SYMMETRIC POLYNOMIALS: THE FUNDAMENTAL THEOREM AND UNIQUENESS

By

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ABSTRACT

KENDER Symmetric Polynomials.

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We explore the Fundamental Theorem of Symmetric Polynomials (FTSP), using a classical proof that is streamlined by rigorously proving lemmas and a corollary prior to tackling the FTSP. This paper provides many examples of the complex ideas in order to provide more clarity and a deeper understanding of the FTSP. We also explore the historical uses of symmetric polynomials, dating as far back as 1782 in Edward Waring's *Meditationes Algebraicæ* and as recently as 2005 with an application of the FTSP on multilevel converters from Chiasson et al.

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I first want to say thank you to my mother and father for bringing me into this world and being such great role models growing up, and teaching me that hard work does in fact pay off. To my sisters and brother for always being there for me. For Papa, who showed me what a man is. But most importantly I want to thank God for the gifts He has given me and for never giving up on me.

Finally, thank you to Jeff Hatley, my advisor, for leading me through this challenging process of writing a mathematics thesis. Yo Jeff, I did it!

Contents

1. History of Symmetric Polynomials

Here we provide a historical context for the study and application of symmetric polynomials. We just take a glimpse of the main ideas from previous sources, but we use the historical context as our motivation to prove the Fundamental Theorem of Symmetric Polynomials.

The first time we see symmetric polynomials dates back to 1629, when Albert Girard published his book *New Inventions in Algebra*, where he provided a clear description on the elementary symmetric polynomials. Next in the early 1700's, Isaac Newton published his work on what are now known as the *Newton Identities*; we explore this concept in Example (2.8) below.

In this paper, we explore two very different uses of symmetric polynomials; first, we see how Edward Waring defines the idea of the elementary symmetric polynomials *Meditationes Algebraicæ*[1] and then we land in 2005 and see an electrical engineering application of symmetric polynomials.

I hope your flux capacitor is pumping out 1.21 gigawatts, because we are going on a voyage through time.

1.1. ***Metitationes Algebraicæ.*** In the first chapter of Edward Waring's 1782 book *Meditationes Algebraicæ*, Waring utilizes what we will prove later in this paper; the idea that a symmetric polynomial can be written in terms of the elementary symmetric polynomials. While we see a formal definition of these concepts in the pages below, first we will consider how Waring applied symmetric polynomials in *Meditationes Algebraicæ*.

Suppose $\alpha, \beta, \gamma, \delta, \epsilon, etc.$ are the roots of a given equation

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + ... = 0,$$

and let $a, b, c, d, e, f, etc.$ be given indices, m, in number $m \leq n$. Then, while not providing a formal definition, he defines a collection of functions that are defined in a similar way to how we define the elementary symmetric polynomials. Waring denotes them as follows:

$$P = \alpha^a + \beta^a + \gamma^a + \delta^a ...$$

$$Q = \alpha^a\beta^b + \beta^a\alpha^b + \alpha^a\gamma^b + \gamma^a\alpha^b + \beta^a\gamma^b ...$$

$$R = \alpha^a\beta^b\gamma^c + \alpha^a\beta^c\gamma^b + \alpha^b\beta^a\gamma^c ...$$

$$T = \alpha^a\beta^b\gamma^c\delta^d + \alpha^b\beta^a\gamma^c\delta^d + \alpha^a\beta^b\gamma^d\delta^c ...$$

$$V = \alpha^a\beta^b\gamma^c\delta^d\epsilon^e + ...$$

We see this later in our definition of the elementary symmetric polynomials, which are crucial to this paper. While Waring uses old notation and a very complex organizational system, it is really amazing to see how far back the idea of symmetric polynomials dates.

1.2. **Symmetric Polynomials and Harmonics.** In a more recently published article (only by about 200 years), J.N. Chiasson, L.M. Tolbert, K.J. McKenzie, and Zhong Du [2] explore the elimination of harmonics in a multilever converter. These machines are used for power conversion from DC voltage to AC voltage for high levels of power, typically drawing from several sources of DC capacitors. A great example of this would be with solar panels, since they provide DC energy that has to be converted to AC to feed into the power grid. Symmetric polynomials are applicable because when attempting to characterize the harmonic characteristics, the degree of the polynomials became too large to compute; however, Chiasson et al. use symmetric polynomials to reduce the degree of the polynomials to compute the harmonic content. Applying the Fundamental Theorem of Symmetric Polynomials, Chiasson et al. were able to take the following set of functions

$$
\begin{aligned}
p_1(x) =& x_1 + x_2 + x_3 - m = 0, \quad m = \frac{\frac{V_1}{4V_{dc}}}{\pi} \\
p_5(x) =& \sum_{i=1}^{3} (5x_i - 20x_i^3 + 16x_i^5) = 0 \\
p_7(x) =& \sum_{i=1}^{3} (-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7) = 0
\end{aligned}
$$

and express them in terms of the elementary symmetric polynomials:

$$
\begin{aligned}
p_1(\sigma) =& \sigma_1 - m \\
p_5(\sigma) =& 5\sigma_1 - 20\sigma_1^3 + 16\sigma_1^5 + 60\sigma_1\sigma_2 - 80\sigma_1^3\sigma_2 + 80\sigma_1\sigma_2^2 \\
& - 60\sigma_3 + 80\sigma_1^2\sigma_3 - 80\sigma_2\sigma_3 \\
p_7(\sigma) =& -7\sigma_1 + 56\sigma_1^3 - 112\sigma_1^5 + 64\sigma_1^7 - 168\sigma_1\sigma_2 + 560\sigma_1^3\sigma_2 \\
& - 448\sigma_1^5\sigma_2 - 560\sigma_1\sigma_2^2 + 896\sigma_1^3\sigma_2^2 - 448\sigma_1\sigma_2^3 \\
& + 168\sigma_3 - 560\sigma_1^2\sigma_3 + 448\sigma_1^4\sigma_3 + 560\sigma_2\sigma_3 \\
& - 1344\sigma_1^2\sigma_2\sigma_3 + 448\sigma_2^2\sigma_3 + 448\sigma_1\sigma_3^2.
\end{aligned}
$$

