathfrak{m}_x$ is the unique maximal ideal of $(\mathcal{O}_X)_x$ says the following: Let f be a function defined on a neighborhood U of x (i.e. $f \in \mathcal{F}(U)$). If f does not vanish at x (i.e. $f_x \notin \mathfrak{m}_x$), then there is a smaller neighborhood $U' \subset U$ of x so that $f|_{U'}$ is invertible. See Lemma 2.38 below.

Example 1.47. Let X be a local ringed space and U an open subset of X. The open ringed subspace U of Example 1.40 is a local ringed space. This follows from the isomorphism $(\mathcal{O}_U)_x \cong \mathcal{O}_x$ for $x \in U$. We call U an open local ringed subspace of X.

Definition 1.48. Let X and Y be ringed spaces, $f : X \to Y$ a map of ringed spaces and $x \in X$. Combining Example 1.33 with Remark 1.36 we obtain a map $(\mathcal{O}_Y)_{f(x)} \cong (f^*\mathcal{O}_Y)_x \to (\mathcal{O}_X)_x$. Unwinding the definitions, this is the unique homomorphism of rings making

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_X(f^{-1}(V)) \\ \downarrow & & \downarrow \\ (\mathcal{O}_Y)_{f(x)} & \dashrightarrow & (\mathcal{O}_X)_x \end{array}$$

commute for V an open subset of Y containing $f(x)$.

Definition 1.49. Let X and Y be local ringed spaces. A map of ringed spaces $f : X \to Y$ is *local* if for all $x \in X$, the map $(\mathcal{O}_Y)_{f(x)} \to (\mathcal{O}_X)_x$ of Definition 1.48 is a local homomorphism (Definition B.12) i.e. the inverse image of $\mathfrak{m}_x$ in $(\mathcal{O}_Y)_{f(x)}$ is $\mathfrak{m}_{f(x)}$.

Example 1.50. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The map of ringed spaces of Example 1.42 is local. This follows since the diagram

$$\begin{array}{ccc} (\mathcal{O}_Y)_{f(x)} & \longrightarrow & (\mathcal{O}_X)_x \\ & \searrow^{\mathrm{ev}_{f(x)}} & \downarrow{\scriptstyle \mathrm{ev}_x} \\ & & \mathbf{R} \end{array}$$

commutes, where ev_x and $\mathrm{ev}_{f(x)}$ are as in Example 1.46.

Let X and Y be local ringed spaces and $f : X \to Y$ a local map. Taking inspiration from Example 1.50, we view the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ as being given by precomposition by f. Locality of $(\mathcal{O}_Y)_{f(x)} \to (\mathcal{O}_X)_x$ says the following: Let g be a function defined on a neighborhood V of $g(x)$ (i.e. $g \in \mathcal{O}_Y(V)$). Then g vanishes at $f(x)$ (i.e. $g_{f(x)} \in \mathfrak{m}_{f(x)}$) if and only if $g \circ f$ vanishes at x (i.e. $(g \circ f)_x \in \mathfrak{m}_x$), where we write $g \circ f$ for the image of g under $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$).

Example 1.51. Let X be a ringed space and U an open subset, viewed as a local ringed space as in Example 1.47. The inclusion $i : U \hookrightarrow X$ of Example 1.43 is local; in fact it is an isomorphism on stalks.

Definition 1.52. The local ringed spaces (Definition 1.45) and local maps (Definition 1.49) define a subcategory **LRS** of the category **RS** of Definition 1.44, which we call the category of local ringed spaces.

2. The Spectrum of a Ring

In this section, we study the analog of an affine variety in the (yet to be defined) theory of schemes, called an *affine scheme*. As a starting point, we seek to generalize the duality between affine varieties over a field K and certain algebras over K to a duality between affine schemes and arbitrary rings. The analog of the coordinate ring of a variety is the global sections of a scheme.

Definition 2.1. We define the *global sections* functor $\Gamma : \mathbf{LRS} \to \mathbf{Ring}^{\mathrm{op}}$ as follows:

- For X a local ringed space, $\Gamma(X)$ is the ring $\mathcal{O}_X(X)$.
- For local ringed spaces X and Y and $f : X \to Y$ a local map, the homomorphism of rings $\Gamma(f) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is the value of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ at Y.

Compare with Example 1.28.

Let R be a ring. The affine scheme corresponding to R should have global sections R. In fact, we can take it to be the universal such local ringed space.

Theorem 2.2. *The functor* Γ *of Definition 2.1 admits a fully faithful right adjoint* $\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$. *Thus, for a local ringed space* X *and ring* R*, local maps*

$$X \to \mathrm{Spec}(R)$$

correspond to homomorphisms of rings

$$R \to \mathcal{O}_X(X).$$

We call $\mathrm{Spec}(R)$ *for* R *a ring the spectrum of* R.

Definition 2.3. A local ringed space X is an *affine scheme* if there exists a ring R and isomorphism of local ringed spaces $X \cong \mathrm{Spec}(R)$. Affine schemes form a full subcategory **AffSch** of **LRS**.

Corollary 2.4. *The functor* Γ *of Definition 2.1 restricts to an equivalence of categories* $\mathbf{AffSch} \to \mathbf{Ring}^{\mathrm{op}}$.

Proof. It follows from fully faithfulness of Spec: $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$ (Theorem 2.2) that the factorization of Spec : $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$ through **AffSch** is an inverse. □

We spend Sections 2.1-2.3 on the construction of the spectrum of a ring and the proof of Proposition 1.24. In Section 2.4, we apply Proposition 1.24 to define affine spaces over a ring and study their basic geometry.

2.1. **The Underlying Space.** Let R be a ring. In this section, we define the underlying topological space of $\mathrm{Spec}(R)$. As usual, this space and its underlying set are also denoted $\mathrm{Spec}(R)$.

Definition 2.5. Let R be a ring. The *spectrum* of R, written $\mathrm{Spec}(R)$ is the set of prime ideals of R.

Let R be a ring. We will view elements of R as functions on $\mathrm{Spec}(R)$. For $f \in R$ and $\mathfrak{p} \in \mathrm{Spec}(R)$, we view the function f as vanishing at $\mathfrak{p}$ if $f \in \mathfrak{p}$. Under this interpretation, the topology on $\mathrm{Spec}(R)$ will be defined the same in the same way as that of the Zariski topology on an affine variety.

Definition 2.6. For R a ring and S a subset of R, let $V(S)$ denote the subset $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid S \subset \mathfrak{p}\}$ of $\mathrm{Spec}(R)$.

For R a ring and S a subset of R, we view $V(S)$ as the set of common zeros of the functions in S.

Lemma 2.7. *Let R be a ring.*

(1) For $(S_i)_{i\in I}$ a family of subsets of R, $V(\bigcup_{i\in I} S_i) = \bigcap_{i\in I} V(S_i)$.

(2) $V(\{1\}) = \varnothing$.

(3) For subsets S and S' of R, $V(SS') = V(S) \cup V(S')$, where SS' is $\{ss' \mid s \in S, s' \in S'\}$.

Proof. (1) is immediate. (2) holds since R is not prime. (3) follows from a standard property of prime ideals. □

Definition 2.8. Let R be a ring. Lemma 2.7 shows the set of subsets of $\mathrm{Spec}(R)$ of the form $V(S)$ for $S \subset R$ are closed under arbitrary intersection and finite union. Hence they define the closed sets of a topology, called the *Zariski topology* on $\mathrm{Spec}(R)$. From now on, we equip $\mathrm{Spec}(R)$ with this topology.

We can now state and prove the analog of Hilbert's Nullstellensatz for the spectrum of a ring (which is easier than the usual Nullstellensatz).

Proposition 2.9. *Let R be a ring. The map from radical ideals (Definition B.14) to closed subsets of $\mathrm{Spec}(R)$ assigning a radical ideal I to the closed subset $V(I)$ is a bijection.*

Proof. For the inverse map, we associate a closed set C of $\mathrm{Spec}(R)$ to the ideal $\cap_{\mathfrak{p}\in C}\mathfrak{p}$ (informally the ideal of $f \in R$ which vanish on all points of C). This ideal is radical because prime ideals are radical and radical ideals are closed under intersection. Let C be a closed set of $\mathrm{Spec}(R)$, say $C = V(S)$ for

$S \subset R$. It is clear that $C = V(I)$ for I the smallest radical ideal containing S i.e. $\sqrt{(S)}$, so the map in the proposition is surjective. It thus suffices to show that $I = \cap_{I \subset \mathfrak{p}} \mathfrak{p}$ for all radical ideals I, which follows from Lemma B.16. $\square$

Remark 2.10. Let R be a ring. The map in Proposition 2.9 and its inverse are order-reversing for the inclusion orders. Thus for subsets S, S' of R, we have $V(S) \subset V(S')$ if and only if $\sqrt{(S')} \subset \sqrt{(S)}$ if and only if $S' \subset \sqrt{(S)}$.

Let R be a ring. It will often be more convenient to work with the open subsets of $\mathrm{Spec}(R)$, especially those in the following definition.

Definition 2.11. For R a ring and $f \in R$, we write $D(f)$ for the subset $\mathrm{Spec}(R) - V(\{f\}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}$ of $\mathrm{Spec}(R)$.

For R a ring and $f \in R$, we view $D(f)$ as the set of points on which f is nonzero.

Lemma 2.12. *Let R be a ring. The subsets of $\mathrm{Spec}(R)$ of the form $D(f)$ for $f \in R$ form a basis for the Zariski topology on $\mathrm{Spec}(R)$ (Definition 2.8) closed under binary intersection. In fact, $D(f) \cap D(f') = D(ff')$ for $f, f' \in R$.*

Proof. It is immediate that $D(f)$ is open for $f \in R$. Let U be an open subset of $\mathrm{Spec}(R)$, say $U = \mathrm{Spec}(R) - V(S)$ for $S \subset R$. By Lemma 2.7 (1), $U = \mathrm{Spec}(R) - V(\bigcup_{f \in S} \{f\}) = \mathrm{Spec}(R) - (\bigcap_{f \in S} V(\{f\})) = \bigcup_{f \in S} (\mathrm{Spec}(R) - V(\{f\})) = \bigcup_{f \in S} D(f)$. The equation $D(f) \cap D(f') = D(ff')$ for $f, f' \in R$ follows from Lemma 2.7 (3). $\square$

Lemma 2.13. *Let R be a ring and $(f)_{i \in I}$ be a family of elements of R. Then $(D(f_i))_{i \in I}$ covers $\mathrm{Spec}(R)$ if and only if $\{f_i \mid i \in I\}$ generates R.*

Proof. Similarly to in the proof of Lemma 2.12, we have $\bigcup_{i\in I} D(f_i) = \mathrm{Spec}(R) - V(\{f_i \mid i \in I\})$. Hence $(D(f_i))_{i\in I}$ covers Spec(R) if and only if $V(\{f_i \mid i \in I\}) \subset \varnothing$. By Lemma 2.7 (2) and Remark 2.10, this is equivalent to $1 \in \sqrt{(f_i \mid i \in I)}$. This is clearly equivalent to $1 \in (f_i \mid i \in I)$ i.e. $\{f_i \mid i \in I\}$ generates R. □

Corollary 2.14. *Let R be a ring and $f \in R$. Then $D(f) = \mathrm{Spec}(R)$ if and only if f is invertible.*

Proof. Apply Lemma 2.13 to the singleton family (f). □

Corollary 2.15. *Let R is a ring. The spectrum* Spec(R) *is compact.*

Proof. Let $(f_i)_{i\in I}$ be a family of elements of R such that $(D(f_i))_{i\in I}$ covers Spec(R). By Lemma 2.12, it suffices to show this family admits a finite subcover. By Lemma 2.13, $\{f_i \mid i \in I\}$ generates R, i.e. there is an expression $1 = \sum_{i\in I'} a_i f_i$ for I' a finite subset of I and $a_i \in R$ for all $i \in I'$. Then $\{f_i \mid i \in I'\}$ also generates R, so $(D(f_i))_{i\in I'}$ is a finite subcover of Spec(R) by Lemma 2.13. □

We next examine how Spec is functorial.

Definition 2.16. Let R and S be rings and $\varphi : R \to S$ a homomorphism of rings. The assignment $\mathfrak{p} \in \mathrm{Spec}(S) \mapsto \varphi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(R)$ defines a map of sets $\mathrm{Spec}(\varphi) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. In this way, $R \mapsto \mathrm{Spec}(R)$ determines a functor Spec : $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$.

Lemma 2.17. *Let R and S be rings and $f \in R$. Then* $\mathrm{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. This is immediate from the definitions. □

Definition 2.18. Let R and S be rings and $\varphi : R \to S$ a homomorphism of rings. By Lemma 2.12 and Lemma 2.17, $\mathrm{Spec}(\varphi) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is continuous when $\mathrm{Spec}(S)$ and $\mathrm{Spec}(R)$ are given the Zariski topologies. Thus, $R \mapsto \mathrm{Spec}(R)$ determines a functor $\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Top}$ lifting the functor of Definition 2.16.

Lemma 2.19. *Let R be a ring and $f \in R$. The homomorphism of rings $R \to R_f$ (Example B.3) induces a homeomorphism from $\mathrm{Spec}(R_f)$ to the open subspace $D(f)$ of $\mathrm{Spec}(R)$.*

Proof. By Lemma B.8, this map induces a bijection $\mathrm{Spec}(R) \to D(f)$ and we know it is continuous. To conclude, it suffices to show any basic open subset of $\mathrm{Spec}(R_f)$ is the inverse image of a basic open subset of $\mathrm{Spec}(R)$. Let $n \in \mathbf{N}$ and $f' \in R$ such that $\frac{f'}{f^n} \in R_f$. Then $D(\frac{f'}{f^n}) = D(\frac{f'}{1}) \cap D(\frac{1}{f^n}) = D(\frac{f'}{1}) = \mathrm{Spec}(R \to R_f)^{-1}(D(f'))$ by Lemma 2.12, Corollary 2.14 and Lemma 2.17, respectively. The result follows. □

Corollary 2.20. *Let R is a ring and $f \in R$. The subset $D(f)$ is compact.*

Proof. Combine Corollary 2.15 and Lemma 2.19. □

Lemma 2.21. *Let R be a ring and $f, f' \in R$. Then $D(f) \subset D(f')$ if and only if $\frac{f'}{1}$ is invertible in R_f.*

Proof. We have the following equivalences:

$$
\begin{aligned}
D(f) \subset D(f') &\Leftrightarrow D(f) \cap D(f') = D(f) \\
&\Leftrightarrow \mathrm{Spec}(R \to R_f)^{-1}(D(f')) = \mathrm{Spec}(R_f) \\
&\Leftrightarrow D(\frac{f'}{1}) = \mathrm{Spec}(R_f) && \text{(By Lemma 2.17)} \\
&\Leftrightarrow \frac{f'}{1} \text{ is invertible in } R_f. && \text{(By Corollary 2.14)}
\end{aligned}
$$

□

2.2. **The Sheaf of Rings.** Let R be a ring. In this section, we define the sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ on $\mathrm{Spec}(R)$ making it into a local ringed space.

Motivation 2.22. Let R be a ring and $f \in R$. We would like $\mathcal{O}_{\mathrm{Spec}(R)}(D(f))$ to be the ring of rational functions defined on $D(f)$. A rational function g should be defined on $D(f)$ if and only if g has no poles on $D(f)$; equivalently every pole of g is a zero of f. Thus we should be able to write g of the form $\frac{f'}{f^n}$ for $f' \in R$ and $n \in \mathbf{N}$ sufficently large. This informal argument shows $\mathcal{O}_{\mathrm{Spec}(R)}(D(f))$ should be R_f.

We now proceed with the technical construction of $\mathcal{O}_{\mathrm{Spec}(R)}$, taking inspiration from Motivation 2.22.

Definition 2.23. Let R be a ring. Define a category $\mathcal{R}$ as follows:

- The objects of $\mathcal{R}$ are elements $f \in R$.
- For $f, f' \in R$, there is a map from f to f' in $\mathcal{R}$ if and only if $D(f) \subset D(f')$ (in which case there is only one).

Remark 2.24. Let R be a ring and $\mathcal{B}$ denote the full subcategory of $\Omega(\mathrm{Spec}(R))$ with objects the basic open subsets of Lemma 2.12. By construction, there is

a fully faithful and essentially surjective functor $D : \mathcal{R} \to \mathcal{B}$ assigning $f \in R$ to $D(f) \in \mathcal{B}$. Hence we may choose an inverse $\mathcal{B} \to \mathcal{R}$.

Definition 2.25. Let R be a ring. We define a functor from $\mathcal{R}^{\mathrm{op}}$ to **Ring** as follows:

- The image of $f \in R$ is the ring R_f.
- For $f, f' \in R$ such that $D(f) \subset D(f')$, Lemma 2.21 provides a unique homomorphism of rings making

$$\begin{array}{ccc} R & \longrightarrow & R_{f'} \\ & \searrow & \downarrow \\ & & R_f \end{array}$$

  commute, which we take to be the image of $f \to f'$.

It is easy to see this definition is compatible with identities and composition.

Definition 2.26. Let R be a ring and $\mathcal{B}$ the base closed under binary intersections of Lemma 2.12. Take a functor $\mathcal{B} \to \mathcal{R}$ inverse to $D : \mathcal{R} \to \mathcal{B}$ as in Remark 2.24. Composing $\mathcal{B}^{\mathrm{op}} \to (\mathcal{R})^{\mathrm{op}}$ with the functor of Definition 2.25, we obtain a presheaf of rings $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{B}$ (Definition 1.22).

Remark 2.27. Let R be a ring and $\mathcal{B}$ the base of Lemma 2.12. By construction, there is an isomorphism between the composite $(\mathcal{R})^{\mathrm{op}} \xrightarrow{D^{\mathrm{op}}} \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{O}_B}$ **Ring** and the functor of Definition 2.25. Explicitly, this means the following:

- For each $f \in R$, we have an isomorphism $\mathcal{O}_{\mathcal{B}}(D(f)) \cong R_f$
- Let $f, f' \in R$ such that $D(f) \subset D(f')$. Under these isomorphisms, the restriction map $\mathcal{O}_{\mathcal{B}}(D(f')) \to \mathcal{O}_{\mathcal{B}}(D(f))$ is the homomorphism of rings $R_{f'} \to R_f$ of Definition 2.25.

Let R be a ring and $\mathcal{B}$ the base of Lemma 2.12. To extend $\mathcal{O}_{\mathcal{B}}$ to a sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ on $\mathrm{Spec}(R)$, we will need the following gluing lemma.

Lemma 2.28. *Let R be a ring, I a finite set and $(f_i \in R)_{i\in I}$ such that $(D(f_i))_{i\in I}$ is a cover of $\mathrm{Spec}(R)$. For $(g_i \in R_{f_i})_{i\in I}$ such that for all $i, j \in I$, the image of g_i in $R_{f_if_j}$ is equal to the image of g_j in $R_{f_if_j}$, there exists a unique $g \in R$ such that the image of g in R_{f_i} is g_i for all $i \in I$.*

Proof. We first note that for all $n \in \mathbf{N}$, there is an expression $1 = \sum_{i\in I} a_i f_i^n$ for $a_i \in R$ for $i \in I$. This follows from Lemma 2.13 when $n = 1$. It is not hard to see this case implies the result for all $n \in \mathbf{N}$; alternatively one could use that $D(f_i^n) = D(f_i)$ for $n \in \mathbf{N}$ positive and $i \in I$.

For each $i \in I$, choose $n_i \in \mathbf{N}$ and $h_i \in R$ so that $f_i^{n_i} g_i$ is the image of $h_i \in R$. Since I is finite, we may assume the n_i are all equal for $i \in I$, say to $n \in \mathbf{N}$. Let $i, j \in I$. Since g_i and g_j are equal in $R_{f_if_j}$, we may choose $m_{ij} \in \mathbf{N}$ so that $(f_if_j)^{m_{ij}} f_j^n h_i = (f_if_j)^{m_{ij}} f_i^n h_j$ in R. Since I is finite, we may assume the m_{ij} are all equal for $i, j \in I$, say to $m \in \mathbf{N}$. Replacing h_i by $f_i^m h_i$ for each $i \in I$ and n by $n + m$, we may assume that $f_j^n h_i = f_i^n h_j$ for all $i, j \in I$. Now take $a_i \in R$ for each $i \in I$ such that $1 = \sum_{i\in I} a_i f_i^n$ and set $g = \sum_{i\in I} a_i h_i$. For $j \in I$, we have $f_j^n g = \sum_{i\in I} a_i f_j^n h_i = \sum_{i\in I} a_i f_i^n h_j = h_j$, so that the image of f_j^n in R_{f_j} is g_j.

For uniqueness, suppose $g, g' \in R$ are such that the images of g and g' are equal in R_{f_i} for all $i \in I$. For each $i \in I$, choose $n_i \in \mathbf{N}$ so that $f_i^{n_i} g = f_i^{n_i} g'$ in R. Since I is finite, we may assume the n_i are all equal for $i \in I$, say to $n \in \mathbf{N}$. Take $a_i \in R$ for each $i \in I$ so that $1 = \sum_{i\in I} a_i f_i^n$. Then $g = \sum_{i\in I} a_i f_i^n g = \sum_{i\in I} a_i f_i^n g' = g'$. □

Proposition 2.29. *Let R be a ring and $\mathcal{B}$ the base of Lemma 2.12. The presheaf of rings $\mathcal{O}_{\mathcal{B}}$ on $\mathcal{B}$ of Definition 2.26 is a sheaf.*

Proof. Let U be in $\mathcal{B}$ and $(U_i)_{i\in I}$ an open covering of U by elements of $\mathcal{B}$. We wish to show the sheaf condition holds for $(U_i)_{i\in I}$.

Case 1: Assume I is finite. Write $U = D(f)$ for $f \in R$ and for each $i \in I$, write $U_i = D(f_i)$ for $f_i \in R$. Under the isomorphisms $\mathcal{O}_{\mathcal{B}}(U) \cong R_f, \mathcal{O}_{\mathcal{B}}(U_i) \cong R_{f_i} \cong (R_f)_{\frac{f_i}{1}}$ and $\mathcal{O}_{\mathcal{B}}(U_i \cap U_j) \cong R_{f_i f_j} \cong (R_f)_{\frac{f_i f_j}{1}}$ (see Lemma B.5), the result reduces to Lemma 2.28 for R_f, I and $(\frac{f_i}{1})_{i\in I}$.