While this may not seem any less complex, it significantly decreases the total degree of each polynomial, thus the computing systems are able to solve these equations.

While we do study symmetric polynomials in this paper, we do not apply it to power conversion (thank goodness!), however it is great to see that symmetric polynomials have very useful applications. In particular, the main focus of this paper, the Fundamental Theorem of Symmetric Polynomials, is relevant in electrical engineering and beyond!

2. Symmetric Polynomials

2.1. **Introduction to Symmetric Polynomials.** Symmetric polynomials are exactly what the name suggests; put in layman's terms, if any of the variables in a polynomial are interchanged, then we get the same polynomial. We will explore some key components of symmetric polynomials, including the *elementary symmetric polynomials*, which have some very useful applications. We start out by giving a formal definition of symmetric polynomials, and then there are some examples to help fully understand the concept.

Definition 2.1 (Symmetric Polynomials)**.** We say a polynomial $f \in F[x_1, ..., x_n]$ is a **symmetric polynomial** if

$$f(x_{\tau(1)}, ..., x_{\tau(n)}) = f(x_1, ..., x_n)$$

for all permutations τ in the symmetric group S_n.

Example 2.2. Suppose we have $\tau = (132)$ and some polynomial $f(x_1, x_2, x_3)$. Then, if $f(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) = f(x_3, x_1, x_2) = f(x_1, x_2, x_3)$, and if this holds for all other permutations, we can say that f is a symmetric polynomial.

While we only showed symmetry for one permutation, it holds for all permutations in the symmetric group S_n which is why we can assert that f is a symmetric polynomial above. A more concrete example of a symmetric polynomial can be found below.

Example 2.3. Suppose we have a polynomial f such that $f = 2x_1 \cdot x_2 \cdot x_3 + 2x_1 \cdot x_3 + 2x_1 \cdot x_2 + 2x_2 \cdot x_3$. Since we can interchange any of the x_i, we say that this is a symmetric polynomial.

To solidify our understanding of what a symmetric polynomial really is, we will consider the following *non*-example.

Example 2.4. Suppose we have a polynomial h where $h = x_1 \cdot x_2^3 + 4x_1 + 4x_2$. Consider the permutation $\tau = (21)$. Then, we would get

$$h(x_{\tau(2)}, x_{\tau(1)}) = x_2 \cdot x_1^3 + 4x_2 + 4x_1 \neq x_1 \cdot x_2^3 + 4x_1 + 4x_2,$$

thus h is not symmetric. Note that if we had $h = x_1^3 \cdot x_2^3 + 4x_1 + 4x_2$, then h would in fact be a symmetric polynomial.

A neat place where symmetric polynomials appear "in nature" is when finding the discriminant of a polynomial. The discriminant is useful because it tells us about the behavior of the roots of a polynomial.

Example 2.5. The discriminant D of a polynomial of degree n can be found using the following equation:

$$D = \sum_{i<j\leq n} (\psi_i - \psi_j)^2$$

where each ψ_i is a root of the polynomial. Consider when we have a polynomial of degree 3 with roots ψ_1, ψ_2, and ψ_3. Then, the discriminant would be

$$
\begin{aligned}
D &= (\psi_1 - \psi_2)^2 + (\psi_2 - \psi_3)^2 + (\psi_1 - \psi_3)^2 \\
&= \psi_1^2 - 2\psi_1\psi_2 + \psi_2^2 + \psi_2^2 - 2\psi_2\psi_3 + \psi_3^2 + \psi_1^2 - 2\psi_1\psi_3 + \psi_3^2 \\
&= 2\psi_1^2 + 2\psi_2^2 + 2\psi_3^2 - 2\psi_1\psi_2 - 2\psi_2\psi_3 - 2\psi_1\psi_3
\end{aligned}
$$

which is in fact a symmetric polynomial.

Think about why that is a symmetric polynomial; understanding that concept is important as we move forward and introduce more ideas that build off of this foundation. For any permutation in the symmetric group, we always get the same polynomial. In simpler terms, we can switch any two of the ψ_i and still get the same polynomial.

The next step on our journey to proving the Fundamental Theorem of Symmetric Polynomials is to introduce and define the elementary symmetric polynomials. Caution, just because the name includes the word "elementary" it is not a concept intended for children. The following is rated 18+ for complex arguments.

2.2. **Elementary Symmetric Polynomials.** There is a specific type of symmetric polynomial that is extremely powerful; the **elementary symmetric polynomials**. These can be thought of as the building blocks of all symmetric polynomials, since we will later prove the Fundamental Theorem that says any symmetric polynomial can be written in terms of these elementary symmetric polynomials.

Definition 2.6 (Elementary Symmetric Polynomials)**.** Let $x_1, ..., x_n$ be variables and consider the polynomial ring $F[x_1, ..., x_n]$. Then,

$$\sigma_1 = x_1 + ... + x_n$$

$$\sigma_2 = \sum_{1 \leq i < j \leq n} x_i x_j$$

$$\vdots$$

$$\sigma_r = \sum_{1 \leq i_1 < ... < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

$$\vdots$$

$$\sigma_n = x_1 x_2 \cdots x_n$$

are the **elementary symmetric polynomials**. Thus $\sigma_1, ..., \sigma_n \in F[x_1, ..., x_n]$.

To help visualize what the elementary symmetric polynomials are, consider the following example.

Example 2.7. Let x_1, x_2, and x_3 be variables over a field F. Then, we would have

$$\sigma_1 = x_1 + x_2 + x_3$$

$$\sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$\sigma_3 = x_1 x_2 x_3$$

Due to the commutative property of addition and multiplication, we can permute σ_1 and σ_3 in any way and still get the same outcome, thus they are symmetric. It is less obvious that σ_2 is symmetric, but rest assured, it is; consider the permutation $\tau = (132)$. We would get

$$\sigma_2 = x_3x_1 + x_3x_2 + x_1x_2,$$

which is equivalent to the original. A similar argument can be used for other permutations to show that the above polynomials are in fact symmetric.

Now we are certain that the elementary symmetric polynomials are, well, symmetric. Never trust a book by its cover, and never trust a math concept by its name.

As promised, we will now explore a classical example that comes from Isaac Newton's book *Arithmetica universalis*, published in 1707. The Newton Identities show the relationship between power sums and elementary symmetric polynomials, so we first must see what the power sums are.

Example 2.8 (Newton Identities)**.** Albert Girard is credited with publishing the first power sums in 1629, which he defined as

$$s_r = x_1^r + \cdots + x_n^r$$

When $r = 1$ we get σ_1, and when $r > 1$ we get the *Newton identities*:

$$s_r = \sigma_1 s_{r-1} - \sigma_2 s_{r-2} + \ldots + (-1)^{r-1} r\sigma_r \qquad \text{if } 1 < r \leq n,$$

$$s_r = \sigma_1 s_{r-1} - \sigma_2 s_{r-2} + \ldots + (-1)^{r-1} \sigma_n s_{r-n} \qquad \text{if } r > n.$$

This is a neat example to consider that symmetric polynomials have been thought about since the 1600s. At this point, I am pretty much Isaac Newton's apprentice.

Now that we have a firm grasp of what a symmetric polynomial looks like and what the elementary symmetric polynomials are, we look forward to the Fundamental Theorem of Symmetric Polynomials (FTSP) which states that *any* symmetric polynomial in $F[x_1, ..., x_n]$ can be written in terms of $\sigma_1, ..., \sigma_n$. However, before we consider the FTSP, we must first state a few definitions, prove a few lemmas, and dare we forget to introduce and prove a corollary!

2.3. **Lexicographic Order and Leading Terms.** The proof of the FTSP involves an inductive process which requires that we order monomials $x_1^{a_1} \cdots x_n^{a_n}$ in $x_1, ..., x_n$. We will use *graded lexicographic order.*

Definition 2.9 (Lexicographic Order)**.** Let $x_1^{a_1} \cdots x_n^{a_n}$ and $x_1^{b_1} \cdots x_n^{b_n}$ be monomials in $F[x_1, ..., x_n]$. Then we declare

$$x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$$

if either

$$a_1 + \cdots + a_n < b_1 + \cdots + b_n$$

or $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and for some $1 \leq j \leq n$, $a_i = b_i$ for all $1 \leq i < j$ and $a_j < b_j$.

To sum this up, if we want to compare monomials according to lexicographic order, the degree of each monomial must be known; if the degree is equal, then we must check the two monomials one exponent at a time, starting with x_1, to find where there is a difference. To illustrate this, consider the example below.

Example 2.10. Suppose we have the following monomials. If we want to properly order them according to lexicographic ordering, then we see the following:

$$x_1^4x_2^2x_3 < x_1^2x_2^3x_3^3 \qquad \text{(smaller total degree)}$$

$$x_1^4x_2^2x_3 > x_1^4x_2x_3^2 \qquad \text{(same total degree, equal } x_1 \text{ exponent, greater } x_2 \text{ exponent).}$$

Another important part of symmetric polynomials that we will need to consider is the leading term. We use the definition of lexicographic order to define the leading term of a polynomial.

Definition 2.11 (Leading Term)**.** The **leading term** (LT) of a polynomial is the monomial of the greatest value, relative to Definition 2.9, times its coefficient.

This is a very literal definition, so it helps to see an example to fully understand the concept of a leading term.

Example 2.12. The leading term of the elementary symmetric polynomial

$$\sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \cdots + x_{n-1}x_n \tag{2.1}$$

is x_1x_2. Since the degree of each term is the same, we must consider the definition of lexicographic order to show why the leading term is x_1x_2. Since we assume there are n terms, we can rewrite this as

$$\sigma_2 = x_1x_2x_3^0\cdots x_n^0 + x_1x_2^0x_3\cdots x_n^0 + ... + x_1^0x_2^0x_3^0\cdots x_{n-1}x_n.$$

Thus, when we apply lexicographic order to find the leading term, it becomes clear that x_1x_2 is in fact the leading term; comparing the first two terms, since the first term $= x_1x_2$ and the second term $= x_1x_2^0x_3$, clearly x_2 has a bigger exponent value than x_2^0, thus the first term is bigger than the second term. A similar argument shows why x_1x_2 is bigger than any other term, thus x_1x_2 is the leading term of σ_r.