Case 2: Let I be arbitrary. By Corollary 2.20, there is a finite subset $I' \subset I$ so that $(U_i)_{i\in I'}$ covers U. Let $(g_i \in \mathcal{O}_{\mathcal{B}}(U_i))_{i\in I}$ be such that for all $i, j \in I$, $g_i|_{U_i\cap U_j} = g_j|_{U_i\cap U_j}$. In particular this holds for each $i, j \in I'$, so Case 1 provides a unique $g \in \mathcal{O}_{\mathcal{B}}(U)$ such that $g|_{U_i} = g_i$ for all $i \in I'$. Let $j \in I$; to finish we show $g|_{U_j} = g_j$. For $i \in I'$, we compute $(g|_{U_j})|_{U_i\cap U_j} = g|_{U_i\cap U_j} = (g|_{U_i})|_{U_i\cap U_j} = g_i|_{U_i\cap U_j} = g_j|_{U_i\cap U_j}$. The result follows from Case 1 applied to the cover $(U_i \cap U_j)_{i\in I'}$ of U_j. □

Definition 2.30. Let R be a ring. By Proposition 2.29 and Proposition 1.24, we obtain a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ on $\mathrm{Spec}(R)$ which restricts to the presheaf of rings of Definition 2.26. This sheaf makes $\mathrm{Spec}(R)$ into a ringed space (Definition 1.38).

Remark 2.31. Let R be a ring and $\mathcal{B}$ the base of Lemma 2.12. By definition, we have an isomorphism $\mathcal{O}_{\mathrm{Spec}(R)}|_{\mathcal{B}} \cong \mathcal{O}_{\mathcal{B}}$. Combining this with Remark 2.27, we obtain an isomorphism between the composite $(\mathcal{R})^{\mathrm{op}} \xrightarrow{D^{\mathrm{op}}} \Omega(\mathrm{Spec}(R))^{\mathrm{op}} \xrightarrow{\mathcal{O}_{\mathrm{Spec}(R)}} \mathbf{Ring}$ and the functor of Definition 2.25. Explicitly, this means the following:

- For each $f \in R$, we have an isomorphism $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) \cong R_f$
- Let $f, f' \in R$ such that $D(f) \subset D(f')$. Under these isomorphisms, the restriction map $\mathcal{O}_{\mathrm{Spec}(R)}(D(f')) \to \mathcal{O}_{\mathrm{Spec}(R)}(D(f))$ is the homomorphism of rings $R_{f'} \to R_f$ of Definition 2.25.

Lemma 2.32. *Let R be a ring and $\mathfrak{p}$ be a prime ideal of R. There is an isomorphism of the stalk $(\mathcal{O}_{\mathrm{Spec}(R)})_{\mathfrak{p}}$ with the localization $R_{\mathfrak{p}}$ of Example B.6.*

Proof. Since the image of $\mathcal{R}$ in $\Omega(\mathrm{Spec})(R)$ is cofinal, we have an isomorphism $(\mathcal{O}_{\mathrm{Spec}(R)})_{\mathfrak{p}} \cong \mathrm{colim}_{f \notin \mathfrak{p}} R_f$. The latter has the universal property of $R_{\mathfrak{p}}$. □

Corollary 2.33. *Let R be a ring. The ringed space of Definition 2.30 is a local ringed space.*

Proof. This follows from Lemma 2.32 and Example B.11. □

We have defined how the functor of Theorem 2.2 behaves on objects. We turn to its value on maps.

Definition 2.34. Let R and S be rings and $\varphi : R \to S$ a homomorphism of rings. We may promote $\mathrm{Spec}(\varphi) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ to a map of ringed spaces $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by defining the map of sheaves $\mathcal{O}_{\mathrm{Spec}(R)} \to \mathrm{Spec}(\varphi)_* \mathcal{O}_{\mathrm{Spec}(S)}$ as follows:

It suffices to define a map upon precomposition with $D^{\mathrm{op}} : (\mathcal{R})^{\mathrm{op}} \to \Omega(\mathrm{Spec}(R))^{\mathrm{op}}$. For this, let $f \in R$. We take $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) \to \mathcal{O}_{\mathrm{Spec}(S)}(\mathrm{Spec}(\varphi)^{-1}(D(f))) = \mathcal{O}_{\mathrm{Spec}(S)}(D(\varphi(f)))$ to be the following homomorphism of rings:

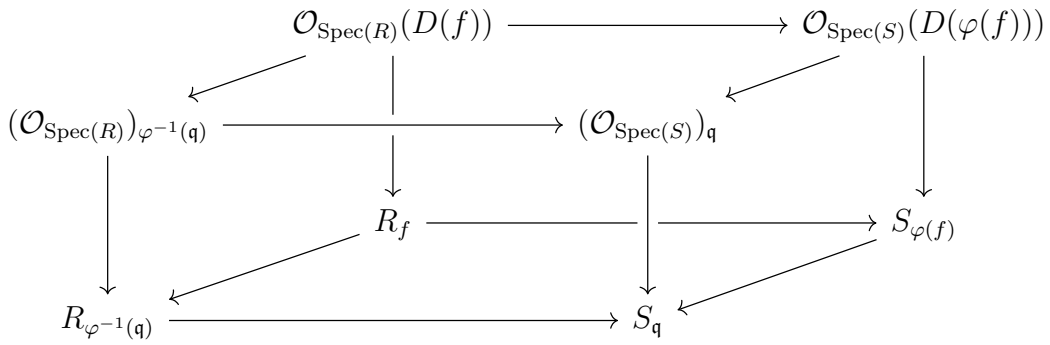
$$\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) \cong R_f \to S_{\varphi(f)} \cong \mathcal{O}_{\mathrm{Spec}(S)}(D(\varphi(f))).$$

Here $R_f \to S_{\varphi(f)}$ is as in Definition B.7.

It is easy to see this is natural in f. In this way, $R \mapsto \mathrm{Spec}(R)$ determines a functor $\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{RS}$ lifting the functor of Definition 2.18.

Lemma 2.35. *Let R and S be rings, $\varphi : R \to S$ a homomorphism of rings and $\mathfrak{q}$ a prime ideal of S. Under the isomorphisms $(\mathcal{O}_{\mathrm{Spec}(R)})_{\varphi^{-1}(\mathfrak{q})} \cong R_{\varphi^{-1}(\mathfrak{q})}$ and $(\mathcal{O}_{\mathrm{Spec}(S)})_{\mathfrak{q}} \cong S_{\mathfrak{q}}$ of Lemma 2.32, the homomorphism of rings $(\mathcal{O}_{\mathrm{Spec}(R)})_{\varphi^{-1}(\mathfrak{q})} \to (\mathcal{O}_{\mathrm{Spec}(S)})_{\mathfrak{q}}$ from Definition 2.34 and Definition 1.48 is given by the homomorphism $R_{\varphi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$ of Example B.13.*

Proof. We wish to show the front face of



commutes. It suffices to show for all $f \in R$ such that $\varphi^{-1}(\mathfrak{q}) \in D(f)$, the other five faces of commute. For such an f, the left and right faces commute by the proof of Lemma 2.32, the bottom face clearly commutes, while the top and back faces commute from the definitions (Definition 1.48, Definition 2.34).

□

Corollary 2.36. *Let R and S be rings and $\varphi : R \to S$ a homomorphism of rings. The map $\mathrm{Spec}(\varphi) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ of Definition 2.34 is local (Definition 1.49).*

Proof. This is immediate from Lemma 2.35 and Example B.13. □

Definition 2.37. By Corollary 2.36, the functor of (Definition 1.49) factors through **LRS** to give a functor Spec : $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$.

2.3. **Maps into the Spectrum.** In this section, we prove Theorem 2.2. We will need some elementary lemmas about invertibility.

Lemma 2.38. *Let X be a ringed space, U an open subset of X, $f \in \mathcal{O}_X(U)$ and $x \in U$. Then f_x is invertible in $(\mathcal{O}_X)_x$ if and only if there exists a neighborhood V of x contained in U such that $f|_V$ is invertible in $\mathcal{O}_X(V)$.*

Proof. The backwards direction is clear. For the forwards direction, suppose f_x is invertible in $(\mathcal{O}_X)_x$. Take an open neighborhood W of x and $g \in \mathcal{O}_X(W)$ such that g_x is inverse to f_x in $(\mathcal{O}_X)_x$. Replacing W by $W \cap U$, we may assume $W \subset U$. Then $(gf|_W)_x = g_x(f|_V)_x = g_x f_x = 1_{(\mathcal{O}_X)_x} = (1_{\mathcal{O}_X(W)})_x$. Thus there is $V \subset W$ such that $(gf|_W)|_V = (1_{\mathcal{O}_X(W)})|_V$. This yields $g|_V f|_V = g|_V(f|_W)|_V = (gf|_W)|_V = (1_{\mathcal{O}_X(W)})|_V = 1_{\mathcal{O}_X(V)}$, showing $g|_V$ is inverse to $f|_V$ in $\mathcal{O}_X(V)$. □

Lemma 2.39. *Let X be a ringed space, U an open subset of X and $f \in \mathcal{O}_X(U)$. The following conditions are equivalent:*

(1) The element f is invertible in $\mathcal{O}_X(U)$.

(2) There is an open covering $(U_i)_{i\in I}$ of U such that $f|_{U_i}$ is invertible in $\mathcal{O}_X(U_i)$ for all $i \in I$.

(3) For all $x \in U$, f_x is invertible in $(\mathcal{O}_X)_x$.

Proof. The implication (1) ⇒ (2) is immediate, while (2) ⇔ (3) follows from Lemma 2.38. To conclude, we show (2) ⇒ (1). Suppose $(U_i)_{i\in I}$ is as in

(2). Consider the family $((f|_{U_i})^{-1} \in \mathcal{O}_X(U_i))_{i\in I}$. For $i, j \in I$, we have $(f|_{U_i})^{-1}|_{U_i\cap U_j} = (f|_{U_i}|_{U_i\cap U_j})^{-1} = (f|_{U_i\cap U_j})^{-1} = (f|_{U_j}|_{U_i\cap U_j})^{-1} = (f|_{U_j})^{-1}|_{U_i\cap U_j}$. Thus the sheaf condition provides $g \in \mathcal{O}_X(U)$ such that $g|_{U_i} = (f|_{U_i})^{-1}$ for all $i \in I$. We claim $gf = 1_{\mathcal{O}_X(U)}$. This follows from the calculation $(gf)|_{U_i} = g|_{U_i}f|_{U_i} = (f|_{U_i})^{-1}f|_{U_i} = 1_{\mathcal{O}_X(U_i)} = 1_{\mathcal{O}_X(U)}|_{U_i}$ for $i \in I$ and the sheaf condition. □

Proof of Theorem 2.2. Take Spec : $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$ to be the functor of Definition 2.37. For a ring R, we have ring isomorphisms $R \cong R_1 \cong \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R))$, giving an isomorphism $\Gamma(\mathrm{Spec}(R)) \cong R$ in $\mathbf{Ring}^{\mathrm{op}}$, natural in R. We show this makes Spec right adjoint to Γ, from which fully faithfulness will follow. Fix a ring R, a local ringed space X and a homomorphism of rings $\varphi : R \to \mathcal{O}_X(X)$. We show there is a unique local map $\psi : X \to \mathrm{Spec}(R)$ making the following diagram in **Ring** commute.

$$\begin{array}{ccc} R & \xrightarrow{\sim} & \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \\ & \overset{\varphi}{\searrow} & \downarrow \\ & & \mathcal{O}_X(X) \end{array}$$

First suppose $\psi : X \to \mathrm{Spec}(R)$ is a map of *ringed spaces* making the above diagram commute. We claim ψ is local if and only its underlying map of topological spaces is given by the formula $\psi(x) = (R \xrightarrow{\varphi} \mathcal{O}_X(X) \to (\mathcal{O}_X)_x)^{-1}(\mathfrak{m}_x)$ for $x \in X$. To see this, let $x \in X$ and consider the following diagram.

$$\begin{array}{ccccccc} R & \xrightarrow{\sim} & R_1 & \xrightarrow{\sim} & \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) & \longrightarrow & \mathcal{O}_X(X) \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & R_{\psi(x)} & \xrightarrow{\sim} & (\mathcal{O}_{\mathrm{Spec}(R)})_{\psi(x)} & \longrightarrow & (\mathcal{O}_X)_x \end{array}$$

Here the right square commutes by Definition 1.48, the middle square by the proof of Lemma 2.32, while the left triangle clearly commutes. By assumption, the top composite is φ. The homomorphism $(\mathcal{O}_{\mathrm{Spec}(R)})_{\psi(x)} \to (\mathcal{O}_X)_x$ is local if and only if the two ideals $((\mathcal{O}_{\mathrm{Spec}(R)})_{\psi(x)} \to (\mathcal{O}_X)_x)^{-1}(\mathfrak{m}_x)$ and $\mathfrak{m}_{\psi(x)}$ of $(\mathcal{O}_{\mathrm{Spec}(R)})_{\psi(x)}$ are equal. By Lemma B.8, this is equivalent to the inverse images of the two ideals under $R \to R_{\psi(x)} \xrightarrow{\sim} (\mathcal{O}_{\mathrm{Spec}(R)})_{\psi(x)}$ being equal. The ideals pull back to $(R \xrightarrow{\varphi} \mathcal{O}_X(X) \to (\mathcal{O}_X)_x)^{-1}(\mathfrak{m}_x)$ and $\psi(x)$, respectively. The claim follows.

Let $\psi : X \to \mathrm{Spec}(R)$ be the map of sets given by the formula above. For $f \in \mathrm{Spec}(R)$ and $x \in X$, we compute

$$\begin{aligned} x \in \psi^{-1}(D(f)) &\Leftrightarrow \psi(x) \in D(f) \\ &\Leftrightarrow f \notin \psi(x) \\ &\Leftrightarrow \varphi(f)_x \notin \mathfrak{m}_x \\ &\Leftrightarrow \varphi(f)_x \text{ is invertible in } (\mathcal{O}_X)_x. \end{aligned}$$

Combining this with Lemma 2.38 and Lemma 2.39, it follows that $\psi^{-1}(D(f))$ for $f \in R$ is the largest open subset U of X such that $\varphi(f)|_U$ is invertible in $\mathcal{O}_X(U)$. In particular, ψ is continuous.

We conclude by showing there is a unique way to promote the continuous map $\psi : X \to \mathrm{Spec}(R)$ to a map of ringed spaces in such a way that the requisite diagram commutes; by the claim above this will automatically be local. Equivalently, there is a unique map from the functor of Definition 2.25 to the composite $(\mathcal{R})^{\mathrm{op}} \xrightarrow{D^{\mathrm{op}}} \Omega(\mathrm{Spec}(R))^{\mathrm{op}} \xrightarrow{\Omega(\psi)^{\mathrm{op}}} \Omega(X)^{\mathrm{op}} \xrightarrow{\mathcal{O}_X} \mathbf{Ring}$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\sim} & R_1 \\ \downarrow{\scriptstyle \varphi} & & \downarrow \\ \mathcal{O}_X(X) & = & \mathcal{O}_X(\psi^{-1}(D(1))) \end{array}$$

commutes.

Let $f \in R$. If such a map exists, the commutative diagram

$$\begin{array}{ccccc} R & \xrightarrow{\sim} & R_1 & \longrightarrow & R_f \\ \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow \\ \mathcal{O}_X(X) & = & \mathcal{O}_X(\psi^{-1}(D(1))) & \longrightarrow & \mathcal{O}_X(\psi^{-1}(D(f))) \end{array}$$

shows $R_f \to \mathcal{O}_X(\psi^{-1}(D(f)))$ is the unique factorization of $R \to \mathcal{O}_X(X) \to \mathcal{O}_X(\psi^{-1}(D(f)))$ through R_f. Conversely, the above calculation of $\psi^{-1}(D(f))$ shows this homomorphism inverts f, so we may define $R_f \to \mathcal{O}_X(\psi^{-1}(D(f)))$ in this way. Letting f vary, it is easy to see that we obtain a natural transformation making the desired diagram commute.

□

2.4. **Affine Space over a Ring.** In this section, we define the analog of affine space over a field K in the theory of affine schemes. We begin some formal consequences of Theorem 2.2.

Proposition 2.40. *The following assertions hold.*

(1) $\mathrm{Spec}(\mathbf{Z})$ *is the terminal object (Definition A.38) of* **LRS**.

(2) Let R *be ring and let* R_1 *and* R_2 *be* R*-algebras. The diagram*

$$\begin{array}{ccc} \mathrm{Spec}(R_1 \otimes_R R_2) & \longrightarrow & \mathrm{Spec}(R_2) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R_1) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

is a pullback square (Definition A.44) in **LRS**. *In symbols,* $\mathrm{Spec}(R_1)\times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_2) \cong \mathrm{Spec}(R_1 \otimes_R R_2)$

Proof. Since Spec: $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{LRS}$ has a left adjoint, it preserves limits Lemma A.53, i.e. transforms colimits into limits. Thus these follow from Example A.40 and Example A.46. □

It will be useful to study not just **LRS**, but the categories $\mathbf{LRS}_{/\mathrm{Spec}(R)}$ for R a ring (Definition A.9). By (1) of Proposition 2.40, we recover **LRS** by taking R to be $\mathbf{Z}$.

Corollary 2.41. *Let R be a ring.*

(1) $\mathrm{Spec}(R)$ *is the terminal object of* $\mathbf{LRS}_{/\mathrm{Spec}(R)}$.

(2) Let R_1 and R_2 be R-algebras. The product of $\mathrm{Spec}(R_1)$ *and* $\mathrm{Spec}(R_2)$ *in* $\mathbf{LRS}_{/\mathrm{Spec}(R)}$ *is* $\mathrm{Spec}(R_1 \otimes_R R_2)$.

Proof. This follows from Example A.41, Example A.48 and Proposition 2.40. □

Remark 2.42. Let R be a ring. Objects of $\mathbf{LRS}_{/\mathrm{Spec}(R)}$ correspond to local ringed spaces X with an R-algebra structure on $\mathcal{O}_X(X)$. This structure uniquely extends to an R-algebra structure on $\mathcal{O}_X(U)$ for each open subset U of X in such a way that for all pairs of open subsets U, V of X such that $V \subset U$, the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is a homomorphism of R-algebras. For local ringed spaces X and Y over $\mathrm{Spec}(R)$, a local map $X \to Y$ determines a map over $\mathrm{Spec}(R)$ if and only if for all open subsets V of Y, $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is a homomorphism of R-algebras.

We have the following relative version of Theorem 2.2:

Proposition 2.43. *Let R be a ring. Write Γ_R for the functor $\mathbf{LRS}_{/\mathrm{Spec}(R)} \to (\mathbf{Ring}_{R/})^{\mathrm{op}}$ given by $X \to \mathrm{Spec}(R) \mapsto R \to \Gamma(X)$ and Spec_R for the functor $(\mathbf{Ring}_{R/})^{\mathrm{op}} \to \mathbf{LRS}_{/\mathrm{Spec}(R)}$ given by $R \to S \mapsto \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Then Spec_R is a fully faithful right adjoint to Γ_R.*

Proof. This is a formal consequence of Theorem 2.2. □

Corollary 2.44. *Let R be a ring. The functor Γ_R of Proposition 2.43 restricts to an equivalence of categories $\mathbf{AffSch}_{/\mathrm{Spec}(R)} \to (\mathbf{Ring}_{R/})^{\mathrm{op}}$. Thus R-algebras are dual to affine schemes over $\mathrm{Spec}(R)$.*

Proof. This follows from Proposition 2.43 as in Corollary 2.4. □

Definition 2.45. Let R be a ring and $n \in \mathbf{N}$. We define *affine n-space* over R, denoted $\mathbf{A}^n_R$ to be $\mathrm{Spec}(R[x_1, \ldots, x_n])$, viewed as an affine scheme over $\mathrm{Spec}(R)$.

Example 2.46. Let R be a ring and $n \in \mathbf{N}$. By construction we have an isomorphism $\Gamma_R(\mathbf{A}^n_R) \cong R[x_1, \ldots, x_n]$ so that global sections of $\mathbf{A}^n_R$ are polynomial functions in n variables.

Example 2.47. Let R be a ring. In the case $n = 0$ of Definition 2.45, $\mathbf{A}^0_R$ is $\mathrm{Spec}(R)$. By Corollary 2.41, $\mathbf{A}^0_R$ is terminal in $\mathbf{LRS}_{/\mathrm{Spec}(R)}$. For X a local ringed space over $\mathrm{Spec}(R)$, we thus call call maps $\mathbf{A}^0_R \to X$ over $\mathrm{Spec}(R)$ *points* of X (not to be confused with points of the underlying space of X).

Example 2.48. Let R be a ring and $n \in \mathbf{N}$. The points of $\mathbf{A}^n_R$, in the sense of Example 2.47, correspond to R-algebra homomorphisms $R[x_1, \ldots, x_n] \to R$, or equivalently to points of R^n, as one would hope.

Example 2.49. Let R be a ring, X a local ringed space over $\mathrm{Spec}(R)$ and U an open subset of X, regarded as a local ringed space as in Example 1.47. By Proposition 2.43, local maps $U \to \mathbf{A}^1_R$ over $\mathrm{Spec}(R)$ correspond to homomorphisms of R-algebras $R[x] \to \mathcal{O}_X(U)$ or equivalently to elements of $\mathcal{O}_X(U)$. Thus formal maps $f \in \mathcal{O}_X(U)$ correspond to actual maps $f : U \to \mathbf{A}^1_R$ in $\mathbf{LRS}_{/\mathrm{Spec}(R)}$. Let U and V be open subsets of X such that $V \subset U$. Under the above correspondence for U and V, the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is given by precomposition by the inclusion $V \hookrightarrow U$ of Example 1.51.

Example 2.50. Let R be a ring and $n, m \in \mathbf{N}$. From the isomorphism $R[x_1, \ldots, x_n] \otimes_R R[y_1, \ldots, y_m] \cong R[z_1, \ldots z_{n+m}]$ and Corollary 2.41, we find that $\mathbf{A}^{n+m}_R$ is the product of $\mathbf{A}^n_R$ and $\mathbf{A}^m_R$ in $\mathbf{LRS}_{/\mathrm{Spec}(R)}$. In symbols, $\mathbf{A}^n_R \times_{\mathrm{Spec}(R)} \mathbf{A}^m_R \cong \mathbf{A}^{n+m}_R$.

Affine spaces are stable under change of base ring:

Example 2.51. Let R and S be rings and $n \in \mathbf{N}$. From the isomorphism $S \otimes_R R[x_1, \ldots, x_n] \cong S[x_1, \ldots, x_n]$ and Proposition 2.40, we obtain a pullback square

$$\begin{array}{ccc} \mathbf{A}^n_S & \longrightarrow & \mathbf{A}^n_R \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

In symbols, $\mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathbf{A}^n_R \cong \mathbf{A}^n_S$.

3. Schemes as Local Ringed Spaces

Classically, an arbitrary variety over a field K is covered by open subsets which are affine varieties. We use an analogous condition to define the schemes amongst the local ringed spaces.

Definition 3.1. Let X be a local ringed space and U an open subset of X. We say U is *affine* if the open local ringed subspace U of X (Example 1.47) is an affine scheme (Definition 2.3).

Definition 3.2. Let X be a local ringed space. We say X is a *scheme* if there exists a covering of X by affine open subsets (Definition 3.1). Schemes form a full subcategory **Sch** of **LRS**.