We now have a good understanding on the make-up of symmetric polynomials, so now we will begin to take a deeper dive into understanding them and prove a few lemmas and a corollary to help us with the proof of the FTSP.

2.4. **Leading Terms are the Best!** We need the following lemmas and corollary in order to provide a rigorous proof of the FTSP. They focus on the usefulness of the leading term of a symmetric polynomial as well as the combination of multiple symmetric polynomials.

Lemma 2.13. *Let $f, g \in F[x_1, ..., x_n]$ be nonzero. Then, LT(fg) = LT(f)LT(g).*

Proof. Let $x = x_1 \cdots x_n$ be a monomial in $F[x_1, ..., x_n]$ and let $\alpha = (a_1, ..., a_n)$, $\beta = (b_1, ..., b_n)$, $\rho = (r_1, ..., r_n)$, and $\delta = (d_1, ..., d_n)$ be exponent vectors where every a_i, b_i, r_i, and d_i are non-negative integers. Then, $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$, $x^\beta = x_1^{b_1} \cdots x_n^{b_n}$, $x^\rho = x_1^{r_1} \cdots x_n^{r_n}$, and $x^\delta = x_1^{d_1} \cdots x_n^{d_n}$. Suppose $x^\alpha > x^\beta$ and let x^ρ be any monomial. Consider $x^\alpha x^\rho$ and $x^\beta x^\rho$. We see that

$$\begin{aligned} x^\alpha x^\rho &= x_1^{a_1} x_1^{r_1} \cdots x_n^{a_n} x_n^{r_n} \\ &= x^{a_1+r_1} \cdots x^{a_n+r_n} \\ &= x^{\alpha+\rho}. \end{aligned}$$

Similarly, $x^\beta x^\rho = x^{\beta+\rho}$. Now, we want to show that $x^{\alpha+\rho} > x^{\beta+\rho}$. Suppose x^α and x^β have different degrees. Since we assumed that $x^\alpha > x^\beta$, if they have different degrees then it follows from the definition of lexicographic order that the degree of x^α is greater than the degree of x^β, therefore the degree of $x^{\alpha+\rho}$ is greater than the degree of $x^{\beta+\rho}$. Thus, we will assume that x^α and x^β have the same degree, thus according to lexicographic order we know that for some $1 \leq i \leq n$, $a_i > b_i$. Thus, since $x^{\alpha+\rho}$ and $x^{\beta+\rho}$ have the same degree, we know that $a_i + r_i > b_i + r_i$, hence lexicographic order tells us that $x^{\alpha+\rho} > x^{\beta+\rho}$.

Now let x^δ be some monomial where $x^\rho > x^\delta$. We also maintain the assumptions from the previous paragraph, namely that $x^\alpha > x^\beta$ where α and β have the same degree but $a_i > b_i$ for some $1 \leq i \leq n$. We want to show that $x^{\alpha+\rho} > x^{\beta+\delta}$. Similarly, we will assume that x^ρ and x^δ have the same degree, otherwise it is trivial that $x^{\alpha+\rho} > x^{\beta+\delta}$. Since x^ρ and x^δ have the same degree, we know that for some $1 \leq j \leq n$, $r_j > d_j$. There are two cases we must consider, when $i \leq j$ and when $i > j$.

Suppose $i \leq j$. Then, we know that $a_i > b_i$ and that $r_i > d_i$, therefore $a_i + r_i > b_i + d_i$ and $x^{\alpha+\rho} > x^{\beta+\delta}$.

Now suppose $i > j$. We have $a_j = b_j$ and $r_j > b_j$, thus $a_j + r_j > b_j + d_j$ and $x^{\alpha+\rho} > x^{\beta+\delta}$.

Since in both cases we have $x^{\alpha+\rho} > x^{\beta+\delta}$, we know that this always holds.

Let $LT(f) = x^\alpha$ and let $LT(g) = x^\rho$. Then, we have shown that $x^\alpha x^\rho = x^{\alpha+\rho} > x^{\beta+\delta}$ for any x^δ, so we know that $LT(f)LT(g) = x^{\alpha+\rho} = LT(fg)$, as desired. $\square$

Understanding what happens to the leading terms of two symmetric polynomials when we multiply them together is a crucial part of proving the FTSP. Thus, we will highlight an example to get a more concrete understanding.

Example 2.14. Suppose we have the following symmetric polynomials

$$f = x_1^2 x_2^2 + x_1^2 + x_2^2$$

$$g = 7x_1x_2x_3 + x_1 + x_2 + x_3.$$

According to lexicographic order, we know that $LT(f) = x_1^2x_2^2$ and $LT(g) = 7x_1x_2x_3$. We want to show that $LT(f \cdot g) = 7x_1^3x_2^3x_3$. We see that

$$\begin{aligned} f \cdot g =& 7x_1^3x_2^3x_3 + x_1^3x_2^2 + x_1^2x_2^3 + x_1^2x_2^2x_3 + 7x_1^3x_2x_3 + \\ & x_1^3 + x_1^2x_2 + x_1^2x_3 + 7x_1x_2^3x_3 + x_1x_2^2 + x_2^3 + x_2^2x_3 \end{aligned}$$

The only monomial in $f \cdot g$ of degree 7 is $7x_1^3x_2^3x_3$ and every other term has a lower degree, thus it is the leading term. So, $LT(f) \cdot LT(g) = LT(f \cdot g)$, as desired.