Example 3.3. Every affine scheme is a scheme.

In order to work with Definition 3.2, we will need to study the category **LRS** in more detail. This study will take up Section 3.1. In Section 3.2, we present a common way for constructing new schemes from old: by *gluing*. We apply this construction in Section 3.3 to define projective space over a ring.

3.1. **Maps Between Schemes.** Many of the results in this and the next section have variants for ringed spaces and are most useful for schemes, but we will state them for local ringed spaces.

Lemma 3.4. *Let X and Y be local ringed spaces and $f : X \to Y$ a local map. The following conditions are equivalent.*

(1) The map $f : X \to Y$ is an isomorphism in **LRS**.

(2) The underlying continuous map $f : X \to Y$ is a homeomorphism and the map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

(3) The underlying continuous map $f : X \to Y$ is a homeomorphism and the map of sheaves $f^*\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

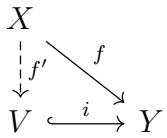
Proof. This is a straightforward consequence of the definitions. □

Remark 3.5. Suppose X is a *ringed space* satisfying the condition analogous to the definition of a scheme:

$(*)$ There exists an open covering $(U_i)_{i\in I}$ of X such that for all $i \in I$, there exists a ring R and an isomorphism of ringed spaces from the open ringed subspace U_i (Example 1.40) to $\mathrm{Spec}(R)$.

Then X is a local ringed space and, in fact, a scheme. This follows by combining Lemma 3.4 with its variant for ringed spaces. It will be important, however, that **Sch** is taken as a full subcategory of **LRS** and not **RS**.

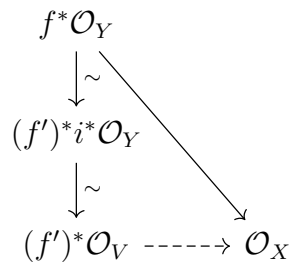
Lemma 3.6. *Let X and Y be local ringed spaces, $f : X \to Y$ a local map and V an open subset of Y. Then there exists a factorization*



of f through the map i of Example 1.51 in **LRS** *is an only if there exists such a factorization in* **Top**. *In this case, the factorization is unique.*

Proof. Unwinding the definitions, a factorization $f' : X \to V$ of f through i in **LRS** amounts to the following data:

- A factorization $f' : X \to V$ of f through i in **Top**.
- A map of sheaves $(f')^*\mathcal{O}_V \to \mathcal{O}_X$ inducing local homomorphisms on all stalks and making the diagram

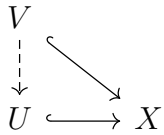


commute.

In presence of the former, the latter uniquely exists. □

Remark 3.7. Let X and Y be local ringed spaces, $f : X \to Y$ a local map and V an open subset of Y. Suppose f factors through the inclusion $i : V \to Y$ in **Top**, say by $f' : X \to V$. Lemma 3.6 promotes f' to a local map $X \to V$. Explicitly, the value of $\mathcal{O}_V \to f'_*\mathcal{O}_X$ at an open subset W of V is the same as the value of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of W.

Example 3.8. Let X be a local ringed space and U, V open subsets of X. Then Lemma 3.6 says there exists a factorization



in **LRS** if and only if V is contained in U. In this case the map $V \to U$ is given by Example 1.51, viewing V as an open subset of U (see Remark 3.7).

Lemma 3.9. *Let R be a ring and $f \in R$. There is an isomorphism of local ringed spaces* $\mathrm{Spec}(R_f) \cong D(f)$ *making the diagram*

$$\begin{array}{ccc} \operatorname{Spec}(R_f) & & \\ \downarrow \sim & \searrow & \\ D(f) & \hookrightarrow & \operatorname{Spec}(R) \end{array}$$

commute.

Proof. (Proof 1): By Lemma 2.19 and Lemma 3.6, there is a map $\psi : \operatorname{Spec}(R) \to D(f)$ making the above diagram commute and inducing a homeomorphism on underlying spaces. By Lemma 3.4, it suffices to show $\mathcal{O}_{D(f)} \to \psi_*(\mathcal{O}_{\operatorname{Spec}(R_f)})$ is an isomorphism of sheaves. By Remark 3.7 and the definition of $\operatorname{Spec}(R_f) \to \operatorname{Spec}(R)$ (Definition 2.34), this reduces to the following:

- For all $g \in R$ such that $D(g) \subset D(f)$, the map $R_g \to (R_f)_{\frac{g}{1}}$ is an isomorphism of rings.

This follows from Lemma 2.21 and Lemma B.5.

(Proof 2): We show the local maps into both are the same (over $\operatorname{Spec}(R)$). Let X be a local ringed space and $\psi : X \to \operatorname{Spec}(R)$ a local map, with $\varphi : R \to \Gamma(X)$ the corresponding homomorphism of rings. We have

$$\begin{aligned} \psi \text{ factors through } D(f) &\Leftrightarrow X \subset \psi^{-1}(D(f)) && \text{(By Lemma 3.6)} \\ &\Leftrightarrow \varphi(f) \text{ is invertible in } \Gamma(X) && \text{(By the proof of Theorem 2.2)} \\ &\Leftrightarrow \varphi \text{ factors through } R_f \\ &\Leftrightarrow \psi \text{ factors through } \operatorname{Spec}(R_f). && \text{(By Theorem 2.2)} \end{aligned}$$

Such a factorization is unique in either case.

□

Corollary 3.10. *Let X be a scheme. The set of affine open subsets of X is a base for X.*

Proof. By definition, X is covered by affine open subsets. Suppose U is an open subset of X and $x \in U$; we wish to find an affine open subset V of X containing x and contained in U. Since X is a scheme, there is an affine open subset W of X containing x. Say R is a ring such that we have an isomorphism $\psi : W \cong \mathrm{Spec}(R)$. Then $\psi(U \cap W)$ is an open neighborhood of $\psi(x)$ in $\mathrm{Spec}(R)$. By Lemma 2.12, there is $f \in R$ such that $D(f)$ is an open neighborhood of $\psi(x)$ contained in $\psi(U \cap W)$. Then $\psi^{-1}(D(f))$ is an open neighborhood of x contained in U, which is affine from the calculation $\psi^{-1}(D(f)) \cong D(f) \cong \mathrm{Spec}(R_f)$ (Lemma 3.9). □

Corollary 3.11. *Let X be a scheme and U an open subset of X. Then the open local ringed subspace U of X is a scheme. We call U an open subscheme of X.*

Proof. This is immediate from Corollary 3.10. □

Lemma 3.12. *Let X and Y be local ringed spaces, $f : X \to Y$ a local map and V an open subset of Y. There is a pullback square*

$$\begin{array}{ccc} f^{-1}(V) & \hookrightarrow & X \\ \downarrow & \ulcorner & \downarrow f \\ V & \hookrightarrow & Y \end{array}$$

in **LRS**.

Proof. Take $f^{-1}(V) \to V$ to be the unique map of **LRS** fitting into a commutative square

$$\begin{array}{ccc} f^{-1}(V) & \longrightarrow & X \\ \vdots & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

(Lemma 3.6). Suppose Z is a local ringed space and

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ V & \longrightarrow & Y \end{array}$$

is a commutative diagram. Since the square of underlying spaces commutes, there is a factorization

$$\begin{array}{ccc} Z & & \\ \vdots & \searrow & \\ f^{-1}(V) & \hookrightarrow & X \end{array}$$

in **Top**. By Lemma 3.6, there is a map $Z \to f^{-1}(V)$ in **LRS** making the top triangle in

$$\begin{array}{ccc} Z & & \\ \vdots & \searrow & \\ f^{-1}(V) & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ V & \hookrightarrow & Y \end{array}$$

commute. To conclude, we show left vertical composite is equal to $Z \to V$. This follows from Lemma 3.6, since both make the diagram

$$\begin{array}{ccc} Z & & \\ \vdots & \searrow & \\ V & \hookrightarrow & Y \end{array}$$

commute.

□

Example 3.13. Let R be a ring, X and Y local ringed spaces over $\mathrm{Spec}(R)$, $f : X \to Y$ a local map over $\mathrm{Spec}(R)$ and V an open subset of Y. Under the correspondence of Example 2.49, the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ induced by f is given by precomposition with the map $f^{-1}(V) \to V$ of Lemma 3.12.

Proposition 3.14. *Let X be a local ringed space and $(U_i)_{i\in I}$ an open covering of X. Suppose given a local ringed space Y and for each $i \in I$, a local map $f_i : U_i \to Y$ such that for all $i, j \in I$, the diagram*

$$\begin{array}{ccc} U_i \cap U_j & \hookrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle f_j} \\ U_i & \xrightarrow{f_i} & Y \end{array}$$

commutes. Then there exists a unique local map $f : X \to Y$ such that for all $i \in I$, the diagram

$$\begin{array}{ccc} U_i & & \\ \downarrow & \searrow^{f_i} & \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

commutes.

Remark 3.15. Let X and Y be local ringed spaces. It follows from Proposition 3.14 that maps from open subsets of X into Y "satisfy the sheaf condition". In the case that Y is $\mathbf{A}^1_{\mathbf{Z}}$, we recover $\mathcal{O}_X$ (see Example 2.49), but in general we do not obtain a sheaf of rings. We will return to this point in Section 4.

Remark 3.16. Let X be a local ringed space, $(U_i)_{i\in I}$ an open covering of X. Define a category $\mathcal{I}$ as follows:

- The set of objects of $\mathcal{I}$ is the disjoint union of I and $I \times I$.
- For all $i, j \in I$ there are maps $i \leftarrow (i,j) \rightarrow j$. These and identity maps make up at the maps of $\mathcal{I}$.

There is a functor $\mathcal{I} \rightarrow \mathbf{LRS}$ assigning $i \in I$ to U_i, $(i,j) \in I \times I$ to $U_i \cap U_j$ and $i \leftarrow (i,j) \rightarrow j$ to $U_i \hookleftarrow U_i \cap U_j \hookrightarrow U_j$. It follows from Example 3.8 that X becomes a cocone of this functor while Proposition 3.14 asserts that X is its colimit. In particular, X is determined by the open local ringed subspaces U_i for $i \in I$ and the ways these intersect.

Proof. Suppose X is a local ringed space, $(U_i)_{i\in I}$ is an open covering of X and $(f_i : U_i \rightarrow Y)$ is a family satisfying the hypotheses. The gluing lemma of topology provides a unique continuous map $f : X \rightarrow Y$ so that for all $i \in I$, the diagram

$$\begin{array}{ccc} U_i & & \\ \downarrow & \searrow^{f_i} & \\ X & \xrightarrow{f} & Y \end{array}$$

in **Top** commutes. We show there is a unique map of sheaves $f^*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ inducing local homomorphisms on all stalks such that for all $i \in I$, the diagram

$$\begin{array}{ccc} (U_i \hookrightarrow X)^* f^* \mathcal{O}_Y & \dashrightarrow & (U_i \hookrightarrow X)^* \mathcal{O}_X \\ \downarrow \sim & & \downarrow \sim \\ (f_i)_* \mathcal{O}_Y & \longrightarrow & (\mathcal{O}_X)|_{U_i} \end{array}$$

commutes, or equivalently that the restriction $f^*\mathcal{O}_Y|_{U_i} \to (\mathcal{O}_X)|_{U_i}$ is equal to α_i, where α_i denotes the composite $(f^*\mathcal{O}_Y)|_{U_i} \cong (U_i \to X)^* f^*\mathcal{O}_Y \cong f_i^*\mathcal{O}_Y \to (\mathcal{O}_X)|_{U_i}$. It is clear that the latter condition implies the former.

Consider the set $\mathcal{B}$ of open subsets U of X contained in some U_i for $i \in I$. It is immediate that $\mathcal{B}$ is a base for X closed under binary intersections. By Proposition 1.24, maps $f^*\mathcal{O}_Y \to \mathcal{O}_X$ are equivalent to maps $(f^*\mathcal{O}_Y)|_{\mathcal{B}} \to (\mathcal{O}_X)|_{\mathcal{B}}$ or to families of maps$((f^*\mathcal{O}_Y)|_{U_i} \to (\mathcal{O}_X)|_{U_i})_{i\in I}$ which agree on intersections. Thus, it suffices to show the family $(\alpha_i)_{i\in I}$ satisfies this condition.

Fix $i, j \in I$; we will show $\alpha_i|_{U_i\cap U_j} = \alpha_j|_{U_i\cap U_j}$. Commutativity of

$$\begin{array}{ccc} U_i \cap U_j & \hookrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle f_j} \\ U_i & \xrightarrow{f_i} & Y \end{array}$$

means that the diagram

$$\begin{array}{ccccc} (U_i \cap U_j \hookrightarrow U_i)^* f_i^*\mathcal{O}_Y & \xleftarrow{\sim} & (U_i \cap U_j \to Y)^*\mathcal{O}_Y & \xrightarrow{\sim} & (U_i \cap U_j \hookrightarrow U_j)^* f_j^*\mathcal{O}_Y \\ \downarrow & & & & \downarrow \\ (U_i \cap U_j \hookrightarrow U_i)^*\mathcal{O}_X|_{U_i} & & & & (U_i \cap U_j \hookrightarrow U_j)^*\mathcal{O}_X|_{U_j} \\ \downarrow\sim & & & & \downarrow\sim \\ \mathcal{O}_X|_{U_i}|_{U_i\cap U_j} & = & \mathcal{O}_X|_{U_i\cap U_j} & = & \mathcal{O}_X|_{U_j}|_{U_i\cap U_j} \end{array}$$

commutes. We will show that

$$\begin{array}{ccccc}
f^*\mathcal{O}_Y|_{U_i}|_{U_i\cap U_j} & = & & = & f^*\mathcal{O}_Y|_{U_i\cap U_j} \\
\downarrow\sim & & & & \downarrow\sim \\
((U_i \hookrightarrow X)^* f^*\mathcal{O}_Y)|_{U_i\cap U_j} & \xrightarrow{\sim} & (U_i\cap U_j \hookrightarrow U_i)^*(U_i \hookrightarrow X)^* f^*\mathcal{O}_Y & \xrightarrow{\sim} & (U_i\cap U_j \hookrightarrow X)^* f^*\mathcal{O}_Y \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
(f_i^*\mathcal{O}_Y)|_{U_i\cap U_j} & \xrightarrow{\sim} & (U_i\cap U_j \hookrightarrow U_i)^* f_i^*\mathcal{O}_Y & \xrightarrow{\sim} & (U_i\cap U_j \to Y)^*\mathcal{O}_Y \\
\downarrow & & \downarrow & & \\
\mathcal{O}_X|_{U_i}|_{U_i\cap U_j} & \xrightarrow{\sim} & (U_i\cap U_j \hookrightarrow U_i)^*(\mathcal{O}_X|_{U_i}) & &
\end{array}$$

commutes. Pasting this and a similar diagram for f_j with the previous diagram will give the desired equality. For this, the top pentagon commutes by Example 1.37 and the left two squares commute by naturality. The commutativity of the right square is a formal consequence of how the isomorphisms were defined in Remark 1.36. □

Let X and Y be local ringed spaces. Proposition 3.14 describes the local maps $X \to Y$ in terms of a given open covering of X. The following proposition describes the local maps $X \to Y$ in terms of a given open covering of Y.

Proposition 3.17. *Let X and Y be local ringed spaces and $(V_i)_{i\in I}$ an open covering of Y. Local maps $f : X \to Y$ correspond to the following data:*

An open covering $(U_i)_{i\in I}$ of X together with a family $(f_i : U_i \to V_i)_{i\in I}$ of local maps satisfying the following conditions:

(1) For all $i, j \in I$, $f_i^{-1}(V_i \cap V_j) = U_i \cap U_j$. Thus f_i pulls back to a map $U_i \cap U_j \to V_i \cap V_j$ as in Lemma 3.12.

(2) For all $i, j \in I$, f_i and f_j pull back to the same map $U_i \cap U_j \to V_i \cap V_j$.

Proof. Suppose given a local map $f : X \to Y$. We obtain the above data by taking the open covering $(f^{-1}(V_i))_{i\in I}$ and for all $i \in I$ taking $f^{-1}(V_i) \to V_i$ to be pulled back from $f : X \to Y$ along $V_i \to Y$ as in Lemma 3.12. Conditions (1) and (2) are clear.

Now suppose given the data in the statement. We wish to show there is a unique local map $f : X \to Y$ such that for all $i \in I$, $f^{-1}(V_i) = U_i$ and the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Consider the family $(U_i \to V_i \hookrightarrow Y)_{i\in I}$. We will apply Proposition 3.14 to this family to obtain the desired local map $f : X \to Y$ (it is clear that we are forced to define f in this way). The hypotheses of Proposition 3.14 are satisfied, from the following diagram for $i, j \in I$.

$$\begin{array}{ccccc} & & U_i \cap U_j & \hookrightarrow & U_j \\ & \swarrow & \downarrow & & \downarrow{\scriptstyle f_j} \\ U_i \cap U_j & \longrightarrow & V_i \cap V_j & \hookrightarrow & V_j \\ \downarrow & & \downarrow & & \downarrow \\ U_i & \xrightarrow{f_i} & V_i & \hookrightarrow & Y \end{array}$$

Here the bottom right square commutes by Example 3.8, the other two squares commute by definition and the top triangle commutes by (2).

Thus we obtain $f : X \to Y$. Let $i \in I$. We compute

$$\begin{aligned} f^{-1}(V_i) &= \bigcup_{j\in I} f^{-1}(V_i)\cap U_j \\ &= \bigcup_{j\in I} f_j^{-1}(V_i\cap V_j) \\ &= \bigcup_{j\in I} U_i\cap U_j \qquad \text{(By (1))} \\ &= U_i. \end{aligned}$$

Thus f is the desired map.

□

3.2. **Gluing Schemes.** Currently the only examples of schemes we have are open subschemes (Corollary 3.11) of affine schemes. To build more interesting schemes, we will need to glue known schemes together.

Definition 3.18. Suppose given the following data:

- A family of local ringed spaces $(X_i)_{i\in I}$
- For each $i,j\in I$ an open subset X_{ij} of X_i.
- For each $i,j\in I$ an isomorphism of local ringed spaces $\psi_{ij}: X_{ij}\cong X_{ji}$.

We say a local ringed space X is *glued* from the $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}$ and $(\psi_{ij})_{i,j\in I}$ if there is an open covering $(U_i)_{i\in I}$ of X and for each $i\in I$ an isomorphism of local ringed spaces $\psi_i: X_i\cong U_i$ satisfying the following conditions:

(1) For each $i,j\in I$, $\psi_i^{-1}(U_i\cap U_j)=X_{ij}$. Thus, for $i,j\in I$ $\psi_i: X_i\cong U_i$ pulls back to an isomorphism $X_{ij}\cong U_i\cap U_j$ (Lemma 3.12).

(2) For each $i,j\in I$ the diagram

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\sim} & X_{ji} \\ & \searrow^{\sim} & \downarrow{\scriptstyle \sim} \\ & & U_i \cap U_j \end{array}$$

commutes.

Example 3.19. Let X be a local ringed space and $(U_i)_{i\in I}$ an open cover of X. Then X is glued from the $(U_i)_{i\in I}, (U_i \cap U_j)_{i,j\in I}$ and $(U_i \cap U_j = U_j \cap U_i)_{i,j\in I}$.

Lemma 3.20. *Suppose given $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}, (\psi_{ij})_{i,j\in I}, X, (U_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ as in Definition 3.18. If for all $i \in I$, X_i is a scheme, then X is a scheme.*

Proof. This is immediate. □

Suppose $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}$ and $(\psi_{ij})_{i,j\in I}$ are as in Definition 3.2. Then a local ringed space glued from the $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}$ and $(\psi_{ij})_{i,j\in I}$ is unique if it exists. In fact, we have the following:

Lemma 3.21. *Suppose $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}, (\psi_{ij})_{i,j\in I}, X, (U_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ are as in Definition 3.18. Define a functor $\mathcal{I} \to \mathbf{LRS}$ where $\mathcal{I}$ is the category of Remark 3.16 by assigning $i \in I$ to X_i, $(i,j) \in I \times I$ to X_{ij} and $i \leftarrow (i,j) \to j$ to $X_i \hookleftarrow X_{ij} \cong X_{ji} \hookrightarrow X_j$. Then X is the colimit of this functor.*

Proof. This is an immediate consequence of Remark 3.16 and Definition 3.18. □

Lemma 3.22. *Suppose $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}, (\psi_{ij})_{i,j\in I}, X, (U_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ are as in Definition 3.18 and Y is a local ringed space. Local maps $f : Y \to X$ correspond to the following data:*

An open covering $(V_i)_{i\in I}$ of Y together with a family $(f_i : V_i \to X_i)_{i\in I}$ of local maps satisfying the following conditions:

(1) For all $i, j \in I$, $f_i^{-1}(X_{ij}) = V_i \cap V_j$. Thus f_i pulls back to a map $V_i \cap V_j \to X_{ij}$ as in Lemma 3.12.

(2) For all $i, j \in I$, the diagram

$$\begin{array}{ccc} V_i \cap V_j & \longrightarrow & X_{ij} \\ & \searrow & \downarrow \sim \\ & & X_{ji} \end{array}$$

commutes, where the horizontal map is pulled back from $f_i : V_i \to X_i$ and the diagonal map is pulled back from $f_j : V_j \to X_j$.

Proof. This is an immediate consequence of Proposition 3.17 and Definition 3.18. □

We spend the rest of the section proving the following criteria for the existence of a glued local ringed space:

Proposition 3.23. *Suppose $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}$ and $(\psi_{ij})_{i,j\in I}$ are as in Definition 3.2. Then there exists a local ringed space X glued from the $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}$ and $(\psi_{ij})_{i,j\in I}$ if and only if the following three conditions are satisfied:*

(1) For all $i \in I$, $X_{ii} = X_i$.

(2) For all $i, j, k \in I$, $\psi_{ij}^{-1}(X_{ji} \cap X_{jk}) = X_{ij} \cap X_{ik}$. Thus $\psi_{ij} : X_{ij} \cong X_{ji}$ pulls back to an isomorphism $X_{ij} \cap X_{ik} \cong X_{ji} \cap X_{jk}$.

(3) For all $i, j, k \in I$ the diagram

$$\begin{array}{ccc} X_{ij} \cap X_{ik} & \xrightarrow{\sim} & X_{ji} \cap X_{jk} \\ & \searrow^{\sim} & \downarrow{\scriptstyle \sim} \\ & & X_{ki} \cap X_{kj} \end{array}$$

commutes.

Proof. First suppose X is glued from the $(X_i)_{i\in I}$, $(X_{ij})_{i,j\in I}$, with this exhibited by $(U_i)_{i\in I}$ and $(\psi_i)_{i\in I}$. We check the conditions in turn.

(1) Let $i \in I$. Then $X_{ii} = \psi_i^{-1}(U_i \cap U_i) = \psi_i^{-1}(U_i) = X_i$.