The leading term of a symmetric polynomial is very helpful information; so, it makes sense to be curious what the leading term of the elementary symmetric polynomials looks like. Well, fear not because the next lemma illustrates that exactly.

Lemma 2.15. *Suppose σ_r is some nonzero elementary symmetric polynomial. Then, $LT(\sigma_r) = x_1x_2 \cdots x_r$.*

Proof. Consider $\sigma_r = x_1x_2x_3 \cdots x_r + x_1x_3 \cdots x_rx_{r+1} + x_1x_2 \cdots x_rx_{r+1}x_{r+2} + \ldots + x_1x_2 \cdots x_n$. We know that each term has the same degree, so we must compare the exponent values according to lexicographic order to determine which term is the leading term. Similar to Example (2.12), we know that the first term is greater than the second term because $\sigma_{r_1} = x_1x_2x_3 \cdots x_r$ has degree 1 for x_2 and $\sigma_{r_2} = x_1x_3 \cdots x_rx_{r+1}$ has degree 0 for x_2. A similar argument shows why σ_{r_1} beats all other terms, thus we know that $LT(\sigma_r) = \sigma_{r_1} = x_1x_1 \cdots x_r$. $\square$

The proof of this lemma is fairly straightforward so there is no need to highlight an example; rather, we take it a step further and introduce the following corollary, that shows us what the leading term of an elementary symmetric polynomial is when raised to some exponent.

Corollary 2.16. *Suppose σ_r^β is some nonzero elementary symmetric polynomial where $\beta \geq 1$ is an integer. Then, LT(σ_r^β) = $x_1^\beta \cdot x_2^\beta \cdots x_r^\beta$.*

Proof. Let $f, g \in F[x_1, ..., x_n]$ be nonzero. Suppose $f = \sigma_r$ and $g = \sigma_r$. We know that $\mathrm{LT}(f) = x_1x_2 \cdots x_r$ and $\mathrm{LT}(g) = x_1x_2 \cdots x_r$ from Lemma 2.15. Then, $\mathrm{LT}(fg) = \mathrm{LT}(f)\mathrm{LT}(g) = x_1^2x_2^2 \cdots x_r^2$ which follows from Lemma 2.13. The result now follows by induction on β. $\mathrm{LT}(\sigma_r^\beta)$ can be written as $\mathrm{LT}(f_1)\mathrm{LT}(f_2) \cdots \mathrm{LT}(f_\beta)$, where each $f_i = x_1x_2 \cdots x_r$. Thus, $\mathrm{LT}(\sigma_r^\beta) = x_1^\beta x_2^\beta \cdots x_r^\beta$, as desired. $\square$

To help fully understand this corollary, consider the following example. We use σ_2 for the sake of simplicity, however rest assured, it works for any σ_r, it just gets more and more complex and messy.

Example 2.17. Suppose we have variables x_1, x_2 and x_3 over a field F. Consider $\sigma_2 = x_1x_2 + x_1x_3 + x_2x_3$. From Lemma 2.15, we know that $LT(\sigma_2) = x_1x_2$.

Now consider σ_2^2. We want to show that $LT(\sigma_2^2) = x_1^2 x_2^2$.

$$\begin{aligned}\sigma_2^2 =&(x_1x_2 + x_1x_3 + x_2x_3)(x_1x_2 + x_1x_3 + x_2x_3)\\ =&x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2x_3 + x_1^2x_3^2+\\ &x_1x_2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2\\ =&x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2\end{aligned}$$

Since all of the monomials have the same degree, we must compare each term. We know that $x_1^2x_2^2 > x_1^2x_3^2$ since the total exponent on x_2 is larger. A similar proof shows why $x_1^2x_2^2$ is the leading term according to lexicographic order. Thus, $LT(\sigma_2) = x_1^2x_2^2$, as desired.

Our next lemma introduces the idea that there are only a finite number of monomials that, according to lexicographic order, are less than the leading term. Essentially, this lemma is saying that there are a finite number of monomials in the infinite ring $F[x_1, ..., x_n]$. Also, please note: 2.18 is my mom's birthday, so if you are reading this Mom, this is your birthday lemma!

Lemma 2.18. *Suppose f is a symmetric polynomial where $LT(f) = x_1^{a_1} \cdots x_n^{a_n}$. Then, there are at most finitely many monomials $x_1^{b_1} \cdots x_n^{b_n}$ such that*

$$x_1^{b_1} \cdots x_n^{b_n} < x_1^{a_1} \cdots x_n^{a_n} \qquad \textit{for fixed } a_1, ..., a_n. \tag{2.2}$$

Proof. We know this because Definition 2.9 and Equation (2.2) imply that $a_1 + \cdots + a_n \geq b_1 + \cdots b_n$. Then, since $N = a_1 + \cdots + a_n$ is fixed and $b_i \geq 0$ for all i, we get the following inequality

$$N = a_1 + \cdots + a_n \geq b_1 + \cdots + b_n \geq b_i$$

for all i. This tells us that there are only $N+1$ possibilities for each b_i, implying that (2.2) can hold for at most finitely many $x_1^{b_1} \cdots x_n^{b_n}$. □

Since this lemma is more fundamental, we will not be going through an example of it; however, it still is an important thing to keep in mind once we begin considering the FTSP. Our penultimate lemma deals with the linear combinations of two symmetric polynomials and states that if you add them together, even if one is multiplied by a constant integer, the resulting polynomial is also symmetric.

Lemma 2.19. *Suppose f and g are symmetric polynomials and c is some constant integer. Then, $f + cg$ is also a symmetric polynomial.*

Proof. Assume f and g are both symmetric. By definition, we know that $f(x_\tau) = f(x)$ and $g(x_\tau) = g(x)$ where τ is some permutation of f. Let $c \in \mathbb{Z}$ be some constant. We know that $cg(x) = c[g(x)]$. Then, we know that

$$(f + cg)(x) = f(x) + cg(x).$$

But, since both f and cg are symmetric, we also know that

$$(f + cg)(x_\tau) = f(x_\tau) + c[g(x_\tau)] = f(x) + c[g(x)] = (f + cg)(x),$$

so by definition this is symmetric. □

We see a nice example of this lemma in action below.

Example 2.20. Suppose $f = x_1 + x_2$ and $g = x_1 x_2$ and $c = 2$. We can clearly see that both f and g are symmetric; thus, Lemma 2.19 tells us that $f + cg$

must also be symmetric. We see that

$$f + cg = x_1 + x_2 + 2x_1x_2,$$

which is symmetric, as desired.

If you have made it this far, I am very thankful. It means a lot that you are willing to read my work, and I promise that the end is near. Just one more lemma and then we really get into the exciting parts, proving the Fundamental Theorem of Symmetric Polynomials- that just sounds awesome, right?

Our last lemma that we need is helping us to understand what the leading term of a symmetric polynomial actually looks like. This is powerful information, so use it wisely.

Lemma 2.21. *Let $f \in F[x_1, ..., x_n]$ be symmetric with some nonzero leading term $cx_1^{a_1} \cdots x_n^{a_n}$. Then, $a_1 \geq a_2 \geq \cdots \geq a_n$*

Proof. By way of contradiction, suppose that $a_i < a_{i+1}$ for some $1 \leq i \leq n-1$. By definition of the leading term, we know that there does not exist a monomial that has a greater degree than $cx_1^{a_1} \cdots x_n^{a_n}$. Due to Definition (2.9), we do not have to consider any monomial of lower degree since it would automatically be disqualified. Thus, we only have to consider monomials of the same degree as $cx_1^{a_1} \cdots x_n^{a_n}$. Since f is a symmetric polynomial, we can permute it and maintain the same polynomial. So, we will switch x_i and x_{i+1}. Then, we have a new monomial in f, namely $cx_1^{a_1} \cdots x_{i+1}^{a_i} x_i^{a_{i+1}} \cdots x_n^{a_n}$. Based on our assumption that $a_i < a_{i+1}$, by Definition (2.9) we see see that $cx_1^{a_1} \cdots x_{i+1}^{a_i} x_i^{a_{i+1}} \cdots x_n^{a_n}$ is a term of f that is greater than $cx_1^{a_1} \cdots x_n^{a_n}$. This is because x_i has an exponent value of a_i in $cx_1^{a_1} \cdots x_i^{a_i} \cdots x_n^{a_n}$ and an exponent value of a_{i+1} in $cx_1^{a_1} \cdots x_{i+1}^{a_i} x_i^{a_{i+1}} \cdots x_n^{a_n}$, and since we assumed that $a_{i+1} > a_i$, by Definition (2.9), we know that $cx_1^{a_1} \cdots x_{i+1}^{a_i} x_i^{a_{i+1}} \cdots x_n^{a_n}$ is a term that is greater than $cx_1^{a_1} \cdots x_n^{a_n}$. Yet $cx_1^{a_1} \cdots x_n^{a_n}$ is our leading term, therefore we have a contradiction, thus $a_{i+1} \leq a_i$, as desired. □

Now that we have ventured through the hardware aisles of Home Depot, we have the necessary tools to tackle the Fundamental Theorem of Symmetric Polynomials.

3. Fundamental Theorem of Symmetric Polynomials

3.1. FTSP:PSTF (Fundamental Theorem of Symmetric Polynomials: Please Sit-down, Think-hard, and Focus).

Sorry, I could not pass up on the opportunity for some beautiful symmetry in my section title. Back to business.

We will be taking a similar approach to proving the FTSP as David Cox in his book *Galois Theory*[3], however we have already done most of the leg work for the proof. A more modern approach to this theorem would be to apply Galois Theory, however it requires a deeper understanding of group theory and field theory, which goes beyond the scope of this thesis. Plus, the proof using Galois Theory is short, so what fun is that? We want a good head-scratcher, and that is exactly what we are going to get.

Theorem 3.1 (Fundamental Theorem of Symmetric Polynomials)**.** *Any symmetric polynomial in $F[x_1, ..., x_n]$ can be written as a polynomial in $\sigma_1, ..., \sigma_n$ with coefficients in F.*

Proof. Suppose $f \in F[x_1, ..., x_n]$ is a symmetric polynomial where $LT(f) = cx_1^{a_1} \cdots x_n^{a_n}$. By Lemma 2.21, we know that $a_1 \leq \cdots \leq a_n$.