(2) Let $i, j, k \in I$. Then $X_{ij} \cap X_{ik} = \psi_i^{-1}(U_i \cap U_j) \cap \psi_i^{-1}(U_i \cap U_k) = \psi_i^{-1}(U_i \cap U_j \cap U_k)$. Similarly, $X_{ji} \cap X_{jk} = \psi_j^{-1}(U_i \cap U_j \cap U_k)$. Hence $\psi_{ij}^{-1}(X_{ji} \cap X_{jk}) = \psi_{ij}^{-1}(\psi_j^{-1}(U_i \cap U_j \cap U_k)) = \psi_i^{-1}(U_i \cap U_j \cap U_k) = X_{ij} \cap X_{ik}$. Note that the pulled back isomorphism $X_{ij} \cap X_{ik} \cong X_{ji} \cap X_{jk}$ is given by $\psi_i^{-1}(U_i \cap U_j \cap U_k) \cong U_i \cap U_j \cap U_k \cong \psi_j^{-1}(U_i \cap U_j \cap U_k)$.

(3) This is immediate from the above.

We now prove the converse. Suppose the three conditions of the proposition are satisfied. We take the underlying space of X to be the colimit of the composite $\mathcal{I} \to \mathbf{LRS} \to \mathbf{Top}$, where the first functor is from Lemma 3.21. As in Example A.37, X is the quotient of the disjoint union $\bigsqcup_{i\in I} X_i$ by the equivalence relation generated by the relation defined as follows:

$(*)$ For $i, j \in I$, $x \in X_i$ and $x' \in X_j$, (i, x) is related to (j, x') if and only if $x \in X_{ij}$ and $\psi_{ij}(x) = x'$.

We claim this is already an equivalence relation.

(Reflexivity): It follows from (1) and (3) that $\psi_{ii} : X_i \cong X_i$ is the identity map $X_i = X_i$ for $i \in I$. Reflexivity follows.

(Symmetry): It follows from (3) and the above that $\psi_{ji} : X_{ji} \cong X_{ij}$ is inverse to $\psi_{ij} : X_{ij} \cong X_{ji}$ for $i, j \in I$. Symmetry follows.

(Transitivity): Let $i, j, k \in I, x \in X_i, x' \in X_j$ and $x'' \in X_k$ be such that (i, x) is related to (j, x') and (j, x') is related to (k, x''). Thus $x \in X_{ij}, x' \in X_{jk}, \psi_{ij}(x) = x'$ and $\psi_{jk}(x') = x''$. In particular $x \in \psi_{ij}^{-1}(X_{ji} \cap X_{jk})$. By (2), $x \in X_{ik}$, while $\psi_{ik}(x) = \psi_{jk}\psi_{ij}(x) = \psi_{jk}(x') = x''$ by (3). Hence (i, x) is related to (k, x'').

For $i \in I$, write ψ_i for the map $X_i \to X$. As a consequence of the above, $\psi_i : X_i \to X$ is injective for all $i \in I$, while for all $i, j \in I$, the diagram of sets

$$\begin{array}{ccc} X_{ij} & \longrightarrow & X_j \\ \downarrow & & \downarrow{\scriptstyle \psi_j} \\ X_i & \xrightarrow{\psi_i} & X \end{array}$$

is a pullback square.

Now let $i \in I$. We claim $\psi_i : X_i \to X$ is an open map of topological spaces. Let U be an open subset of X and $j \in J$. By the above pullback square, $\psi_j^{-1}(\psi_i(U)) = \psi_{ij}(U \cap X_{ij})$. The claim follows from the definition of the topology on X.

For $i \in I$, we take U_i to be the open subset $\psi_i(X_i)$ of X, so that we have a homeomorphism $\psi_i : X_i \cong U_i$. By construction $(U_i)_{i \in I}$ is an open cover of X. For $i, j \in I$, we have $\psi_i^{-1}(U_i \cap U_j) = \psi_i^{-1}(\psi_i(X_i) \cap \psi_j(X_j)) = \psi_i^{-1}(\psi_j(X_j)) = \psi_{ij}^{-1}(X_j) = X_{ij}$, while the diagram of spaces

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\sim} & X_{ji} \\ & \searrow^{\sim} & \downarrow{\scriptstyle \sim} \\ & & U_i \cap U_j \end{array}$$

commutes by construction. It remains to promote X to a local ringed space so that for each $i \in I$, $\psi_i : X_i \cong U_i$ becomes an isomorphism of local ringed spaces and for all $i, j \in I$ the lift of the above diagram to **LRS** commutes.

Write $\mathcal{B}$ for the base of X (closed under binary intersections) consisting of open subsets U of X contained in U_i for some $i \in I$, as in the proof of Proposition 3.14. We will take $\mathcal{O}_X$ to be extended from $\mathcal{O}_\mathcal{B}$ (Proposition 1.24). As in Section 2.2, it will be convenient to work with an equivalent auxiliary category.

Define a category $\mathcal{X}$ as follows:

- The objects of $\mathcal{X}$ are pairs (i, U) for $i \in I$ and $U \in \Omega(X_i)$.
- For $i, j \in I$, $U \in \Omega(X_i)$ and $U' \in \Omega(X_j)$, there is a map from $(i, U) \to (j, U')$ in $\mathcal{X}$ if and only if $\psi_i(U) \subset \psi_j(U')$ (in which case there is only one).

Let $i, j \in I$, $U \in \Omega(X_i)$ and $U' \in \Omega(X_j)$. It follows from the above pullback square that there is a map $(i, U) \to (j, U')$ in $\mathcal{X}$ if and only if $U \subset X_{ij}$ and $\psi_{ji}^{-1}(U) = \psi_{ij}(U) \subset U'$. We may thus define a functor $\mathcal{X}^{\mathrm{op}} \to \mathbf{Ring}$ as follows:

- For $i \in I$ and $U \in \Omega(X_i)$, the image of (i, U) is the ring $\mathcal{O}_{X_i}(U)$.
- Let $i, j \in I$, $U \in \Omega(X_i)$ and $U' \in \Omega(X_j)$ such that $\psi_i(U) \subset \psi_j(U')$. The isomorphism of local ringed spaces $\psi_{ji} : X_{ji} \cong X_{ij}$ induces an isomorphism $\mathcal{O}_{X_{ij}} \cong (\psi_{ji})_* \mathcal{O}_{X_{ji}}$ (Lemma 3.4), whose value on

$U \subset X_{ij}$ is an isomorphism $\mathcal{O}_{X_i}(U) \cong \mathcal{O}_{X_j}(\psi_{ji}^{-1}(U))$. We take the composite $\mathcal{O}_{X_j}(U') \to \mathcal{O}_{X_j}(\psi_{ji}^{-1}(U)) \cong \mathcal{O}_{X_i}(U)$ of its inverse with the restriction map $\mathcal{O}_{X_j}(U') \to \mathcal{O}_{X_j}(\psi_{ji}^{-1}(U))$ to be the image of $(i, U) \to (j, U')$.

That this assignment preserves identities is clear. To show it preserves composition, let $i, j, k \in I$, $U \in \Omega(X_i), U' \in \Omega(X_j)$ and $U'' \in \Omega(X_k)$ such that $\psi_i(U) \subset \psi_j(U') \subset \psi_k(U'')$. Consider the following diagram.

$$\begin{array}{ccccc}
 & & \mathcal{O}_{X_k}(U'') & & \\
 & & \downarrow & & \\
 & \mathcal{O}_{X_k}(\psi_{kj}^{-1}(U')) & \xrightarrow{\sim} & \mathcal{O}_{X_j}(U') & \\
 & \downarrow & & \downarrow & \\
\mathcal{O}_{X_k}(\psi_{kj}^{-1}(\psi_{ji}^{-1}(U))) & \xrightarrow{\sim} & \mathcal{O}_{X_j}(\psi_{ji}^{-1}(U)) & \xrightarrow{\sim} & \mathcal{O}_{X_i}(U)
\end{array}$$

Here the square commutes by naturality of $\mathcal{O}_{X_{jk}} \cong (\psi_{kj})_* \mathcal{O}_{X_{kj}}$. The result follows from (3).

By construction there is a fully faithful and essentially surjective functor $\mathcal{X} \to \mathcal{B}$; we choose an inverse $\mathcal{B} \to \mathcal{X}$ and take $\mathcal{O}_{\mathcal{B}}$ to be the composite $\mathcal{B}^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}} \to \mathbf{Ring}$. Thus there is an isomorphism between $\mathcal{X}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{O}_B} \mathbf{Ring}$ and the functor defined above.

Unwinding how that functor was defined, this means the following:

- For each $i \in I$, we have an isomorphism between $\Omega(X_i)^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{O}_{\mathcal{B}}} \mathbf{Ring}$ and $\mathcal{O}_{X_i}$, or equivalently between $\Omega(U_i)^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{O}_{\mathcal{B}}} \mathbf{Ring}$ and $(\psi_i)_* \mathcal{O}_{X_i}$.
- For each $i, j \in I$ and U an open subset of $U_i \cap U_j$, the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{B}}(U) & & \\ \downarrow \sim & \searrow^{\sim} & \\ \mathcal{O}_{X_i}(\psi_i^{-1}(U)) & \xrightarrow{\sim} & \mathcal{O}_{X_j}(\psi_j^{-1}(U)) \end{array}$$

commutes.

From the first of these it follows that $\mathcal{O}_{\mathcal{B}}$ is a sheaf on $\mathcal{B}$. Let $\mathcal{O}_X$ be a sheaf on X which restricts to $\mathcal{O}_{\mathcal{B}}$. It is then immediate that $\mathcal{O}_X$ makes X into a local ringed space, ψ_i becomes an isomorphism of local ringed spaces for all $i \in I$ and the appropriate diagrams of local ringed spaces commute for all $i, j \in I$.

□

3.3. **Projective Space over a Ring.** Let K be a field and $n \in \mathbf{N}$. Recall that the variety $\mathbf{P}_K^n$ has points $(k_0, \ldots, k_n) \in K^{n+1} - \{(0, \ldots, 0)\}$ modulo the relation that $(k_0, \ldots, k_n) \in K^{n+1} - \{(0, \ldots, 0)\}$ is equivalent to $(k_0', \ldots, k_n') \in K^{n+1} - \{(0, \ldots, 0)\}$ if and only if there exists $\lambda \in K$ nonzero such that $\lambda k_i = k_i'$ for all $0 \leq i \leq n$. We write $[k_0, \ldots, k_n]$ for the equivalence class of $(k_0, \ldots, k_n) \in K^{n+1} - \{(0, \ldots, 0)\}$. There is an open covering $(U_i)_{0 \leq i \leq n}$ of $\mathbf{P}_K^n$ where for $(k_0, \ldots, k_n) \in K^{n+1}$ and $0 \leq i \leq n$, we have $[k_0, \ldots, k_n] \in U_i$ if and only if $k_i \neq 0$. For $0 \leq i \leq n$, there is an isomorphism of varieties $\psi_i : \mathbf{A}_K^n \cong U_i$; viewing $\mathbf{A}_K^n$ as the set of tuples $(k_0, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)$ with no ith coordinate, the inverse of ψ_i is given by $(k_0, \ldots, k_n) \mapsto [k_1, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_n]$. For $0 \leq i, j \leq n$ distinct, $\psi_i^{-1}(U_i \cap U_j)$ is the set $D(x_j)$ of points $(k_0, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \in K^n$ with k_j nonzero. For such i, j, the isomorphism $D(x_j) \cong U_i \cap U_j \cong D(x_i)$ is given by $(k_0, \ldots, k_n) \mapsto (k_0 k_j^{-1}, \ldots k_{i-1} k_j^{-1}, k_j^{-1}, k_{i+1} k_j^{-1}, \ldots k_n k_j^{-1})$. We can

use analogous maps between schemes to define projective n-space over a ring by gluing.

Definition 3.24. Let R be a ring and $n \in \mathbf{N}$. We define *projective n-space* over R, denoted $\mathbf{P}^n_R$, to be the scheme glued from the following data (Definition 3.18 and Lemma 3.20):

- The family of local ringed spaces $(\mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]))_{0 \leq i \leq n}$. Here for $0 \leq i \leq n$, $R[x_0, \ldots, \hat{x_i}, \ldots, x_n]$ is a polynomial ring in the variables x_j for $0 \leq j \leq n$ distinct from i.
- For $0 \leq i, j \leq n$, the following open subset of $(\mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]))_{0 \leq i \leq n}$:
  - If $i = j$, then $\mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n])$ itself.
  - If $i \neq j$, then $D(x_j)$.
- For each $0 \leq i, j \leq n$, the following isomorphism of schemes:
  - If $i = j$, then the identity map $\mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]) = \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n])$.
  - If $i \neq j$, the isomorphism $D(x_j) \cong \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j}) \cong \mathrm{Spec}(R[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i}) \cong D(x_i)$ where the first and last isomorphism are from Lemma 3.9 and the middle isomorphism is induced by the isomorphism of rings $\varphi_{ij} : R[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i} \cong R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j}$ defined as follows:

    It is the unique homomorphism of R-algebras such that for $0 \leq k \leq n$ distinct from j,

$$\varphi_{ij}(\frac{x_k}{1}) = \begin{cases} \frac{x_k}{x_j} & \text{if } k \neq i. \\ \frac{1}{x_j} & \text{if } k = i. \end{cases}$$

This is an isomorphism because it is inverse to $\varphi_{ji} : R[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i} \to R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j}$.

It remains to check the conditions of Proposition 3.23. Condition (1) holds by definition. Let $0 \leq i, j, k \leq n$. Condition (2) for i, j and k is immediate if any two are equal. If i, j and k are all distinct, this condition is a consequence of Lemma 2.17 and the definitions. Note in this case, $D(x_j) \cong D(x_i)$ pulls back to $D(x_j x_k) \cong \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j x_k}) \cong \mathrm{Spec}(R[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i x_k}) \cong D(x_i x_k)$, where the first and third isomorphism are from Lemma 3.9 and the middle isomorphism is induced by an isomorphism of rings, defined by a similar formula to φ_{ij}. Let $0 \leq i, j, k \leq n$. Condition (3) for i, j and k is immediate if $i = j$ or $j = k$, while is the above remark that φ_{ij} and φ_{ji} are inverse if $i = k$. If i, j and k are all distinct, this condition reduces to commutativity of a diagram of rings as below.

$$\begin{array}{ccc} R[x_0, \ldots, \hat{x_k}, \ldots, x_n]_{x_i x_j} & \xrightarrow{\sim} & R[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i x_k} \\ & \searrow^{\sim} & \downarrow \sim \\ & & R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j x_k} \end{array}$$

This is a straightforward case analysis.

For $0 \leq i \leq n$, we write U_i for the open subset of $\mathbf{P}^n_R$ as in Definition 3.18.

Remark 3.25. Let R be a ring and $n \in \mathbf{N}$. Then for each $0 \leq i \leq n$, $\mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n])$ is a scheme over $\mathrm{Spec}(R)$ and each of the isomorphisms in Definition 3.24 is a map over $\mathrm{Spec}(R)$. It follows from Lemma 3.21 that $\mathbf{P}^n_R$ may be uniquely promoted to a scheme over $\mathrm{Spec}(R)$ in such a way that for all $0 \leq i \leq n$, $\mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]) \cong U_i \hookrightarrow \mathbf{P}^n_R$ is a map over $\mathrm{Spec}(R)$. We equip $\mathbf{P}^n_R$ with this structure.

Example 3.26. Let K be a field and $n \in \mathbf{N}$. Let us compute the points of the scheme $\mathbf{P}^n_K$ in the sense of Example 2.47. The underlying topological space of $\mathrm{Spec}(K)$ has only one point, namely the zero ideal. It thus follows by Lemma 3.6 that every point $\mathrm{Spec}(K) \to \mathbf{P}^n_K$ factors through $\mathrm{Spec}(K[x_0, \ldots, \hat{x_i}, \ldots x_n])$ for some $0 \leq i \leq n$, corresponding to $(k_0, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \in K^n$ (Example 2.48). For $0 \leq i, j \leq n$, $(k_0, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \in K^n$ and $(k'_0, \ldots, k'_{j-1}, k'_{j+1}, \ldots, k'_n) \in K^n$ determine the same point as above if and only if either $i = j$ and $(k_0, \ldots, k_n) = (k'_0, \ldots, k'_n)$ or $i \neq j$, k_j is invertible and $(k_0 k_j^{-1}, \ldots k_{i-1} k_j^{-1}, k_j^{-1}, k_{i+1} k_j^{-1}, \ldots k_n k_j^{-1}) = (k'_0, \ldots, k'_n)$. This is the same condition for $[k_0, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_n]$ and $[k_0, \ldots, k_{j-1}, 1, k_{j+1}, \ldots, k_n]$ to determine the same point of $\mathbf{P}^n_K$. It follows that the points of $\mathbf{P}^n_K$ correspond to points of $\mathbf{P}^n_K$ in the classical sense.

Remark 3.27. The argument of Example 3.26 relied heavily on the fact that a field has a single prime ideal. In Section 4.3, we will calculate the points of $\mathbf{P}^n_R$ for R an arbitrary ring (see Corollary 4.37).

A problem with Definition 3.24 is that it does not make clear which "function" on $\mathbf{P}^n_R$ a rational function on one of the affine spaces corresponds to. Our next goal is to give an equivalent definition which solves this problem.

Motivation 3.28. Let K be a field and $n \in \mathbf{N}$. For a rational function $f(x_0, \ldots, x_n)$ to be well defined on the variety $\mathbf{P}^n_K$, f should be homogeneous of degree 0 (at least assuming K is infinite). Obviously this means every rational function $\mathbf{P}^n_K \to \mathbf{A}^1_K$ should be constant. Let $0 \leq i \leq n$. In constrast, rational functions on U_i can have denominator a power of x_i. Thus $\Gamma(U_i)$ should be $(K[x_0, \ldots, x_n]_{x_i})_0$, the subring of $K[x_0, \ldots, x_n]_{x_i}$ on the homogeneous elements

of degree 0. In other words, U_i should be the affine variety corresponding to this K-algebra.

Definition 3.29. Let R be a ring and $n \in \mathbf{N}$. We define $(\mathbf{P}^n_R)'$ to be the scheme glued from the following data (Definition 3.18 and Lemma 3.20):

- The family of local ringed spaces $(\mathrm{Spec}((R[x_0, \ldots, x_n]_{x_i})_0))_{0 \leq i \leq n}$.
- For $0 \leq i, j \leq n$ the open subset $D(\frac{x_j}{x_i})$ of $\mathrm{Spec}((R[x_0, \ldots, x_n]_{x_i})_0)$.
- For each $0 \leq i, j \leq n$, the isomorphism of schemes: $D(\frac{x_j}{x_i}) \cong \mathrm{Spec}(((R[x_0, \ldots, x_n]_{x_i})_0)_{\frac{x_j}{x_i}}) \cong \mathrm{Spec}(((R[x_0, \ldots, x_n]_{x_j})_0)_{\frac{x_i}{x_j}}) \cong D(\frac{x_i}{x_j})$, where the first and last isomorphism are from Lemma 3.9 and the middle isomorphism is induced by the isomorphism of rings $((R[x_0, \ldots, x_n]_{x_j})_0)_{\frac{x_i}{x_j}} \cong (R[x_0, \ldots, x_n]_{x_i x_j})_0 \cong ((R[x_0, \ldots, x_n]_{x_i})_0)_{\frac{x_j}{x_i}}$

It remains to check the conditions of Proposition 3.23. Condition (1) is immediate. Condition (2) is a consequence of Lemma 2.17 and the definitions. Note that for $0 \leq i, j, k \leq n$, the isomorphism $D(\frac{x_j}{x_i}) \cong D(\frac{x_i}{x_j})$ pulls back to $D(\frac{x_j x_k}{x_i^2}) \cong \mathrm{Spec}(((R[x_0, \ldots, x_n]_{x_i})_0)_{\frac{x_j x_k}{x_i^2}}) \cong \mathrm{Spec}(((R[x_0, \ldots, x_n]_{x_j})_0)_{\frac{x_i x_k}{x_j^2}}) \cong D(\frac{x_i x_k}{x_j^2})$, where the first and last isomorphisms are from Lemma 3.9 and the middle isomorphism is induced by the isomorphism of rings $((R[x_0, \ldots, x_n]_{x_j})_0)_{\frac{x_i x_k}{x_j^2}} \cong (R[x_0, \ldots, x_n]_{x_i x_j x_k})_0 \cong ((R[x_0, \ldots, x_n]_{x_i})_0)_{\frac{x_j x_k}{x_i^2}}$. Condition (3) follows immediately.

Remark 3.30. Let R be a ring and $n \in \mathbf{N}$. As in Remark 3.25, the scheme $(\mathbf{P}^n_R)'$ becomes a scheme over $\mathrm{Spec}(R)$.

Proposition 3.31. *Let R be a ring and $n \in \mathbf{N}$. There is an isomorphism $(\mathbf{P}^n_R)' \cong \mathbf{P}^n_R$ of schemes over* $\mathrm{Spec}(R)$.

Proof. Let $0 \leq i \leq n$ and consider the homomorphisms of R-algebras $R[x_0, \ldots, \hat{x_i}, \ldots, x_n] \to (R[x_0, \ldots, x_n]_{x_i})_0$ given by $x_j \mapsto \frac{x_j}{x_i}$ for $0 \leq k \leq n$ distinct from i. This homomorphism maps the basis $(x_0^{k_0} \ldots x_{i-1}^{k_{i-1}} x_{i+1}^{k_{i+1}} \ldots x_n^{k_n})_{k_0,\ldots,k_{i-1},k_{i+1},\ldots,k_n \in \mathbf{N}}$ of the first ring to a basis of the second, so is an isomorphism. This induces an isomorphism of schemes $\mathrm{Spec}((R[x_0, \ldots, x_n]_{x_i})_0) \cong \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n])$ over $\mathrm{Spec}(R)$. For $0 \leq i, j \leq n$, this isomorphism pulls back to an isomorphism $D(\frac{x_j}{x_i}) \cong D(x_j)$. Let $0 \leq i, j \leq n$ distinct. The result will follow from Lemma 3.21 once we show the diagram

$$\begin{array}{ccc} D(\frac{x_j}{x_i}) & \xrightarrow{\sim} & D(\frac{x_i}{x_j}) \\ \downarrow \sim & & \downarrow \sim \\ D(x_j) & \xrightarrow{\sim} & D(x_i) \end{array}$$

commutes, where the horizontal maps are given in Definition 3.29 and Definition 3.24. This reduces to the commutativity of the a diagram of rings as below.

$$\begin{array}{ccc} R[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i} & \xrightarrow{\sim} & R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j} \\ & \searrow^{\sim} & \downarrow \sim \\ & & (R[x_0, \ldots, x_n]_{x_i x_j})_0 \end{array}$$

Both ways around the diagram map $\frac{x_k}{1}$ for $0 \leq k \leq n$ distinct from j to $\frac{x_k x_i}{x_i x_j}$. □

This alternative approach will allow us to compute the global sections of projective space.

Proposition 3.32. *Let R be a ring and $n \in \mathbf{N}$. Then the map $R \to \Gamma(\mathbf{P}_R^n)$ (corresponding to the map $\mathbf{P}_R^n \to \mathrm{Spec}(R)$ of Remark 3.25) is an isomorphism.*

Proof. By Proposition 2.43, this is equivalent to the assertion that every map $\mathbf{P}^n_R \to \mathbf{A}^1_R$ factors uniquely as

$$\begin{array}{ccc} \mathbf{P}^n_R & & \\ \downarrow & \searrow & \\ \mathbf{A}^0_R & \xrightarrow{r} & \mathbf{A}^1_R \end{array}$$

for some $r \in R$ (here we use the correspondence of Example 2.48 between points $\mathbf{A}^0_R \to \mathbf{A}^1_R$ and elements $r \in R$). It suffices to show the corresponding result for $(\mathbf{P}^n_R)'$ (Proposition 3.31). By Lemma 3.21 and Example 2.49, maps $(\mathbf{P}^n_R)' \to \mathbf{A}^1_R$ correspond to families $(f_i \in (R[x_0, \ldots, x_n]_{x_i})_0)_{0 \leq i \leq n}$ such that for all $0 \leq i \leq n$, the image of f_i in $(R[x_0, \ldots, x_n]_{x_i x_j})_0$ is equal to the image of f_j in $(R[x_0, \ldots, x_n]_{x_i x_j})_0$. For $r \in R$ and $0 \leq i \leq n$, the family induced by $(\mathbf{P}^n_R)' \to \mathbf{A}^0_R \xrightarrow{r} \mathbf{A}^1_R$ has ith element the image of r in $(R[x_0, \ldots, x_n]_{x_i})_0$.

Suppose given a family $(f_i)_{0 \leq i \leq n}$ as above; we wish to show it arises uniquely from $r \in R$. Let $f \in (R[x_0, \ldots, x_n]_{x_0 \cdots x_n})_0$ be the common image of the f_i for $0 \leq i \leq n$ in $(R[x_0, \ldots, x_n]_{x_0 \cdots x_n})_0$. By definition, there are no nonzero terms with a positive exponent in the expansion of f as a sum of monomials. Since f is homogeneous of degree 0, it is in the image of $r \in R$. This is the desired element, while uniqueness is clear. □

Projective space is evidently functorial.

Definition 3.33. Let $n \in \mathbf{N}$, R and S rings and $\varphi : R \to S$ a homomorphism of rings. We let $\mathbf{P}^n_\varphi : \mathbf{P}^n_S \to \mathbf{P}^n_R$ denote the map of schemes corresponding to the following data (Section 4.3):

- The open covering $(U_i)_{0 \leq i \leq n}$ of $\mathbf{P}^n_S$.
- For $0 \leq i \leq n$, the map $U_i \cong \operatorname{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots, x_n]) \to \operatorname{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n])$.

Here condition (1) holds by Lemma 2.17 and Definition 3.18. For $0 \leq i, j \leq n$ distinct, the map $U_i \to \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n])$ pulls back to $U_i \cap U_j \cong D(x_j) \cong \mathrm{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j}) \to \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j})$. Condition (3) thus follows from the definitions.

Lemma 3.34. *Let $n \in \mathbf{N}$, R and S rings and $\varphi : R \to S$ a homomorphism of rings. Then the diagram*

$$\begin{array}{ccc} \mathbf{P}^n_S & \longrightarrow & \mathbf{P}^n_R \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

is a pullback square in **Sch**, *where the vertical maps are given by Remark 3.25 and the top horizontal map is given by Definition 3.33.*

Proof. Let Y be a scheme and

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{P}^n_R \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

be a commutative diagram. Let $(V_i)_{0 \leq i \leq n}, (f_i : V_i \to \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots x_n]))_{0 \leq i \leq n}$ be the corresponding data of Lemma 3.22. By that lemma (and Proposition 3.14), a map $Y \to \mathbf{P}^n_R$ making

$$\begin{array}{ccc} Y & & \\ \vdots\downarrow & \searrow & \\ \mathbf{P}^n_S & \longrightarrow & \mathbf{P}^n_R \end{array}$$

and

$$\begin{array}{ccc} Y & \dashrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(S) \end{array}$$

commute is tantamount to for each $0 \leq i \leq n$, a map $V_i \to \mathrm{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots x_n])$ making

$$\begin{array}{ccc} V_i & & \\ \downarrow & \searrow & \\ \mathrm{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots x_n]) & \longrightarrow & \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots x_n]) \end{array}$$

and

$$\begin{array}{ccc} V_i & \dashrightarrow & \mathrm{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots x_n]) \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(S) \end{array}$$

commute in such a way that for all $0 \leq i, j \leq n$ distinct the diagram

$$\begin{array}{ccc} V_i \cap V_j & \dashrightarrow & D(x_j) \\ & \searrow & \downarrow \sim \\ & & D(x_i) \end{array}$$

commutes, where the horizontal map is pulled back from $V_i \to \mathrm{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots x_n])$, the diagonal map is pulled back from $V_j \to \mathrm{Spec}(S[x_0, \ldots, \hat{x_j}, \ldots x_n])$ and the vertical map is as in Definition 3.24.

For $0 \leq i \leq n$ we have a pullback square

$$\begin{array}{ccc} \mathrm{Spec}(S[x_0, \ldots, \hat{x_i}, \ldots x_n]) & \longrightarrow & \mathrm{Spec}(R[x_0, \ldots, \hat{x_i}, \ldots x_n]) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

by Example 2.51. It follows that there is a unique map $V_i \to \mathrm{Spec}(S[x_0, \ldots, \hat{x_j}, \ldots x_n])$ satisfying the first two conditions.

To show the family of such maps satisfies the third condition, let $0 \leq i, j \leq n$ be distinct. We have a pullback square

$$\begin{array}{ccc} D(x_i) & \longrightarrow & D(x_i) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(S[x_0, \ldots, \hat{x_j}, \ldots x_n]) & \longrightarrow & \mathrm{Spec}(R[x_0, \ldots, \hat{x_j}, \ldots x_n]) \end{array}$$

by Lemma 2.17 and Lemma 3.12. Pasting with the above pullback square for j yields the following pullback square.

$$\begin{array}{ccc} D(x_i) & \longrightarrow & D(x_i) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

We conclude by observing that the diagram

$$\begin{array}{ccc} V_i \cap V_j & \longrightarrow & D(x_j) \\ & \searrow & \downarrow \sim \\ & & D(x_i) \end{array}$$

commutes upon composing with $D(x_i) \to \mathrm{Spec}(S)$ and by $D(x_i) \to D(x_i)$. $\square$

4. Schemes as Functors

In Section 3, we defined a scheme X to be a topological space X together with the data of what maps from all open subsets of X into $\mathbf{A}^1$ should be, satisfying conditions that ensure X behaves like a variety. In this section, we give an equivalent definition of a scheme: a scheme X consists of the data of

what maps from all affine schemes into X should be, satisfying conditions that ensure X behaves like a variety.

Definition 4.1. Let X be a scheme. We write h_X for the composite functor $\mathbf{Ring} \xrightarrow{\mathrm{Spec}} \mathbf{Sch}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbf{Sch}}(-,X)} \mathbf{Set}$ (see Example A.19). The assignment $X \mapsto h_X$ becomes a functor $h : \mathbf{Sch} \to [\mathbf{Ring}, \mathbf{Set}]$.

Example 4.2. Let X be a scheme. The values of h_X at fields parametrize the points of the underlying space of X. Namely, let K be a field. Unwinding the definitions, elements of $h_X(K)$ correspond to points $x \in X$ together with a local homomorphism $(\mathcal{O}_X)_x \to K$, or equivalently to points $x \in X$ together with a homomorphism of fields $(\mathcal{O}_X)_x/\mathfrak{m}_x \to K$.

Let X be a scheme. It follows from Example 4.2 that the underlying set of X is determined by its value at fields. More generally, h_X contains enough data to recover the whole scheme X.

Proposition 4.3. *The functor $h : \mathbf{Sch} \to [\mathbf{Ring}, \mathbf{Set}]$ of Definition 4.1 is fully faithful.*

Proof. The functor h factors as $\mathbf{Sch} \to [\mathbf{AffSch}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\sim} [\mathbf{Ring}, \mathbf{Set}]$ where the latter functor is an equivalence of categories (Corollary 2.4). It thus suffices to show $\mathbf{Sch} \to [\mathbf{AffSch}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. Let X and Y be schemes and $\alpha : \mathrm{Hom}_{\mathbf{Sch}}(-, X)|_{\mathbf{AffSch}^{\mathrm{op}}} \to \mathrm{Hom}_{\mathbf{Sch}}(-, Y)|_{\mathbf{AffSch}^{\mathrm{op}}}$ a natural transformation. We wish to show there is a unique $f : X \to Y$ so that α is given by postcomposition by f. Let $(U_i)_{i \in I}$ be a covering of X by affine open sets. By Proposition 3.14, a desired $f : X \to Y$ is determined by the requirement that for all $i \in I$,

$$\begin{array}{ccc} U_i & & \\ \downarrow & \searrow^{\alpha_{U_i}(U_i \hookrightarrow X)} & \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Fix $i, j \in I$; we next show the diagram

$$\begin{array}{ccc} U_i \cap U_j & \hookrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle \alpha_{U_j}(U_j \hookrightarrow X)} \\ U_i & \xrightarrow{\alpha_{U_i}(U_i \hookrightarrow X)} & Y \end{array}$$

commutes. By Corollary 3.10, we may cover $U_i \cap U_j$ by affine open sets $(V_{ijk})_{k \in K_{ij}}$. Applying Proposition 3.14, it suffices to show for all $k \in K_{ij}$ that

$$\begin{array}{ccc} V_{ijk} & \hookrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle \alpha_{U_j}(U_j \hookrightarrow X)} \\ U_i & \xrightarrow{\alpha_{U_i}(U_i \hookrightarrow X)} & Y \end{array}$$

commutes. This follows from naturality of α.

Thus we may take $f : X \to Y$ such that

$$\begin{array}{ccc} U_i & & \\ \downarrow & \searrow^{\alpha_{U_i}(U_i \hookrightarrow X)} & \\ X & \xrightarrow{f} & Y \end{array}$$

commutes for all $i \in I$. Let Z be an affine scheme and $g : Z \to X$; to conclude we show the diagram

$$\begin{array}{ccc} Z & & \\ \downarrow{\scriptstyle g} & \searrow^{\alpha_Z(g)} & \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. By Corollary 3.10, for each $i \in I$, there is a covering $(W_{ij})_{j \in J_i}$ of $g^{-1}(U_i)$ by affine open sets. By Proposition 3.14 it suffices to show the above diagram commutes upon precomposition with $W_{ij} \hookrightarrow Z$ for all $i \in I$ and $j \in J_i$. For $i \in I$ and $j \in J_j$, we have a diagram

$$\begin{array}{ccccc} W_{ij} & \hookrightarrow & Z & & \\ \downarrow & & \downarrow{\scriptstyle g} & \searrow^{\alpha_Z(g)} & \\ U_i & \hookrightarrow & X & \xrightarrow{f} & Y \end{array}$$

where the left square commutes and the bottom composite is given by $\alpha(U_i \hookrightarrow X)$. The result follows from naturality of α. $\square$

In Section 4.1, we give an elementary description of the essential image of h. We apply this description in Section 4.2 to show the category **Sch** has all pullbacks and to construct affine/projective space over an arbitrary scheme. In Section 4.3, we compute which functor projective space over a scheme corresponds to.

4.1. **The Topology of Functors.**

Definition 4.4. Let R be a ring. We write h_R for the functor $\mathrm{Hom}_{\mathbf{Ring}}(R, -) : \mathbf{Ring} \to \mathbf{Set}$ (Example A.19).

Remark 4.5. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor. By Lemma A.26, natural transformations from $h_R \to F$ for R a ring correspond to elements of $F(R)$. We may thus view F as a functor of maps from the "spaces" h_R for R a ring into the "space" F. For rings R, S, a homomorphism $\varphi : R \to S$ and $f \in F(R)$, we thus write $f|_S$ for $F(\varphi)(f) \in F(S)$.

Definition 4.6. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor. We say F is *affine* if there exists a ring R an an isomorphism $F \cong h_R$.

Example 4.7. The forgetful functor $\mathbf{A}^1 : \mathbf{Ring} \to \mathbf{Set}$ of Example A.16 is affine in the sense of Definition 4.6: we can take R to be $\mathbf{Z}[x]$. We call $\mathbf{A}^1$ the affine line.

Lemma 4.8. *Let X be a scheme. Then X is an affine scheme if and only if h_X is affine in the sense of Definition 4.6.*

Proof. For R a ring, we have an isomorphism $h_{\mathrm{Spec}(R)} \cong h_R$ by Theorem 2.2. The result follows from Proposition 4.3. □

Definition 4.9. Let R be a ring and S a subset of R. The assignment $T \mapsto \{\varphi : R \to T \mid \varphi(S) \text{ generates } T\}$ determines a subfunctor of h_R, which we denote by h_S.

Definition 4.10. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor. We say a subfunctor $F' \hookrightarrow F$ of F is *open* if for all rings R and $f \in F(R)$, there exists a subset S of R and an isomorphism $h_R \times_F F' \cong h_S$ over h_R. Here $h_R \times_F F'$ is the pullback (Example A.33) of $F' \hookrightarrow F$ along the map $f : h_R \to F$ corresponding to f (Remark 4.5).

Remark 4.11. Let $F, G : \mathbf{Ring} \to \mathbf{Set}$ be functors, $G \to F$ a natural transformation and $F' \hookrightarrow F$ an open subfunctor of F. Then $G \times_F F' \hookrightarrow G$ is an open subfunctor of G. This is immediate from Definition 4.10.

Lemma 4.12. *Let X be a scheme. The open subfunctors $F \hookrightarrow h_X$ of h_X are precisely those of the form $h_U \hookrightarrow h_X$ for U an open subset of X viewed as an open subscheme of X as in Corollary 3.11.*

Proof. We first note the following:

($*$) Let R be a ring and S a subset of R. Then there is an isomorphism $h_S \cong h_{(\bigcup_{f\in S} D(f))}$ making

$$\begin{array}{ccc} h_S & \hookrightarrow & h_R \\ \downarrow\sim & & \downarrow\sim \\ h_{(\bigcup_{f\in S} D(f))} & \hookrightarrow & h_{\mathrm{Spec}(R)} \end{array}$$

commute.

To see this, let T be a ring and $\varphi : R \to T$ a homomorphism of rings. Then

$\varphi \in h_S(T) \Leftrightarrow \varphi(S)$ generates T

$$\Leftrightarrow \mathrm{Spec}(T) \subset \bigcup_{f\in S} D(\varphi(f)) \qquad \text{(By Lemma 2.13)}$$

$$\Leftrightarrow \mathrm{Spec}(T) \subset \mathrm{Spec}(\varphi)^{-1}(\bigcup_{f\in S} D(f)) \qquad \text{(By Lemma 2.17)}$$

$$\Leftrightarrow \mathrm{Spec}(\varphi) \text{ factors through } \bigcup_{f\in S} D(f) \qquad \text{(By Lemma 3.6)}$$

$$\Leftrightarrow \mathrm{Spec}(\varphi) \text{ is in } h_{(\bigcup_{f\in S} D(f))}(T) \hookrightarrow h_{\mathrm{Spec}(R)}(T).$$

It follows from ($*$) that a subfunctor $F \hookrightarrow h_X$ of h_X is open if and only if for all affine schemes Y and maps $Y \to X$, there is an open subset V of Y and an isomorphism $h_Y \times_{h_X} F \cong h_V$ over h_Y. By Lemma 3.12, it is thus clear that any subfunctor of h_X of the form $h_U \hookrightarrow h_X$ for U an open subscheme of X is open.

Conversely, suppose $F \hookrightarrow h_X$ is an open subfunctor of F. Let Y be an affine scheme and $f : Y \to X$. By assumption there is an open subset V of Y and an isomorphism $h_Y \times_{h_X} F \cong h_V$ over h_R. It follows from Proposition 4.3

(or Example 4.2) that V is unique, call it $V_{Y,f}$. By Lemma 3.12, for all affine schemes Y and Z and maps $g : Z \to Y$ and $f : Y \to X$, we have $g^{-1}(V_{Y,f}) = V_{Z,f\circ g}$.

Now let $(U_i)_{i\in I}$ be a covering of X by affine open sets and take U to be the open subset $\bigcup_{i\in I} V_{U_i,U_i\hookrightarrow X}$ of X. To show this works, let R be a ring and $f : \mathrm{Spec}(R) \to X$. Then $f \in F(R)$ if and only if $1_{\mathrm{Spec}(R)}$ is in $h_{\mathrm{Spec}(R)}(R)\times_{h_X(R)} F(R) \hookrightarrow h_{\mathrm{Spec}(R)}(R)$ if and only if $1_{\mathrm{Spec}(R)}$ is in $h_{V_{\mathrm{Spec}(R),f}}(R) \hookrightarrow h_{\mathrm{Spec}(R)}(R)$. Similarly, f is in $h_U(R) \hookrightarrow h_X(R)$ if and only if $1_{\mathrm{Spec}(R)}$ is in $h_{f^{-1}(U)}(R) \hookrightarrow h_{\mathrm{Spec}(R)}(R)$. It thus suffices to show $f^{-1}(U) = V_{\mathrm{Spec}(R),f}$.

Let Y be an affine scheme and $f : Y \to X$. By Corollary 3.10, for each $i \in I$, there is a covering $(W_{ij})_{j\in J_i}$ of $f^{-1}(U_i)$ by affine open sets. We compute

$$\begin{aligned}
f^{-1}(U) &= \bigcup_{i\in I} f^{-1}(V_{U_i,U_i\hookrightarrow X}) \\
&= \bigcup_{i\in I}\bigcup_{j\in I_j} f^{-1}(V_{U_i,U_i\hookrightarrow X}) \cap W_{ij} \\
&= \bigcup_{i\in I}\bigcup_{j\in I_j} (W_{ij} \to U_i)^{-1}(V_{U_i,U_i\hookrightarrow X}) \\
&= \bigcup_{i\in I}\bigcup_{j\in I_j} V_{W_{ij},W_{ij}\to X} \\
&= \bigcup_{i\in I}\bigcup_{j\in I_j} V_{Y,f} \cap W_{ij} \\
&= V_{Y,f}.
\end{aligned}$$

□

Remark 4.13. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor, $F' \hookrightarrow F$ an open subfunctor of F and $F'' \hookrightarrow F'$ an open subfunctor of F'. Then the composite $F'' \hookrightarrow F' \hookrightarrow$

F is an open subfunctor of F. To see this, we may assume F is of the form h_X for X a scheme. In this case, the result is immediate from Lemma 4.12.

Remark 4.14. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor, $F' \hookrightarrow F$ a subfunctor of F, R a ring and $f \in F(R)$. By assumption, there exists a subset S of R and an isomorphism $h_R \times_F F' \cong h_S$ over h_R. Beware that such an S is not unique. In fact, for subsets S and S' of R, the first part of the proof of Lemma 4.12 shows there is an isomorphism $h_S \cong h_{S'}$ over h_R if and only if $\bigcup_{f \in S} D(f) = \bigcup_{f \in S'} D(f)$. The proof of Proposition 2.9 shows there this condition is equivalent to the equality $\sqrt{(S)} = \sqrt{(S')}$.

Definition 4.15. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor and $(F_i \hookrightarrow F)_{i \in I}$ a family of open subfunctors of F. We say $(F_i \hookrightarrow F)_{i \in I}$ is an *open covering* of F if for all $f \in F(R)$ and any family $(S_i)_{i \in I}$ of subsets of R with for each $i \in I$ an isomorphism $h_R \times_F F_i \cong h_{S_i}$ over h_R, the union $\bigcup_{i \in I} S_i$ generates R. By Remark 4.14, this is independent of choice of $(S_i)_{i \in I}$.

Lemma 4.16. *Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor and $(F_i \hookrightarrow F)_{i \in I}$ a family of open subfunctors of F. The following are equivalent.*

(1) $(F_i \hookrightarrow F)_{i \in I}$ is an open covering of F in the sense of Definition 4.15.

(2) For all rings R and $g \in F(R)$, there exists a subset S of R such that S generates R and for all $f \in S$, $g|_{F(R_f)}$ is in the image of $F_i(R_f) \hookrightarrow F(R_f)$ for some $i \in I$.

(3) For all fields K, the family $(F_i(K) \hookrightarrow F(K))_{i \in I}$ covers $F(K)$.

Proof. (1) $\Rightarrow$ (2). Assume $(F_i \hookrightarrow F)_{i \in I}$ is an open covering of F. For R a ring and $g \in F(R)$, take a family $(S_i)_{i \in I}$ of subsets of R with for each $i \in I$ an

isomorphism $h_R \times_F F_i \cong h_{S_i}$ over h_R. By assumption, $S = \bigcup_{i \in I} S_i$ generates R. Let $f \in S$, say $f \in S_i$ for $i \in I$. Then $R \to R_f \in h_{S_i}(R_f)$ by definition. The commutative square

$$\begin{array}{ccc} h_{S_i} & \hookrightarrow & h_R \\ \downarrow & & \downarrow g \\ F_i & \hookrightarrow & F \end{array}$$

shows the image of $R \to R_f$ in $F_i(R_f)$ works.

(2) $\Rightarrow$ (3). Assume (2) holds. Let K be a field and $g \in F(K)$; we show g is in the image of $F_i(K)$ for some $i \in I$. Take a subset S as in (2). Since K is a field there is $f \in S$ invertible. We conclude since $K \to K_f$ is an isomorphism.

(3) $\Rightarrow$ (1). Assume (3) holds. For R a ring and $g \in F(R)$, take a family $(S_i)_{i \in I}$ of subsets of R with for each $i \in I$ an isomorphism $h_R \times_F F_i \cong h_{S_i}$ over h_R. Let $\mathfrak{m}$ be a maximal ideal of R; it suffices to show $\bigcup_{i \in I} S_i \not\subset \mathfrak{m}$. It follows from (3) that $R \to R/\mathfrak{m}$ is in the image of $h_{S_i}(R/\mathfrak{m}) \hookrightarrow h_R(R/\mathfrak{m})$ for some $i \in I$. By definition this means there exists $f \in S_i - \mathfrak{m}$.