Now consider

$$g = \sigma_1^{a_1 - a_2} \sigma_2^{a_2 - a_3} \cdots \sigma_{n-1}^{a_{n-1} - a_n} \sigma_n^{a_n}. \tag{3.1}$$

Since we know that each $a_i \geq a_{i+1}$, we know that this is a polynomial. Then, by Lemma 2.18 and Corollary 2.16 we know that

$$\begin{aligned} LT(g) &= x_1^{a_1 - a_2}(x_1 x_2)^{a_2 - a_3}(x_1 x_2 x_3)^{a_3 - a_4} \cdots (x_1 \cdots x_{n-1})^{a_{n-a} - a_n}(x_1 \cdots x_n)^{a_n} \\ &= x_1^{a_1 - a_2 + a_2 + a_3 + \ldots + a_n} x_2^{a_2 - a_3 + \ldots + a_n} \cdots x_{n-1}^{a_n - 1 - a_n + a_n} x_n^{a_n} \qquad (3.2) \\ &= x_1^{a_1} \cdots x_n^{a_n} \end{aligned}$$

Thus, we see that $LT(cg) = LT(f)$, so we define $f_1 = f - cg$ as a polynomial with strictly smaller leading term than f due to Definition (2.9), and we know that f_1 is symmetric due to Lemma (2.19).

By a similar method, we can show that there is a $f_2 = f_1 - c_1 g_1$ with a leading term strictly smaller than f_1. But, we can rewrite this as $f_2 = f - cg - c_1 g_1$. By continuing this method, we get we get the following polynomials:

$$f, \quad f_1 = f - cg, \quad f_2 = f - cg - c_1 g_1, \quad f_3 = f - cg - c_1 g_1 - c_2 g_2, \ ...,$$

By Lemma (2.18), we know that there is eventually some m such that $f_m = 0$. Since $f_m = 0$, there is no leading term, therefore the process terminates and we obtain

$$f = cg + c_1 g_1 + \cdots + c_{m-1} g_{m-1}. \tag{3.3}$$

Since each g_i is a product of the σ_j to various powers, we know by Lemma 2.19 that f is a polynomial composed of the elementary symmetric polynomials, as desired. □

Now that we have proven the FTSP, we will consider a few examples to see it in action. It is really a beautiful thing, so don't just take my word for it; see it for yourself!

In the following examples, we will use the notation

$$\sum_n x_1^{a_1} \cdots x_n^{a_n}$$

to denote the sum of all distinct monomials obtained from $x^{a_1} \cdots x_n^{a_n}$ by permuting $x_1, ..., x_n$.

Example 3.2. Let $f = \sum_2 x_1^2 x_2$. Then, we see that

$$\begin{aligned} f &= x^2 x_1 + x_1 x^2 \\ &= (x_1 x_2)(x_1 + x_2) \\ &= \sigma_2 \cdot \sigma_1. \end{aligned}$$

Thus, $f = \sum_2 x_1^2 x_2 = \sigma_1 \cdot \sigma_2$

That one was too easy. Let's take it up a notch and consider variables x_1, x_2 and x_3.

Example 3.3. Let $g = \sum_3 x_1^2 x_2$. Then, we see that

$$g = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

Now consider

$$(x_1 + x_2 + x_3)(x_1x_2 + x_1x_3 + x_2x_3) = \sigma_1 \cdot \sigma_2 =$$

$$= x_1^2x_2 + x^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3 + x_1x_2x_3 +$$

$$x_1x_3^2 + x_2x_3^2$$

$$= x_1^2x_2 + x^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 3x_1x_2x_3$$

$$= g + 3x_1x_2x_3$$

Since $3x_1x_2x_3 = 3 \cdot \sigma_3$, we see that $g = \sigma_1 \cdot \sigma_2 - 3\sigma_3$, thus we have found g in terms of the elementary symmetric polynomials, as desired.

The final exciting theorem that we explore here takes the FTSP one step further and says that there is only one way in which we can express a symmetric polynomial in terms of the elementary symmetric polynomials.

3.2. **A Symmetric Polynomial is Like a Snowflake.** Indeed, while every snowflake is unique in shape and pattern, each symmetric polynomial is unique in that there is only one way to write it in terms of the elementary symmetric polynomials. Since we know that every symmetric polynomial can be written in terms of the elementary symmetric polynomials, we now want to assert that this is unique.

Theorem 3.4. *A given symmetric polynomial can be expressed as a polynomial in the elementary symmetric polynomials in only one way.*

Proof. We will use the polynomial ring $F[u_1, ..., u_n]$, where $u_1, ..., u_n$ are new variables. We know the map sending u_i to $\sigma_i \in F[x_1, ..., x_n]$ defines a ring homomorphism

$$\phi : F[u_1, ..., u_n] \to F[x_1, ..., x_n]$$

In other words, if $h = h(u_1, ..., u_n)$ is a polynomial in $u_1, ...u_n$ with coefficients in F, then $\phi(h) = h(\sigma_1, ..., \sigma_n)$. The image of ϕ is the set of all polynomials in the σ_i with coefficients in F. We denote this image by

$$\text{Im}(\phi) = F[\sigma_1, ..., \sigma_n] \subset F[x_1, ..., x_n].$$

Note that $F[\sigma_1, ..., \sigma_n]$ is a subring of $F[x_1, ..., x_n]$. In this notation, we can write ϕ as a map

$$\phi : F[u_1, ..., u_n] \to F[\sigma_1, ..., \sigma_n]. \tag{3.4}$$

This map is onto by the definition of $F[\sigma_1, ..., \sigma_n]$, and uniqueness will be proved by showing that ϕ is one-to-one. In order to show that ϕ is one-to-one, we will show that its kernel is $\{0\}$.