□

Lemma 4.17. *Let X be a scheme and $(U_i)_{i \in I}$ a family of open subsets of X. Then $(U_i)_{i \in I}$ is an open covering of X if and only if $(h_{U_i} \hookrightarrow h_X)_{i \in I}$ is an open covering in the sense of Definition 4.15.*

Proof. This follows from condition (3) of Lemma 4.16 and Example 4.2. □

Definition 4.18. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a functor. We say F is a *Zariski-sheaf* if for all rings R, families $(f_i \in R)_{i \in I}$ with $\{f_i \mid i \in I\}$ generating R and families $(g_i \in F(R_{f_i}))_{i \in I}$, such that for all $i, j \in I$, $g_i|_{R_{f_i f_j}} = g_j|_{R_{f_i f_j}}$, there is a unique $g \in F(R)$ such that $g|_{R_{f_i}} = g_i$ for all $i \in I$.

Lemma 4.19. *Let X be a scheme. Then h_X is a Zariski-sheaf.*

Proof. Let R be a ring, $(f_i \in R)_{i\in I}$ $(g_i \in F(R_{f_i}))_{i\in I}$ as in the statement. Under the isomorphisms $\mathrm{Spec}(R_{f_i}) \cong D(f_i)$ for $i \in I$ and $\mathrm{Spec}(R_{f_if_j}) \cong D(f_if_j)$ for $i, j \in I$ of Lemma 3.9, this reduces to an instance of Proposition 3.14. □

Lemma 4.20. *Let F : **Ring** $\to$ **Set** be a functor and $(F_i \hookrightarrow F)_{i\in I}$ an open covering of F. Suppose given G : **Ring** $\to$ **Set** a Zariski-sheaf and for each $i \in I$, a map $\alpha_i : F_i \to G$ such that for all $i, j \in I$, the diagram*

$$\begin{array}{ccc} F_i \times_F F_j & \hookrightarrow & F_j \\ \downarrow & & \downarrow{\scriptstyle \alpha_j} \\ F_i & \xrightarrow{\alpha_i} & G \end{array}$$

commutes. Then there exists a unique map $\alpha : F \to G$ such that for all $i \in I$, the diagram

$$\begin{array}{ccc} F_i & & \\ \downarrow & \searrow^{\alpha_i} & \\ F & \overset{\alpha}{\dashrightarrow} & G \end{array}$$

commutes.

Proof. Let R be a ring and $g \in F(R)$. Let S be a subset of R as in Lemma 4.16 (2). Consider the family $(h_f \in G(R_f))_{f\in S}$ defined as follows:

$(*)$ If $f \in S$, $i \in I$ and $g_f \in F_i(R_f)$ are such that the image of g_f in $F(R_f)$ is $g|_{R_f}$ then $h_f = (\alpha_i)_{R_f}(g_f)$.

By the assumption on S such a g_f exists and the assumed commutative diagrams shows h_f is well defined. It also follows from these diagrams that $(h_f)_{f\in S}$ satisfies the condition of Definition 4.18, so that there is a unique

$h \in G(R)$ with $h|_{R_f} = h_f$ for all $f \in S$. It is not hard to see that this h is independent of S; we take it to be $\alpha_R(f)$ (it is also clear that we are forced to define α in this way).

To show the above assignment is natural, let R and T be rings, $\varphi : R \to T$, $g \in F(R)$ and S be a subset of R as in Lemma 4.16 (2) for R and g. Then $\varphi(S)$ is a subset of T as in Lemma 4.16 (2) for T and $g|_T$: if $f \in S$, $i \in I$ and $g_f \in F_i(R_f)$ are such that the image of g_f in $F(R_f)$ is $g|_{R_f}$ then the image of $g_f|_{T_{\varphi(f)}}$ in $F(T_{\varphi(f)})$ is $g|_{R_f}|_{T_{\varphi(f)}} = g|_{T_{\varphi(f)}} = g|_T|_{T_{\varphi(f)}}$. Moreover for $f \in S$, $i \in I$ and $g_f \in F_i(R_f)$ such that the image of g_f in $F(R_f)$ is $g|_{R_f}$, we have $\alpha_R(g)|_T|_{T_\varphi(f)} = \alpha_R(g)|_{T_\varphi(f)} = \alpha_R(g)|_{R_f}|_{T_\varphi(f)} = (\alpha_i)_{R_f}(g_f)|_{T_\varphi(f)} = (\alpha_i)_{T_\varphi(f)}(g_f|_{T_\varphi(f)})$. It follows that $\alpha_R(g)|_T = \alpha_T(g|_T)$.

The required diagrams commute by construction.

□

Theorem 4.21. *The functor h of Definition 4.1 determines an equivalence of categories between* **Sch** *and the full subcategory of* $[\mathbf{Ring}, \mathbf{Set}]$ *with objects the Zariski-sheaves F (Definition 4.18) such that there exists an open covering $(F_i \hookrightarrow F)_{i \in I}$ of F (Definition 4.15) by affine functors (Definition 4.6).*

Proof. By Lemma 4.19, Lemma 4.17 and Lemma 4.8, h takes values in this full subcategory. By Proposition 4.3, the factorization of h through this full subcategory is fully faithful. Let $F : \mathbf{Ring} \to \mathbf{Set}$ be a Zariski-sheaf and $(F_i \hookrightarrow F)_{i \in I}$ an open covering of F by affine functors. To conclude, we show there exists a scheme X and an isomorphism $h_X \cong F$.

For each $i \in I$, take an affine scheme X_i and an isomorphism $h_{X_i} \cong F_i$. For all $i, j \in I$, $F_i \times_F F_j \hookrightarrow F_i$ is a pullback of the open subfunctor $F_j \hookrightarrow F$ of

F and hence is open (Remark 4.11). By Lemma 4.12, there is an open subset X_{ij} of X_i and an isomorphism $h_{X_{ij}} \cong F_i \times_F F_j$ making the following diagram commute.

$$\begin{array}{ccc} h_{X_{ij}} & \hookrightarrow & h_{X_i} \\ \downarrow\sim & & \downarrow\sim \\ F_i \times_F F_j & \hookrightarrow & F_i \end{array}$$

For $i, j \in I$, both $h_{X_{ij}}$ and $h_{X_{ji}}$ are pullbacks of the below diagram.

$$\begin{array}{ccc} & & F_j \\ & & \downarrow \\ F_i & \hookrightarrow & F \end{array}$$

Let $\psi_{ij} : X_{ij} \cong X_{ji}$ be the unique isomorphism of schemes lifting the isomorphism $h_{X_{ij}} \cong h_{X_{ji}}$. The conditions of Proposition 3.23 follow by construction. Take X to be the scheme glued from this data with open covering $(U_i)_{i \in I}$. By definition, we have for each $i \in I$, an isomorphism $h_{U_i} \cong F_i$ and for each $i, j \in I$ an isomorphism $h_{U_i \cap U_j} \cong F_i \times_F F_j$ such that for all $i, j \in I$,

$$\begin{array}{ccc} h_{U_i \cap U_j} & \hookrightarrow & h_{U_i} \\ \downarrow\sim & & \downarrow\sim \\ F_i \times_F F_j & \hookrightarrow & F_i \end{array}$$

commutes. It follows from Lemma 4.20 that there is a unique isomorphism $h_X \cong F$ making

$$\begin{array}{ccc} h_{U_i} & \hookrightarrow & h_X \\ \downarrow\sim & & \downarrow\sim \\ F_i & \hookrightarrow & F \end{array}$$

commute for all $i \in I$. This finishes the proof. $\square$

4.2. **Pullbacks of Schemes.** So far we have seen several examples of pullbacks of schemes (Proposition 2.40, Lemma 3.12, Lemma 3.34). Using the results of Section 4.1, we can show the existence of pullbacks in general:

Proposition 4.22. *The category* **Sch** *has all pullbacks.*

Proof. By Theorem 4.21, it suffices to show that the full subcategory in that theorem has pullbacks. We show it is closed under limits in $[\mathbf{Ring}, \mathbf{Set}]$. Let F, G and H be in this subcategory and

$$\begin{array}{ccc} & & G \\ & & \downarrow \\ F & \longrightarrow & H \end{array}$$

a diagram of functors. It is immediate that $F \times_H G$ is a Zariski-sheaf since F, G and H are Zariski-sheaves. Let $(H_i \hookrightarrow H)_{k \in K}$ be an open covering of H by affine functors. Let $k \in K$. Then $F \times_H H_k \hookrightarrow F$ is an open subfunctor of F and $G \times_H H_k \hookrightarrow G$ is an open subfunctor of G (Remark 4.11). It follows from Lemma 4.12 that these also lie in the subcategory. Thus we may choose open coverings $(F_{ik} \hookrightarrow F \times_H H_k)_{i \in I_k}$ and $(G_{jk} \hookrightarrow G \times_H H_k)_{j \in J_k}$ of F and G, respectively, by affine functors. Consider the family $(F_{ik} \times_{H_k} G_{jk} \hookrightarrow F \times_H G)_{k \in K, i \in I_k, j \in J_k}$ of subfunctors of $F \times_H G$. This satisfies the desired conditions:

- Let $k \in K, i \in I_k$ and $j \in J_k$. Then $F_{ik} \times_{H_k} G_{jk} \hookrightarrow F \times_H G$ factors $F_{ik} \times_{H_k} G_{jk} \cong F_{ik} \times_H G_{jk} \hookrightarrow F \times_H G_{jk} \hookrightarrow F \times_H G$, where the first map is an isomorphism, the second is a pullback of $F_{ik} \hookrightarrow F \times_H H_k \hookrightarrow F$ and the third is a pullback of $G_{jk} \hookrightarrow G \times_H H_k \hookrightarrow G$. By Remark 4.13, $F_{ik} \times_{H_k} G_{jk} \hookrightarrow F \times_H G$ is an open subfunctor of $F \times_H G$.

- Let K be a field, $f \in F(K)$ and $g \in G(K)$ such that $(f,g) \in F(K) \times_{H(K)} G(K)$. Let $h \in H(K)$ be the common image of f and g. By Lemma 4.16, there is $k \in K$ and $h_k \in H_k(K)$ with image h in $H(K)$. Then $(f, h_k) \in F(K) \times_{H(K)} H_k(K)$ and $(g, h_k) \in G(K) \times_{H(K)} H_k(K)$. Applying Lemma 4.16 again, we obtain $i \in I_k$, $f_{ik} \in F_{ik}(K)$ with image (f, h_k) in $F(K) \times_{H(K)} H_k(K)$, $j \in J_k$ and $g_{jk} \in G_{jk}(K)$ with image (g, h_k) in $G(K) \times_{H(K)} H_k(K)$. Then $(f_{ik}, g_{ik}) \in F_{ik} \times_{H_k} G_{jk}$ has image (f, g) in $F(K) \times_{H(K)} G(K)$. By Lemma 4.16, $(F_{ik} \times_{H_k} G_{jk} \hookrightarrow F \times_H G)_{k \in K, i \in I_k, j \in J_k}$ is an open covering of $F \times_H G$.
- Let $k \in K, i \in I_k$ and $j \in J_k$. Take rings R, R' and R'' and isomorphisms $H_k \cong h_R$, $F_{ik} \cong h_{R'}$ and $G_{jk} \cong h_{R''}$. Then we have an isomorphism $F_{ik} \times_{H_k} G_{jk} \cong h_{R'} \times_{h_R} h_{R''} \cong h_{R' \otimes_R R''}$ for certain homomorphisms $R \to R', R \to R''$. Hence $F_{ik} \times_{H_k} G_{jk}$ is affine.

□

In principal one may extract an explicit description of the pullback of schemes using Proposition 4.22. It is more convenient, however, to use the following observation.

Lemma 4.23. *Let X, Y and Z be schemes and*

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

a diagram of schemes and $(U_i)_{i \in I}$ an open covering of X. For each $i \in I$, then image V_i of $U_i \times_Z Y \to X \times_Z Y$ is an open subset of $X \times_Z Y$ and $U_i \times_Z Y \to V_i$

is an isomorphism of schemes. Moreover for $i, j \in I$, $(U_i \cap U_j) \times_Z Y \to X \times_Y Z$ has image $V_i \cap V_j$ and $(U_i \cap U_j) \times_Z Y \to V_i \cap V_j$ is an isomorphism of schemes.

Proof. For $i \in I$ the diagram

$$\begin{array}{ccc} U_i \times_Y Z & \longrightarrow & X \times_Y Z \\ \downarrow & & \downarrow \\ U_i & \hookrightarrow & X \end{array}$$

is a pullback square, so by Lemma 3.12, V_i is open and $U_i \times_Y Z \to V_i$ is an isomorphism. Similarly, for $i, j \in I$, $(U_i \cap U_j) \times_Y Z \to X \times_Y Z$ induces an isomorphism from $(U_i \cap U_j) \times_Y Z$ onto its open image, which is $V_i \cap V_j$ by the below pullback square.

$$\begin{array}{ccc} (U_i \cap U_j) \times_Y Z & \longrightarrow & U_j \times_Y Z \\ \downarrow & & \downarrow \\ U_i \times_Y Z & \longrightarrow & X \times_Y Z \end{array}$$

□

Definition 4.24. Let X be a scheme and $n \in \mathbf{N}$. We define *affine n-space* over X to be the pullback $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{A}^n_{\mathbf{Z}}$ where $X \to \mathrm{Spec}(\mathbf{Z})$ is the unique map (Proposition 2.40) and $\mathbf{A}^n_{\mathbf{Z}}$ is affine space over $\mathbf{Z}$ (Definition 2.45).

Example 4.25. Let R be a ring and $n \in \mathbf{N}$. Then by Example 2.51, we obtain an isomorphism $\mathbf{A}^n_{\mathrm{Spec}(R)} \cong \mathbf{A}^n_R$. We may thus view affine space over a scheme as a generalization of affine space over a ring.

Example 4.26. Let X be a scheme, $n \in \mathbf{N}$ and $(U_i)_{i \in I}$ an open affine covering of X. By Lemma 4.23 and Example 4.25, we obtain a description of $\mathbf{A}^n_X$ in terms of affine spaces over the $\Gamma(U_i)$ for $i \in I$ and $\Gamma(U_i \cap U_j)$ for $i, j \in I$.

Example 4.27. Let X be a scheme and $n \in \mathbf{N}$. Then we have an isomorphism $h_{\mathbf{A}^n_X} \cong h_X \times h_{\mathbf{A}^n_{\mathbf{Z}}} \cong h_X \times (\mathbf{A}^1)^n$, where $\mathbf{A}^1$ is the affine line (Example 4.7).

Definition 4.28. Let X be a scheme and $n \in \mathbf{N}$. We define *projective n-space* over X to be the pullback $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{P}^n_{\mathbf{Z}}$ where $X \to \mathrm{Spec}(\mathbf{Z})$ is the unique map (Proposition 2.40) and $\mathbf{P}^n_{\mathbf{Z}}$ is projective space over $\mathbf{Z}$ (Definition 3.24).

Example 4.29. Let R be a ring and $n \in \mathbf{N}$. Then by Lemma 3.34, we obtain an isomorphism $\mathbf{P}^n_{\mathrm{Spec}(R)} \cong \mathbf{P}^n_R$. We may thus view projective space over a scheme as a generalization of projective space over a ring.

Example 4.30. Let X be a scheme, $n \in \mathbf{N}$ and $(U_i)_{i \in I}$ an open affine covering of X. By Lemma 4.23 and Example 4.29, we obtain a description of $\mathbf{P}^n_X$ in terms of projective spaces over the $\Gamma(U_i)$ for $i \in I$ and $\Gamma(U_i \cap U_j)$ for $i, j \in I$.

Example 4.31. Let X be a scheme and $n \in \mathbf{N}$. By Lemma 4.23, $\mathbf{P}^n_X$ is covered by open sets U_i for $0 \leq i \leq n$, with isomorphisms $U_i \cong \mathbf{A}^n_X$ over $\mathbf{P}^n_X$ for all $0 \leq i \leq n$.

4.3. **Maps into Projective Space.** Let R be a ring, X a scheme and $n \in \mathbf{N}$. To describe the maps from $\mathrm{Spec}(R)$ into $\mathbf{P}^n_X$, we need a definition from commutative algebra.

Definition 4.32. Let R be a ring and M an R-module. We say M is *locally free of rank 1* if there exists a family $(f_i \in R)_{i \in I}$ with $\{f_i \mid i \in I\}$ generating R such that for all $i \in I$, there exists an isomorphism of R_{f_i}-modules $M_{f_i} \cong R_{f_i}$. Here M_{f_i} is the localization of M at f_i (Example B.19).

Remark 4.33. Let R be a ring and M an R-module, locally free of rank 1 and $\mathfrak{p} \in \mathrm{Spec}(R)$. Then there exists an isomorphism $M_\mathfrak{p} \cong R_\mathfrak{p}$ of $R_\mathfrak{p}$-modules. To see this, let $(f_i \in R)_{i \in I}$ be as in Definition 4.32. By Lemma 2.13, $(D(f_i))_{i \in I}$ is an open cover of $\mathrm{Spec}(R)$. Hence there is $i \in I$ such that $\mathfrak{p} \in \mathrm{Spec}(R)$. We obtain an isomorphism $M_\mathfrak{p} \cong (M_{f_i})_{\mathfrak{p}R_\mathfrak{p}} \cong (R_{f_i})_{\mathfrak{p}R_\mathfrak{p}} \cong R_\mathfrak{p}$.

Lemma 4.34. *Let R and S be rings, $\varphi : R \to S$ a homomorphism of rings and M an R-module, locally free of rank 1. Then the S-module $S \otimes_R M$ is locally free of rank 1.*

Proof. Let $(f_i \in R)_{i \in I}$ be as in Definition 4.32. For $i \in I$, let $M_{f_i} \cong R_{f_i}$ be an isomorphism. We obtain $(S \otimes_R M)_{\varphi(f_i)} \cong S_{\varphi(f_i)} \otimes_S (S \otimes_R M) \cong S_{\varphi(f_i)} \otimes_R M \cong S_{\varphi(f_i)} \otimes_{R_{f_i}} (R_{f_i} \otimes_R M) \cong S_{\varphi(f_i)} \otimes_{R_{f_i}} M_{f_i} \cong S_{\varphi(f_i)} \otimes_{R_{f_i}} R_{f_i} \cong S_{\varphi(f_i)}$ using Lemma B.18. Thus $(\varphi(f_i) \in S)_{i \in I}$ is the desired family. □

Definition 4.35. Let R be a ring and $n \in \mathbf{N}$. We write $\mathrm{Line}^n(R)$ for the set of quotients $R^n \twoheadrightarrow M$ (i.e. R-modules M with a surjective homomorphism of R-modules $R^n \twoheadrightarrow M$ modulo isomorphism under R^n) such that M is locally free of rank 1. Let R and S be rings, $\varphi : R \to S$ a homomorphism of rings and $n \in \mathbf{N}$. It follows from Lemma 4.34 that we obtain a function $\mathrm{Line}^n(R) \to \mathrm{Line}^n(S)$ by mapping a quotient $R^n \twoheadrightarrow M$ to $S^n \cong S \otimes_R R^n \twoheadrightarrow S \otimes_R M$. For $n \in \mathbf{N}$, this is easily seen to make $R \mapsto \mathrm{Line}^n(R)$ into a functor $\mathrm{Line}^n : \mathbf{Ring} \to \mathbf{Set}$.

Theorem 4.36. *Let X be a scheme and $n \in \mathbf{N}$. Then there is an isomorphism $h_{\mathbf{P}^n_X} \cong h_X \times \mathrm{Line}^{n+1}$ over h_X where Line^{n+1} is the functor of Definition 4.35.*

Corollary 4.37. *Let R be a ring and $n \in \mathbf{N}$. Then the points of $\mathbf{P}^n_R$, in the sense of Example 2.47 correspond to elements of $\mathrm{Line}^{n+1}(R)$ (Definition 4.35).*

Proof. Combining Example 4.29 and Theorem 4.36 for $\mathrm{Spec}(R)$, we obtain an isomorphism $h_{\mathbf{P}^n_R}(R) \cong h_{\mathrm{Spec}(R)}(R) \times \mathrm{Line}^{n+1}(R)$ over $h_{\mathrm{Spec}(R)}$. Now use that a point of $\mathbf{P}^n_R$ is exactly an element of $h_{\mathbf{P}^n_R}(R)$ with image the identity map $\mathrm{Spec}(R) = \mathrm{Spec}(R)$ in $h_{\mathrm{Spec}(R)}(R)$. □

Example 4.38. Let K be a field and $n \in \mathbf{N}$. Then a K-vector space V is free of rank 1 if and only if it has dimension 1 i.e. there exists an isomorphism of K-vector spaces $V \cong K$. Thus $\mathrm{Line}^{n+1}(K)$ is the set of K-linear surjections $K^{n+1} \twoheadrightarrow K$ modulo K-linear automorphisms of K. Maps $K^{n+1} \to K$ of K-vector spaces correspond to points $(k_0, \ldots, k_n) \in K^{n+1}$, with a map $K^{n+1} \to K$ being a surjection if and only if $(k_0, \ldots, k_n) \neq (0, \ldots, 0)$. Two surjections $K^{n+1} \twoheadrightarrow K$ corresponding to $(k_0, \ldots, k_n) \in K^{n+1} - \{(0, \ldots, 0)\}$ and $(k'_0, \ldots, k'_n) \in K^{n+1} - \{(0, \ldots, 0)\}$ differ by an automorphism of K if and only if there exists $\lambda \in K$ nonzero such that $\lambda k_i = k'_i$ for all $0 \leq i \leq n$. Thus elements of $\mathrm{Line}^{n+1}(K)$ correspond to points of the variety $\mathbf{P}^n_K$. Combining this with Corollary 4.37, we recover the result of Example 3.26 (unwinding the proof of Theorem 4.36, one may see the two correspondences are the same).

Remark 4.39. Let R be a ring. Taking inspiration from Example 4.38, we may view $\mathrm{Line}^{n+1}(R)$ as the set of lines through the origin in $\mathbf{A}^{n+1}_R$.

The proof of Theorem 4.36 will require some preliminaries.

Lemma 4.40. *Let R be a ring and M a finitely generated R-module. Then $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \text{ is zero}\}$ is open in $\mathrm{Spec}(R)$.*

Proof. In fact this set is equal to $\mathrm{Spec}(R) - V(\mathrm{Ann}(M))$ for $\mathrm{Ann}(M)$ the annihilator ideal $\{f \in R \mid \text{for all } m \in M, fm = 0\}$ of M. Let $(m_i)_{i \in I}$ be a finite generating set of M. Then for $\mathfrak{p} \in \mathrm{Spec}(R)$, we have

$M_{\mathfrak{p}}$ is zero $\Leftrightarrow$ for all $i \in I, \dfrac{m_i}{1} = 0$ in $M_{\mathfrak{p}}$

$\Leftrightarrow$ for all $i \in I,$ there exists $s_i \notin \mathfrak{p}$ such that $s_i m_i = 0$ in M

$\Leftrightarrow$ there exists $s \notin \mathfrak{p}$ such that $sm_i = 0$ for all $i \in I$

$\Leftrightarrow$ there exists $s \notin \mathfrak{p}$ such that $sm = 0$ for all $m \in M$

$\Leftrightarrow \mathrm{Ann}(M) \not\subset \mathfrak{p}$

$\Leftrightarrow \mathfrak{p} \notin V(\mathrm{Ann}(M))$.

□

Lemma 4.41. *Let R be a ring, M an R-module and $(f_i \in R)_{i \in I}$ a family such that $\{f_i \mid i \in I\}$ generates R. Suppose given for each $i \in I$, an R_{f_i}-module quotient $M_{f_i} \twoheadrightarrow N_i$ of M_{f_i} such that for all $i, j \in I$, the two $R_{f_i f_j}$-module quotients $M_{f_i f_j} \cong (M_{f_i})_{\frac{f_j}{1}} \twoheadrightarrow (N_i)_{\frac{f_j}{1}}$ and $M_{f_i f_j} \cong (M_{f_j})_{\frac{f_i}{1}} \twoheadrightarrow (N_j)_{\frac{f_i}{1}}$ of $M_{f_i f_j}$ are equal. Then there exists a unique R-module quotient $M \twoheadrightarrow N$ such that for all $i \in I$, N_i is the R_{f_i}-module quotient $M_{f_i} \twoheadrightarrow N_{f_i}$ of M_{f_i}.*

Proof. It suffices to prove the dual result for submodules (Lemma B.22). Suppose given for each $i \in I$, an R_{f_i}-submodule $K_i \hookrightarrow M_{f_i}$ of M_{f_i} such that for all $i, j \in I$, the two $R_{f_i f_j}$-submodules $(K_i)_{\frac{f_j}{1}} \hookrightarrow (M_{f_i})_{\frac{f_j}{1}} \cong M_{f_i f_j}$ and $(K_j)_{\frac{f_i}{1}} \hookrightarrow (M_{f_j})_{\frac{f_i}{1}} \cong M_{f_i f_j}$ of $M_{f_i f_j}$ are equal.

For existence, take K to be the R-submodule $\{m \in M \mid$ the image of m in M_{f_i} is in $K_{f_i} \hookrightarrow M_{f_i}$ for all $i \in I\}$ of M. Fix $i \in I$. By definition, $K_{f_i} \hookrightarrow M_{f_i}$ is contained in $K_i \hookrightarrow M_{f_i}$. To show the other containment, let $m_i \in M_{f_i}$ be in $K_i \hookrightarrow M_{f_i}$; we wish to show m_i is in $K_{f_i} \hookrightarrow M_{f_i}$. We may assume m_i is the image of $m \in M$. Let I' be a finite subset so $(f_j)_{j \in I'}$ generates R. For each $j \in I'$ the image of

m in $M_{f_if_j}$ is in $(K_j)_{\frac{f_i}{1}} \hookrightarrow (M_{f_j})_{\frac{f_i}{1}} \cong M_{f_if_j}$ by assumption, so we may choose $n_j \in \mathbf{N}$ so the image of $f_i^{n_j}m$ in M_{f_j} is in $K_j \hookrightarrow M_{f_j}$. Since I' is finite, we may assume the n_j are all equal for all $j \in I'$; say to $n \in \mathbf{N}$. By replacing m with $f_i^n m$, we may assume that the image of m in M_{f_j} is in $K_j \hookrightarrow M_{f_j}$ for all $j \in I'$. We claim $m \in K$. Let $k \in I$; we show the image of M in M_{f_k} is in $K_k \hookrightarrow M_{f_k}$. For each $j \in I'$, the image of m in $M_{f_jf_k}$ is in $(K_k)_{\frac{f_j}{1}} \hookrightarrow (M_{f_k})_{\frac{f_j}{1}} \cong M_{f_jf_k}$ by assumption, so we may choose $l_j \in \mathbf{N}$ so the image of $f_j^{l_j}m$ in M_{f_k} is in $K_k \hookrightarrow M_{f_k}$. Since I' is finite, we may assume the l_j are all equal for $j \in I'$; say to $l \in \mathbf{N}$. As in the proof of Lemma 2.28, we may take $a_j \in R$ for $j \in I'$ so that $1 = \sum_{j \in I'} a_j f_j^l$. The result follows since $m = \sum_{j \in I'} a_j f_j^l m$.

For uniqueness, let $K' \hookrightarrow M$ be another R-submodule such that $K_{f_i} \hookrightarrow M_{f_i}$ equal to K_i for all $i \in I$. By definition, $K' \hookrightarrow M$ is contained in $K \hookrightarrow M$. To show the other containment, let $m \in M$ be contained in $K \hookrightarrow M$. Let I' be a finite subset so that $(f_i)_{i \in I'}$ generates R. For each $i \in I'$, the image of m in M_{f_i} is in $K'_{f_i} \hookrightarrow M_{f_i}$ by assumption, so we may choose $n_i \in \mathbf{N}$ so that $f^{n_i}m$ is in $K' \hookrightarrow M$. Since I' is finite, we may assume the n_i are all equal for $i \in I'$, say to $n \in \mathbf{N}$. As in the proof of Lemma 2.28, we may take $a_i \in R$ for $i \in I'$ so that $1 = \sum_{i \in I'} a_i f_i^n$. The result follows since $m = \sum_{i \in I'} a_i f_i^n m$. $\square$

Proof of Theorem 4.36. Let X be a scheme and $n \in \mathbf{N}$. We will produce an isomorphism $h_{\mathbf{P}^n_X} \cong h_X \times \mathrm{Line}^{n+1}$. By definition of $\mathbf{P}^n_X$ (Definition 4.28), it suffices to give an isomorphism $h_{\mathbf{P}^n_{\mathbf{Z}}} \cong \mathrm{Line}^{n+1}$.

Let R be a ring. By Lemma 3.22, maps of schemes $\mathrm{Spec}(R) \to X$ correspond to the following data:

An open covering $(U_i)_{0 \leq i \leq n}$ of $\mathrm{Spec}(R)$ together with a family $(f_i : U_i \to \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_i}, \ldots, x_n]))_{0 \leq i \leq n}$ of maps satisfying the following conditions:

(1) For all $0 \leq i, j \leq n$ distinct, $f_i^{-1}(D(x_j)) = U_i \cap U_j$. Thus the restriction of $U_i \to \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_i}, \ldots, x_n])$ to $U_i \cap U_j$ factors through $\mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j}) \to \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_i}, \ldots, x_n])$.

(2) For all $0 \leq i, j \leq n$ distinct, the diagram

$$\begin{array}{ccc} U_i \cap U_j & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_i}, \ldots, x_n]_{x_j}) \\ & \searrow & \downarrow \sim \\ & & \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_j}, \ldots, x_n]_{x_i}) \end{array}$$

commutes, where the vertical map is as in Definition 3.24.

Applying Theorem 2.2 and its proof, this corresponds to the following data:

An open covering $(U_i)_{0 \leq i \leq n}$ of $\mathrm{Spec}(R)$ together with a family $(g_{ij} \in \mathcal{O}_{\mathrm{Spec}(R)}(U_i))_{0 \leq i,j \leq n \text{ distinct}}$ of elements satisfying the following conditions:

(1) For all $0 \leq i, j \leq n$ distinct, $U_i \cap U_j$ is the largest open set U contained in U_i such that $g_{ij}|_U$ is invertible in $\mathcal{O}_X(U)$. Equivalently, for all $0 \leq i, j \leq n$ distinct and $\mathfrak{p} \in U_i$, $(g_{ij})_{\mathfrak{p}}$ is invertible in $(\mathcal{O}_{\mathrm{Spec}(R)})_{\mathfrak{p}}$ if and only if $\mathfrak{p} \in U_j$.

(2) For all $0 \leq i, j, k \leq n$ distinct, $g_{ik}|_{U_i \cap U_j}(g_{ij}|_{U_i \cap U_j})^{-1} = g_{jk}|_{U_i \cap U_j}$ in $\mathcal{O}_{\mathrm{Spec}(R)}(U_i \cap U_j)$.

(3) For all $0 \leq i, j \leq n$ distinct, $g_{ij}|^{-1}_{U_i \cap U_j} = g_{ji}|_{U_i \cap U_j}$ in $\mathcal{O}_{\mathrm{Spec}(R)}(U_i \cap U_j)$.

This uniquely extends to the following data:

An open covering $(U_i)_{0 \leq i \leq n}$ of $\mathrm{Spec}(R)$ together with a family $(g_{ij} \in \mathcal{O}_{\mathrm{Spec}(R)}(U_i))_{0 \leq i,j \leq n}$ of elements satisfying the following conditions:

(1) For all $0 \leq i, j \leq n$, $U_i \cap U_j$ is the largest open set U contained in U_i such that $g_{ij}|_U$ is invertible in $\mathcal{O}_X(U)$. Equivalently, for all $0 \leq i, j \leq n$ and $\mathfrak{p} \in U_i$, $(g_{ij})_{\mathfrak{p}}$ is invertible in $(\mathcal{O}_{\mathrm{Spec}(R)})_{\mathfrak{p}}$ if and only if $\mathfrak{p} \in U_j$.

(2) For all $0 \leq i, j, k \leq n$, $g_{ij}|_{U_i \cap U_j} g_{jk}|_{U_i \cap U_j} = g_{ik}|_{U_i \cap U_j}$ in $\mathcal{O}_{\mathrm{Spec}(R)}(U_i \cap U_j)$. In particular $g_{ii} = 1_{\mathcal{O}_{\mathrm{Spec}(R)}(U_i)}$ for all $0 \leq i \leq n$.

Using the the definition of $\mathcal{O}_{\mathrm{Spec}(R)}$ and Lemma 2.32, this corresponds to the following data:

An open covering $(U_i)_{0 \leq i \leq n}$ of $\mathrm{Spec}(R)$ of $\mathrm{Spec}(R)$ together with for each $0 \leq i, j \leq n$ and $f \in R$ such that $D(f) \subset U_i$, $g_{ij}^f \in R_f$ satisfying the following conditions:

(1) For all $0 \leq i, j \leq n$ and $f, f' \in R$ such that $D(f) \subset D(f') \subset U_i$, the image of $g_{ij}^{f'}$ in R_f is g_{ij}^f.
(2) For all $0 \leq i, j \leq n$, $f \in R$ such that $D(f) \subset U_i$ and $\mathfrak{p} \in D(f)$, the image of g_{ij}^f in $R_{\mathfrak{p}}$ is invertible if and only $\mathfrak{p} \in U_j$. In particular, for all $0 \leq i, j \leq n$ and $f \in R$ such that $D(f) \subset U_i$ and $D(f) \subset U_j$, g_{ij}^f is invertible.
(3) For all $0 \leq i, j, k \leq n$ and $f \in R$ such that $D(f) \subset U_i$ and $D(f) \subset U_j$, $g_{ij}^f g_{jk}^f = g_{ik}^f$ in R_f. In particular $g_{ii}^f = 1_{R_f}$ for all $0 \leq i \leq n$.

We will show such data corresponds to an element of $\mathrm{Line}^{n+1}(R)$.

Let M be an R-module, locally free of rank 1 and $\alpha : R^{n+1} \twoheadrightarrow M$ a surjection. Write the basis for R^{n+1} as $(e_i)_{0 \leq i \leq n}$. For $0 \leq i \leq n$, let U_i be the set of primes $\mathfrak{p} \in \mathrm{Spec}(R)$ such that the following equivalent conditions hold:

(*a*) The $R_{\mathfrak{p}}$-module $(M/R\alpha(e_i))_{\mathfrak{p}}$ is zero.
(*b*) The $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is generated by $\frac{\alpha(e_i)}{1}$.
(*c*) For some (or equivalently any) isomorphism $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}$, the image of $\frac{\alpha(e_i)}{1}$ in $R_{\mathfrak{p}}$ is invertible. Such an isomorphism exists by Remark 4.33.
(*d*) The localization of $R \xrightarrow{\alpha(e_i)} M$ at $\mathfrak{p}$ is an isomorphism of $R_{\mathfrak{p}}$-modules.

By Lemma 4.40, (a) shows U_i are open for $i \in I$. By locality of $R_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$, (c) shows $(U_i)_{0\leq i\leq n}$ is an open cover of R. Now let $i \in I$ and $f \in R$ be such that $D(f) \subset U_i$. By Lemma B.23, (d) shows the localization of $R \xrightarrow{\alpha(e_i)} M$ at f is invertible. Thus there is a unique factorization

$$\begin{array}{ccc} (R_f)^{n+1} & \xrightarrow{\sim} & (R^{n+1})_f \\ \vdots\!\!\twoheadrightarrow & & \downarrow\!\!\alpha \\ R_f & \xrightarrow{\sim} & M_f \end{array}$$

corresponding to a family $(g_{ij}^f \in R_f)_{0\leq j\leq n}$. The defining property of this family is that $g_{ij}^f \frac{\alpha(e_i)}{1} = \frac{\alpha(e_j)}{1}$ for $0 \leq j \leq n$. Letting i and f vary, it is immediate that this data satisfies the required conditions. This assignment is well defined on $\operatorname{Line}^{n+1}(R)$.

Conversely, suppose we have an open covering $(U_i)_{0\leq i\leq n}$ of $\operatorname{Spec}(R)$ of $\operatorname{Spec}(R)$ together with for each $0 \leq i, j \leq n$ and $f \in R$ such that $D(f) \subset U_i$, $g_{ij}^f \in R_f$ satisfying the required conditions. We wish to show this data arises uniquely from an element of $\operatorname{Line}^{n+1}(R)$. Since $(U_i)_{0\leq i\leq n}$ covers $\operatorname{Spec}(R)$, Lemma 2.13 shows $\{f \mid i \in I, f \in R \text{ such that } D(f) \subset U_i\}$ generates R. For all $i \in I$ and $f \in R$ such that $D(f) \subset U_i$, $(g_{ij}^f)_{0\leq j\leq n} : (R_f)^{n+1} \twoheadrightarrow R_f$ is surjective since $g_{ii}^f = 1_{R_f}$. Consider the family of such quotients $(R_f)^{n+1} \twoheadrightarrow R_f$ for $i \in I$ and $f \in R$ such that $D(f) \subset U_i$. We show this satisfies the condition of Lemma 4.41. Let $0 \leq i, j \leq n$, and $f, f' \in R$ such that $D(f) \subset U_i$ and $D(f') \subset U_j$ and take $(R_f)_{f'} \cong R_{ff'} \xrightarrow{g_{ji}^{ff'}} R_{ff'} \cong (R_{f'})_f$. It follows from (2) that this is an isomorphism and from (1) and (3) that it makes requisite diagram commute. Take $\alpha : R^{n+1} \twoheadrightarrow M$ to be the quotient provided by

Lemma 4.41. By construction M is locally free of rank 1. We recover the original data:

- Fix $0 \leq j \leq n$ and $\mathfrak{p} \in \mathrm{Spec}(R)$; we wish to show $\mathfrak{p} \in U_j$ if and only if $M_{\mathfrak{p}}$ is generated by $\frac{\alpha(e_i)}{1}$. By (2) this is immediate if $\mathfrak{p} \in U_i$ for $0 \leq i \leq n$. The result follows since $(U_i)_{0 \leq i \leq n}$ covers $\mathrm{Spec}(R)$.
- Let $0 \leq i \leq n$ and $f \in R$ such that $D(f) \subset U_i$. By construction we have a commutative diagram

$$\begin{array}{ccc} (R_f)^{n+1} & \xrightarrow{\sim} & (R^{n+1})_f \\ \downarrow{\scriptstyle (g_{ij}^f)_{0 \leq j \leq n}} & & \downarrow{\scriptstyle \alpha} \\ R_f & \xrightarrow{\sim} & M_f \end{array}$$

  where the bottom map is an isomorphism. Since $g_{ii}^f = 1_{R_f}$, the bottom map corresponds to $\alpha(e_i) : R_f \to M_f$. We recover g_{ij}^f for $0 \leq j \leq n$ by definition.

It remains to show the correspondence thus defined is natural. Let R and S be rings, $\varphi : R \to S$ a homomorphism of rings and $\mathrm{Spec}(R) \to X$ a map of schemes. Use the above notation for the various data corresponding to $\mathrm{Spec}(R) \to X$.

- The composite $\mathrm{Spec}(S) \to \mathrm{Spec}(R) \to X$ corresponds to the open covering $(\mathrm{Spec}(\varphi)^{-1}(U_i))_{0 \leq i \leq n}$ of $\mathrm{Spec}(S)$ and the family of maps $(\mathrm{Spec}(\varphi)^{-1}(U_i) \to U_i \to \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, \hat{x_i}, \ldots, x_n]))_{0 \leq i \leq n}$, where for $0 \leq i \leq n$, $\mathrm{Spec}(\varphi)^{-1}(U_i) \to U_i$ is the pulled back from $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ (Lemma 3.12).
- This corresponds to the open covering $(\mathrm{Spec}(\varphi)^{-1}(U_i))_{0 \leq i \leq n}$ of $\mathrm{Spec}(S)$ and the family of images g_{ij} under $\mathcal{O}_{\mathrm{Spec}(R)}(U_i) \to \mathcal{O}_{\mathrm{Spec}(S)}(\varphi^{-1}(U_i))$ for $0 \leq i, j \leq n$ distinct.

- This corresponds to the open covering $(\mathrm{Spec}(\varphi)^{-1}(U_i))_{0\leq i\leq n}$ of $\mathrm{Spec}(S)$ and the family of images g_{ij} under $\mathcal{O}_{\mathrm{Spec}(R)}(U_i) \to \mathcal{O}_{\mathrm{Spec}(S)}(\varphi^{-1}(U_i))$ for $0 \leq i, j \leq n$.
- This corresponds to the open covering $(\mathrm{Spec}(\varphi)^{-1}(U_i))_{0\leq i\leq n}$ of $\mathrm{Spec}(S)$ and the unique family that assigns i, j and $\varphi(f)$ to $\varphi(g_{ij}^f)$ for $0 \leq i, j \leq n$ and $f \in R$ such that $D(f) \subset U_i$.

By Lemma 4.41, it suffices to show that the diagram

$$\begin{array}{ccc} (S_{\varphi(f)})^{n+1} & \xrightarrow{\sim} & (S^{n+1})_{\varphi(f)} \\ \downarrow{\scriptstyle (\varphi(g_{ij}^f))_{0\leq j\leq n}} & & \downarrow \\ S_{\varphi(f)} & \xrightarrow{\sim} & (S \otimes_R M)_{\varphi(f)} \end{array}$$

commutes for all $0 \leq i \leq n$ and $f \in R$ such that $D(f) \subset U_i$. Here the bottom horizontal map is from Lemma 4.34 and the right horizontal map is induced by $S^n \cong S \otimes_R R^n \twoheadrightarrow S \otimes_R M$ (Definition 4.35). This follows from commutativity of

$$\begin{array}{ccc} (R_f)^{n+1} & \xrightarrow{\sim} & (R^{n+1})_f \\ \downarrow{\scriptstyle (g_{ij}^f)_{0\leq j\leq n}} & & \downarrow \\ R_f & \xrightarrow{\sim} & M_f \end{array}$$

and the definitions.

□

Appendix A. Category Theory

Definition A.1. A category $\mathcal{C}$ consists of the following data:

- A collection, called the *objects* of $\mathcal{C}$.

- For all objects X, Y of $\mathcal{C}$ a collection $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, whose elements are called *maps* between X and Y. We write $f : X \to Y$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- For all objects X of $\mathcal{C}$, an element $1_X : X \to X$ called an *identity map*.
- For all objects X, Y and Z of $\mathcal{C}$ and for all $f : X \to Y$ and $g : Y \to Z$ an element $g \circ f : X \to Z$, called the *composite* of g and f.

This data is required to satisfy the following conditions:

- For all objects X and Y of $\mathcal{C}$ and $f : X \to Y$, $1_Y \circ f = f = f \circ 1_X$ as elements of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- For all objects X, Y, Z and W of $\mathcal{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$ as elements of $\mathrm{Hom}_{\mathcal{C}}(X, W)$.

Example A.2. Sets, functions and the usual identities and composition laws define a category **Set**.

Example A.3. Rings, homomorphisms of rings and the usual identities and composition laws define a category **Ring**.

Example A.4. Topological spaces, continuous maps and the usual identities and composition laws define a category **Top**.

Example A.5. Let P be a poset. Then P determines a category with objects the elements of P. For all $p, p' \in P$, there is a map $p \to p'$ in this category if and only if $p \leq p'$ (in which case there is only one). We also write P for this category.

Example A.6. Let $\mathcal{C}$ be a category. We define the *opposite* category $\mathcal{C}^{\mathrm{op}}$ to have same objects, but with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ for objects X and

Y of $\mathcal{C}$. The identities and composition laws on $\mathcal{C}^{\mathrm{op}}$ are induced by those of $\mathcal{C}$. Using opposite categories, definitions in category theory give rise to *dual* definitions where all the arrows are reversed.

Definition A.7. Let $\mathcal{C}$ be a category, S a sub-collection of the objects of $\mathbf{C}$ and for all $X, Y \in S$, $T_{X,Y}$ a sub-collection of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that the following conditions are satisfied:

- For all $X \in S$, we have $1_X \in T_{X,X}$.
- For all $X, Y, Z \in S$, $f \in T_{X,Y}$ and $g \in T_{Y,Z}$, we have $g \circ f \in T_{X,Z}$.

Then S and the $T_{X,Y}$ for $X, Y \in S$ evidently form a category, which we call a *subcategory* of $\mathcal{C}$. If S contains every object of $\mathcal{C}$, we call it a *full subcategory* of $\mathcal{C}$.

Remark A.8. Throughout this paper, we display equalities between maps in a category using *commutative diagrams*. A *diagram* in a category $\mathcal{C}$ displays some objects of $\mathcal{C}$ and maps between them. A diagram is *commutative* if any two parallel paths compose to the same map. For example, for objects X, Y, Z and W of $\mathcal{C}$ and maps $f : X \to Y$, $g : X \to Z$, $h : Y \to W$ and $k : Z \to W$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{k} & W \end{array}$$

commutes if and only if $h \circ f = k \circ g$ as elements of $\mathrm{Hom}_{\mathcal{C}}(X, W)$.

Definition A.9. Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. We may define new categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$. The objects of $\mathcal{C}_{/X}$ are objects Y of $\mathcal{C}$ equipped

with a map $f : Y \to X$. For objects Y, Z of $\mathcal{C}$ together with $g : Y \to X$ and $h : Z \to X$, a map between the associated objects of $\mathcal{C}_{/X}$ is a map $f : Y \to Z$ in $\mathcal{C}$ making the following diagram in $\mathcal{C}$ commute.

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & \searrow^{g} & \downarrow{\scriptstyle h} \\ & & X \end{array}$$

We refer to such a map as a map *over* X. The category $\mathcal{C}_{X/}$ is defined dually.

Example A.10. Let R be a ring. The category $\mathbf{Ring}_{R/}$ is the category of R-algebras.