Let's assume $h \in F[u_1, \ldots, u_n]$ is nonzero. To show uniqueness, we must show that $\phi(h) = h(\sigma_1, ..., \sigma_n)$ is not the zero polynomial in $x_1, ..., x_n$.

Suppose $cu_1^{b_1} \cdots u_n^{b_n}$ is a term of h. By applying ϕ to $cu_1^{b_1} \cdots u_n^{b_n}$, we get $c\sigma_1^{b_1} \cdots \sigma_n^{b_n}$. According to Lemma 2.13, we know that

$$LT(c\sigma_1^{b_1} \cdots \sigma_n^{b_n}) = c \cdot (LT(\sigma_1^{b_1}) \cdot LT(\sigma_2^{b_2}) \cdots LT(\sigma_n^{b_n}))$$

By Corollary 2.16 we see that

$$LT(c\sigma_1^{b_1} \cdots \sigma_n^{b_n}) = cx_1^{b_1+...+b_n} x_2^{b_2+...+b_n} \cdots x_n^{b_n}.$$

In order to complete the proof, we must show that

$$(b_1, b_2, ..., b_n) \to (b_1 + b_2 + ... + b_n, b_2 + ... + b_n, ..., b_n)$$

is one-to-one. In order to show this, let's set

$$(c_1, c_2, ..., c_n) = (b_1 + b_2 + ... + b_n, b_2 + ... + b_n, ..., b_n)$$

and we will define a map ψ where

$$\psi(b_1, b_2, ..., b_n) = (c_1, c_2, ..., c_n).$$

By construction, we see that for all $k \in \{1, ..., n-1\}$,

$$c_k = b_k + c_{k+1}$$

Assume $(b_1, ..., b_n) \neq (0, ..., 0)$. Then, for some $i \in \{1, ..., n\}$, $b_i \neq 0$. Let j be the largest integer where $b_j \neq 0$, so we know that for all t such that $j < t \leq n$, $b_t = 0$. Then, $c_{j+1} = 0$ thus $c_j = b_j + c_{j+1} = b_j \neq 0$. This proves that ψ is one-to-one which means that the leading terms can't all cancel. Hence, $\phi(h)$ can't be the zero polynomial and uniqueness follows. □

Now we know that each symmetric polynomial can only be written in terms of the elementary symmetric polynomials in one unique way; this is a neat idea, and one that is important to keep in mind the next time you find yourself daydreaming about symmetric polynomials. Or maybe that would be a daynightmare. Is that a thing? Anyway, let's see an example of this theorem to really hammer in this idea.

Example 3.5. Recall from Example (3.2) we had $f = \sum_2 x_1^2 x_2$. We found that $f = \sigma_1 \cdot \sigma_2$, but is there another way that we can write f in terms of the elementary symmetric polynomials? According to Lemma 2.13, we know that $LT(\sigma_1)LT(\sigma_2) = LT(\sigma_1 \cdot \sigma_2)$. So, we want to find out if there is another way to write $LT(\sigma_1 \cdot \sigma_2)$ in terms of the elementary symmetric polynomials. Consider

$$LT(\sigma_1 \cdot \sigma_2) = x_1^2 x_2.$$

The only combination of symmetric polynomials that would give us $x_1^2 x_2$ as the leading term is $\sigma_1 \cdot \sigma_2$, thus it is unique which supports Theorem 3.4, as desired.

4. Cheeseburgers

I told Jeff, my advisor, that I would do my best to find a way to incorporate a section on cheeseburgers. So, buckle your seat belts, because we are getting funky. Suppose you are making a burger with varying amounts of burger meat, bun, lettuce, and tomatoes. Let's let x_1 =bottom bun, x_2 =meat, x_3 =cheese, x_4 =top bun. Now, the natural order for a cheeseburger would be bun, meat, cheese, bun.

$$f_1 = x_1 + x_2 x_3 + x_4$$

We multiply x_2 and x_3 since the cheese really does stick to the meat. This equation lacks symmetry, so we will have to get creative to resurrect that. Naturally, the course of action we must take is to fuse each component together. We will have burger meat fused with the bottom bun ($x_1 x_2$), cheese fused with the bottom bun ($x_1 x_3$), the bottom bun and top bun fused together ($x_1 x_4$), burger meat fused with the top bun ($x_2 x_4$), and cheese fused with the top bun

(x_3x_4). So, combining these together we get

$$f_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

which is symmetric! Nice. However, the order of this cheeseburger is as follows: fusions of bottom bun and meat, bottom bun and cheese, bottom bun and top bun, meat and cheese, meat and top bun, and cheese and top bun. That is going to be one thick burger. The things we do for our love of mathematics and symmetric polynomials!

Anyway, if you have made it this far I really appreciate you reading this. The code word is sea breeze- if you tell me this verbally, through email, or through text, then I owe you a firm handshake and a thank you.

References

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[3] D. Cox, Galois Theory, John Wiley and Sons, Inc., New Jersey, 2012.