Definition A.11. Let $\mathcal{C}$ be a category, X, Y objects of $\mathcal{C}$ and $f : X \to Y$. We say f is an isomorphism if there exists $g : Y \to X$ such that $g \circ f = 1_X$ as elements of $\mathrm{Hom}_{\mathcal{C}}(X, X)$ and $f \circ g = 1_Y$ as elements of $\mathrm{Hom}_{\mathcal{C}}(Y, Y)$. In this case we say such a g is inverse to f and write $f : X \cong Y$.

Example A.12. An isomorphism in **Set** is a bijection. An isomorphism in **Ring** is an isomorphism of rings in the usual sense. An isomorphism in **Top** is a homeomorphism.

Definition A.13. Let $\mathcal{C}$ be a category. We say $\mathcal{C}$ is *locally small* if for all objects X and Y of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set. We say $\mathcal{C}$ is *small* if it is locally small and the objects of $\mathcal{C}$ form a set.

Example A.14. The categories **Set**, **Ring** and **Top** are locally small. The category P of Example A.5 for P a poset is small.

Definition A.15. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- For each object X of $\mathcal{C}$, an object $F(X)$ of $\mathcal{D}$.
- For each pair of objects X and Y of $\mathcal{C}$ and $f : X \to Y$, a map $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$.

This data is required to satisfy the following conditions:

- For all objects X of $\mathcal{C}$, we have $F(1_X) = 1_{F(X)}$ as elements of $\mathrm{Hom}_{\mathcal{D}}(F(X), F(X))$.
- For all objects X, Y and Z of $\mathcal{C}$ and for all $f : X \to Y$ and $g : Y \to Z$ we have $F(g \circ f) = F(g) \circ F(f)$ as elements of $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Z))$.

We write $F : \mathcal{C} \to \mathcal{D}$ and say the assignment of an object X of $\mathcal{C}$ to $F(X)$ is *functorial*.

Example A.16. There are evident functors **Ring** $\to$ **Set** and **Top** $\to$ **Set** by taking underlying sets. We call these *forgetful* functors.

Example A.17. Let $\mathcal{C}$ be a category and, S a sub-collection of the objects of $\mathbf{C}$ and for all $X, Y \in S$, $T_{X,Y}$ a sub-collection of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ satisfying the conditions of Definition A.7. Then there is an inclusion functor from the associated subcategory to $\mathcal{C}$.

Example A.18. Let P and Q be posets, viewed as categories as in Example A.5. Then functors from P to Q correspond to order preserving maps $P \to Q$.

Example A.19. Let $\mathcal{C}$ be a locally small category. Let X be an object of $\mathcal{C}$. Then there are functors $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set}$. For Y an object of $\mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(-, X)(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, -)(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The action of these functors on maps are given by precomposition and postcomposition, respectively.

Example A.20. Let $\mathcal{C}$ be a category. Then there is an evident *identity* functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

Example A.21. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. Then there is an evident *composite* functor $G \circ F : \mathcal{C} \to \mathcal{E}$.

Definition A.22. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. We say F is full (resp. faithful, fully faithful) if for all objects X, Y of $\mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ assigning $f : X \to Y$ to $F(f) : F(X) \to F(Y)$ is surjective (resp. injective, bijective).

Definition A.23. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. The *essential image* of F is the full subcategory of $\mathcal{D}$ with objects those objects Y of $\mathcal{D}$ such that there exists an object X of $\mathcal{C}$ and an isomorphism $F(X) \cong Y$. We say F is essentially surjective if every object of $\mathcal{D}$ belongs to the essential image of F.

Definition A.24. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then the collection of functors from $\mathcal{C}$ to $\mathcal{D}$ is the objects of a category $[\mathcal{C}, \mathcal{D}]$. For functors $F, G : \mathcal{C} \to \mathcal{D}$, a map $\alpha : F \to G$ in $[\mathcal{C}, \mathcal{D}]$ assigns to each object X of $\mathcal{C}$ a map $\alpha_X : F(X) \to G(X)$ in D such that for all objects X, Y of $\mathcal{C}$ and $f : X \to Y$, the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

We refer such a map a *natural transformation* and refer to the above condition as *naturality.*

Lemma A.25. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories $F, G : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \to G$ a natural transformation. Then α is an isomorphism in the category $[\mathcal{C}, \mathcal{D}]$ if and only if for all objects X of $\mathcal{C}$, $\alpha_X : F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$. In this case we call α a natural isomorphism.* □

Lemma A.26. *Let $\mathcal{C}$ be a locally small category (Definition A.13). For X an object of $\mathcal{C}$ and $F : \mathcal{C} \to$ **Set** a functor, natural transformations $\mathrm{Hom}(X, -) \to F$ correspond to elements of the set $F(X)$. A natural transformation $\alpha : \mathrm{Hom}(X, -) \to F$ corresponds to $\alpha_X(1_X) \in F(X)$.* □

Definition A.27. Let $\mathcal{C}$ and $\mathcal{D}$ be a categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. We say F is an equivalence of categories if there exists $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$. In this case we say G is inverse to F.

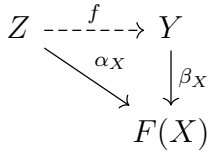
Lemma A.28. *Let $\mathcal{C}$ and $\mathcal{D}$ be a categories and $F : \mathcal{C} \to \mathcal{D}$. Then F is an equivalence of categories if and only if it is fully faithful and essentially surjective.* □

Definition A.29. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ a functor. A *cone* of F is an object Y in $\mathcal{D}$ equipped with for each object X of $\mathcal{C}$ a map $\alpha_X : Y \to F(X)$ such that for all objects X, Y of $\mathcal{C}$ and maps $f : X \to Y$, the diagram

$$\begin{array}{ccc} Y & & \\ \downarrow{\scriptstyle \alpha_X} & \searrow^{\alpha_Y} & \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

commutes. Such a cone Y of F is a *limit* of F if for any other cone consisting of Z and $(\beta_X : Z \to F(X))_{X \text{ an object of } \mathcal{C}}$, there exists a unique map $f : Z \to Y$

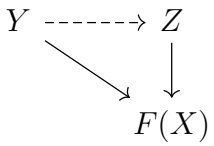
such that for all objects X of $\mathcal{C}$, the diagram



commutes. We write a chosen limit of F as $\lim_X F(X)$.

Dually, we define a *cocone* of F and a *colimit* of F. We write a chosen colimit of F as $\operatorname{colim}_X F(X)$.

Remark A.30. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ a functor and Y, Z two limits of F. Then there is a unique map $Y \to Z$ making



commute for all objects X of $\mathcal{C}$. Moreover this map is an isomorphism.

Example A.31. Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then the set of families $(a_X \in F(X))_{X \text{ an object of } \mathcal{C}}$ such that for all objects X, Y of $\mathcal{C}$ and $f : X \to Y$, $F(f)(a_X) = a_Y$ becomes a limit of F.

Example A.32. Let $\mathcal{C}$ be a a small category and $F : \mathcal{C} \to \mathbf{Top}$ a functor. Equipping the limit $\lim_X F(X)$ of the composite functor $\mathcal{C} \to \mathbf{Top} \to \mathbf{Set}$ of Example A.31 with the smallest topology such that for all objects X of $\mathcal{C}$, $\lim_X F(X) \to F(X)$ is continuous, $\lim_X F(X)$ becomes a limit of F.

Example A.33. Let $\mathcal{C}$ be a small category, $\mathcal{D}$ a category and $F : \mathcal{C} \to [\mathcal{D}, \mathbf{Set}]$ a functor. The limits $\lim_X (F(X)(D))$ of Example A.31 for D an object of $\mathcal{D}$ assemble into a functor which becomes a limit of F.

Example A.34. Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then the quotient of the disjoint union $\bigsqcup_{X \text{ an object of } \mathcal{C}} F(X)$ by the smallest equivalence relation relating (X, a) and $(Y, F(f)(a))$ for all objects X, Y in $\mathcal{C}$, $f : X \to Y$ and $a \in F(X)$ becomes a colimit of F.

Example A.35. Let P be a poset, viewed and $F : P \to \mathbf{Set}$ a functor. Assume that P is directed. For $p', p'' \in P$ in P elements $a \in F(p')$ and $b \in F(p'')$, the image of a under $F(p') \to \mathrm{colim}_p F(p)$ is equal to the image of b under $F(p'') \to \mathrm{colim}_p F(p)$ if and only if there exists $p''' \in P$ such that $p''' \leq p', p''' \leq p''$ and $c \in F(p''')$ such that $F(p''' \leq p')(c) = a$ and $F(p''' \leq p'')(c) = b$.

Example A.36. Let P be a poset and $F : P \to \mathbf{Ring}$ a functor. Assume that P is directed. Then the colimit of the composite functor $F : P \to \mathbf{Ring} \to \mathbf{Set}$ of Example A.34 admits a ring structure. The resulting ring becomes a colimit of F.

Example A.37. Let $\mathcal{C}$ be a a small category and $F : \mathcal{C} \to \mathbf{Top}$ a functor. Equipping the colimit $\mathrm{colim}_X F(X)$ of the composite functor $\mathcal{C} \to \mathbf{Top} \to \mathbf{Set}$ of Example A.34 with the smallest topology such that for all objects X of $\mathcal{C}$, $F(X) \to \mathrm{colim}_X F(X)$ is continuous, $\mathrm{colim}_X F(X)$ becomes a colimit of F.

Definition A.38. Let $\mathcal{C}$ be a category. A cone of the unique functor from the empty category to $\mathcal{C}$ is tantamount to an object X of $\mathcal{C}$. We say X is *terminal* in $\mathcal{C}$ if the corresponding cone is a limit cone. Explicitly, X is terminal if for all objects Y of $\mathcal{C}$, there exists a unique map $Y \to X$. We write 1 for a chosen terminal object of $\mathcal{C}$. Dually, we define an *initial* object of $\mathcal{C}$.

Example A.39. The one element set 1 is terminal in **Set**. The one point space 1 is terminal in **Top**.

Example A.40. The ring $\mathbf{Z}$ is initial in **Ring**.

Example A.41. Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. Then the identity map $X = X$ makes X a terminal object of $\mathcal{C}_{/X}$.

Example A.42. Let $\mathcal{C}$ be a category. A functor from the category with two objects and only identity maps between them to $\mathcal{C}$ corresponds to a pair of objects X, Y in $\mathcal{C}$. A cone of such a functor consists of an object Z of $\mathcal{C}$ equipped with maps $Z \to X$ and $Z \to Y$. We say such a cone makes Z the *product* of X and Y. We write $X \times Y$ for a chosen Z as above.

Example A.43. Products in **Set** are given by Cartesian products of sets. Products in **Top** are given by Cartesian products of topological spaces.

Definition A.44. Let $\mathcal{C}$ be a category. A functor from the category with three objects x, y, z, maps $x \to z$ and $y \to z$ and identity maps corresponds to a triple of objects X, Y, Z of $\mathcal{C}$ and a diagram as below.

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

A cone of such a functor is tantamount to an object W equipped with maps $W \to X$ and $W \to Z$ making

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

commute. We say the above diagram is a *pullback square* if it corresponds to a limit cone. We write $X \times_Z Y$ for a chosen W as above. Dually, we define a *pushout square*.

Example A.45. Let X, Y and Z be sets and

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

a diagram in **Set**. Then the subset of $X \times Y$ consisting of (x, y) for $x \in X$, $y \in Y$ such that $f(x) = g(y)$ becomes a pullback in **Set**. Pullbacks in **Top** may be described similarly.

Example A.46. Let R be a ring and $R \to R_1$, $R \to R_2$ homomorphsims of rings, giving R_1 and R_2 the structure of R-algebras. Then the evident diagram

$$\begin{array}{ccc} R & \longrightarrow & R_2 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_1 \otimes_R R_2 \end{array}$$

is a pushout square in **Ring**.

Example A.47. Let $\mathcal{C}$ be a category, X, Y objects of $\mathcal{C}$ and 1 a terminal object of $\mathcal{C}$. Then a pullback of the unique diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

is tantamount to a product of X and Y.

Example A.48. Let $\mathcal{C}$ be a category, X, Y, Z objects of $\mathcal{C}$ and

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

a diagram in $\mathcal{C}$. Then a pullback for this diagram is tantamount to a product of $X \to Z$ and $Y \to Z$ in $\mathcal{C}_{/Z}$.

Definition A.49. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $G : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{E}$ be a category, $F : \mathcal{E} \to \mathcal{C}$ a functor and Y a limit of F. Then $(G(Y) \to G(F(X)))_{X \text{ an object of } \mathcal{E}}$ makes $G(Y)$ into a cone of $G \circ F$. We say G *preserves* the limit Y if this cone is a limit of $G \circ F$. We say G *preserves* all limits if this condition holds for all such $\mathcal{E}$, F and Y.

Example A.50. Let $\mathcal{C}$ be a locally small category and X an object of $\mathcal{C}$. Then the functor $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set}$ preserves all limits.

Lemma A.51. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ functors. The following pieces of data are equivalent:*

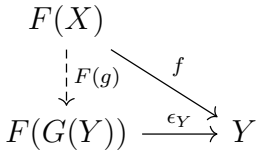
(1) A natural transformation $\eta : 1_{\mathcal{C}} \to G \circ F$ such that for all objects X of $\mathcal{C}$, objects Y of $\mathcal{D}$, and maps $g : X \to G(Y)$ in $\mathcal{C}$, there exists a unique map $f : F(X) \to Y$ in $\mathcal{D}$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(F(X)) \\ & \searrow^{g} & \downarrow{\scriptstyle G(f)} \\ & & G(Y) \end{array}$$

commutes.

(2) A natural transformation $\epsilon : F \circ G \to 1_{\mathcal{D}}$ such that for all objects X of $\mathcal{C}$, objects Y of $\mathcal{D}$, and maps $f : F(X) \to Y$ in $\mathcal{D}$, there exists a unique

map $g : X \to G(Y)$ in $\mathcal{C}$ such that



commutes.

(3) For each object X of $\mathcal{C}$ and object Y of $\mathcal{D}$ an isomorphism $\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$, which is natural in both X and Y, in an evident sense.

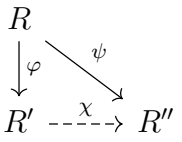
We say this data makes F left adjoint to G and G right adjoint to F. □

Example A.52. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $G : \mathcal{D} \to \mathcal{C}$ and $F, F' : \mathcal{C} \to \mathcal{D}$ two left adjoints of G as in Lemma A.51. Then it follows from Lemma A.26 that we obtain a natural isomorphism $F \cong F'$.

Lemma A.53. *Let $\mathcal{C}, \mathcal{D}$ be categories and $G : \mathcal{C} \to \mathcal{D}$ a functor. If G admits a left adjoint as in Lemma A.51, G preserves all limits.* □

Appendix B. Commutative Algebra

Definition B.1. Let R be a ring and S a subset of R. We say a ring R' equipped with a homomorphism of rings $\varphi : R \to R'$ is a *localization* of R at S if for all $s \in S$, $\varphi(s)$ is invertible in R' and for all rings R'' equipped with a homomorphism of rings $\psi : R \to R''$ such that for all $s \in S$, $\psi(s)$ is invertible in R'', there exists a unique homomorphism of rings $\chi : R' \to R''$ such that the diagram



commutes.

Lemma B.2. *Let R be a ring and S a subset of R. There is a ring $S^{-1}R$, and a homomorphism $R \to S^{-1}R$ which makes $S^{-1}R$ a localization of R at S.*

Proof. We may assume S is closed under multiplication. Take $S^{-1}R$ to be the quotient of the set of pairs (f, s) where $f \in R$ and $s \in S$ by the following equivalence relation:

$(*)$ For $f, f' \in R$ and $s, s' \in S$, (f, s) is related to (f', s') if and only if there exists $s'' \in S$ such that $s''s'f = s''sf'$.

For $f \in R$ and $s \in S$, we denote the equivalence class of (f, s) by $\frac{f}{s}$. Then the ring structure on $S^{-1}R$ are defined in the evident way. The homomorphism $R \to S^{-1}R$ assigns $f \in R$ to $\frac{f}{1} \in S^{-1}R$. One may verify this satisfies Definition B.1. □

Example B.3. Let R be a ring and $f \in R$. We write R_f for the ring $\{f\}^{-1}R$. The elements of R_f are quotients $\frac{g}{f^n}$ for $g \in R$ and $n \in \mathbf{N}$. For $g, g' \in R$ and $n, n' \in \mathbf{N}$, $\frac{g}{f^n} = \frac{g'}{f^{n'}}$ if and only if there exists $n'' \in \mathbf{N}$ such that $f^{n'+n''}g = f^{n+n''}g'$.

Example B.4. Let R be a ring. The the homomorphism of rings $R \to R_1$ is an isomorphism.

Lemma B.5. *Let R be a ring and $f, g \in R$. Then there are isomorphisms of R-algebras $(R_f)_{\frac{g}{1}} \cong R_{gf} \cong (R_g)_{\frac{f}{1}}$. If $\frac{f}{1}$ is invertible in R_g, there is an R-algebra isomorphism between any of these and R_g . If $\frac{g}{1}$ is invertible in R_f, there is an R-algebra isomorphism between any of these and R_f.* □

Example B.6. Let R be a ring and $\mathfrak{p}$ a prime ideal of R. We write $R_\mathfrak{p}$ for the ring $(R-\mathfrak{p})^{-1}R$. Note that $R-\mathfrak{p}$ is closed under multiplication, so it admits a description as in Lemma B.2.

Definition B.7. Let R and R' be rings, S a subset of R, S' a subset of R' and φ a homomorphism of rings such that $\varphi(S) \subset S'$. Then there is a unique homomorphism of rings $S^{-1}R \to S'^{-1}R'$ making the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ \downarrow & & \downarrow \\ S^{-1}R & \dashrightarrow & S'^{-1}R' \end{array}$$

commute.

Lemma B.8. *Let R be a ring and S a subset of R. Then prime ideals of $S^{-1}R$ correspond to prime ideals of R disjoint from S.*

Proof. In one direction we map a prime ideal $\mathfrak{q}$ of $S^{-1}R$ to its inverse image under $R \to S^{-1}R$. In the other we map a prime ideal $\mathfrak{p}$ of R to the ideal generated by the image under $R \to S^{-1}R$. One may check both these assignments are well-defined and mutually inverse. □

Example B.9. Let R be a ring and $f \in R$. By Lemma B.8, prime ideals of R_f correspond to prime ideals of R not containing f.

Lemma B.10. *Let R be a ring. The following are equivalent:*

(1) The ring R has a unique maximal ideal.

(2) The set of elements $f \in R$ which are not invertible forms an ideal.

(3) The ring R is nontrivial and for all $f \in R$, f is invertible or $1-f$ is invertible.

A ring satisfying these equivalent conditions is called local. □

Example B.11. Let R be a ring and $\mathfrak{p}$ a prime ideal of R. Then the ring $R_{\mathfrak{p}}$ of Example B.6 is a local ring. The unique maximal ideal of $R_{\mathfrak{p}}$ corresponds to $\mathfrak{p}$ under the bijection of Lemma B.8.

Definition B.12. Let R and S be local rings and $\varphi : R \to S$ a homomorphism of rings. We say φ is a *local* homomorphism if the inverse image of the unique maximal ideal of S under φ is the unique maximal ideal of R. Equivalently, for all $f \in R$, if $\varphi(f)$ is invertible in S, then f is invertible in R.

Example B.13. Let R and S be rings, $\varphi : R \to S$ a homomorphism of rings and $\mathfrak{q}$ a prime ideal of S. Then the homomorphism $R_{\varphi^{-1}\mathfrak{q}} \to S_{\mathfrak{q}}$ of Definition B.7 is local.

Definition B.14. Let R be a ring. The *radical* $\sqrt{I}$ of an ideal I of R is the ideal of $f \in R$ such that there exists $n \in \mathbf{N}$ positive such that $f^n \in I$. An ideal I of R is called *radical* if $\sqrt{I} = I$.

Example B.15. Let R be a ring and $\mathfrak{p}$ a prime ideal of R. Then $\mathfrak{p}$ is a radical ideal.

Lemma B.16. *Let R be a ring and I an ideal of R. Then $\sqrt{I}$ is the intersection $\bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$ of prime ideals $\mathfrak{p}$ of R containing I.*

Proof. Let $f \in R$ and $\bar{f}$ denote the image of f in R/I. We compute

$$
\begin{aligned}
f \notin \sqrt{I} &\Leftrightarrow f^n \notin I \text{ for all } n \in \mathbf{N} \text{ positive} \\
&\Leftrightarrow \bar{f}^n \neq 0 \text{ in } R/I \text{ for all } n \in N \text{ positive} \\
&\Leftrightarrow (R/I)_{\bar{f}} \neq 0 \\
&\Leftrightarrow (R/I)_{\bar{f}} \text{ has a prime ideal} \\
&\Leftrightarrow \text{there exists a prime ideal } \mathfrak{p} \text{ of } R \text{ containing } I \text{ and not } f \\
&\Leftrightarrow f \notin \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}.
\end{aligned}
$$

□

The rest of this section requires some knowledge of module theory and will only be used in Section 4.3.

Definition B.17. Let R be a ring, S a subset of R and M an R-module. We define an $S^{-1}R$-module $S^{-1}M$ in an evident way.

Lemma B.18. *Let R be a ring, TS a subset of R and M an R-module. Then there is an isomorphism $S^{-1}M \cong S^{-1}R \otimes_R M$ of $S^{-1}R$-modules.* □

Example B.19. Let R be a ring, $f \in R$ and M an R-module. We write M_f for the R_f module $f^{-1}M$. The elements of M_f are quotients $\frac{m}{f^n}$ for $m \in M$ and $n \in \mathbf{N}$. For $m, m' \in R$ and $n, n' \in \mathbf{N}$, $\frac{m}{f^n} = \frac{m'}{f^{n'}}$ if and only if there exists $n'' \in \mathbf{N}$ such that $f^{n'+n''}m = f^{n+n''}m'$.

Example B.20. Let R be a ring, $\mathfrak{p}$ a prime ideal of R and M an R-module. We write $M_{\mathfrak{p}}$ for the $R_{\mathfrak{p}}$-module $(R - \mathfrak{p})^{-1}M$.

Definition B.21. Let R be a ring, S a subset of R, M and N be R-modules and $f : M \to N$ a homomorphism of R-modules. Then there is an evident homomorphism of $S^{-1}R$-modules $S^{-1}f : S^{-1}M \to S^{-1}N$.

Lemma B.22. *Let R be a ring, S a subset of R, M, M' and M'' be R-modules and $M' \to M$, $M \to M''$ homomorphisms of R-modules. If the sequence of R-modules $0 \to M' \to M \to M'' \to 0$ is exact, then sequence of $S^{-1}R$-modules $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is exact.* □

Lemma B.23. *Let R be a ring, $f \in R$, M, M' and M'' be R-modules and $M \to N$ a homomorphism of R-modules. Then $M_f \to N_f$ is an isomorphism of R_f-modules if and only if for all prime ideals $\mathfrak{p}$ of R not containing f, $M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomorphism of $R_\mathfrak{p}$ modules.* □

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