THE RELATIONSHIP BETWEEN ZEROS OF THE RIEMANN ZETA FUNCTION AND PRIMES

By

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ABSTRACT

THOMAS FARINA The Relationship Between Zeros of the Riemann Zeta Function and Primes.

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This thesis looks at the connection between the distribution of the prime numbers and the zeros of the Riemann zeta function. To study this connection, functions that count prime numbers will be used, such as $\pi(x)$ and $\Lambda(x)$. On top of using these functions, techniques from calculus and complex analysis will be utilized to see the connection further. The main idea that connects the zeros of the zeta function and the prime numbers is through the Riemann hypothesis.

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1. INTRODUCTION

In 1859, mathematician Bernhard Riemann gave a talk about the most recent paper he published, which translated is "On the Number of Primes Less Than a Given Magnitude". This paper focused on a function called the zeta function, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Although this function was defined previously, Riemann used techniques from complex analysis to look for a pattern in the distribution of primes. While talking about it he casually mentions a conjecture about the zeros of the zeta function. The conjecture is known as the **Riemann hypothesis** and is stated below.

Riemann Hypothesis. *The **Riemann hypothesis** states that the zeros of the zeta function that lie between* $0 \leq s \leq 1$ *have* $\mathrm{Re}(s) = \frac{1}{2}$.

The language used in the Riemann hypothesis may not make sense right now but this statement will be explained and looked at later on. While this conjecture was made nonchalantly by Riemann at the time, it has still yet to be proven to this day, even having a $1,000,000 prize for the person who solves it. While this paper does not attempt to solve the Riemann hypothesis, it explores the connection between the zeros of the zeta function and the distribution of primes. The purpose is to give a better understanding of how the two ideas are connected.

2. PRELIMINARIES

This section will define important functions that will be used later on.

2.1. **RIEMANN ZETA FUNCTION.** We start by defining the main focus of the paper.

Definition 1. The **Riemann Zeta Function** is defined by (the analytic continuation of) the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s \in \mathbb{C}$.

We write s as $s = a + bi$, where $a, b \in \mathbb{R}$, so that the real part of s is $\mathrm{Re}(s) = a$ and the imaginary part is $\mathrm{Im}(s) = b$. This function has appeared in problems in the past, such as the Basel problem which looked to compute $\zeta(2)$. We will look at different properties of the zeta function later on in the paper.

2.2. **GAMMA FUNCTION.**

Definition 2. The **gamma function**, denoted as $\Gamma(s)$, is defined as

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt.$$

As an example, we can calculate $\Gamma(1)$, which is

$$\Gamma(1) = \int_0^{\infty} e^{-t} t^{1-1} dt = \int_0^{\infty} e^{-t} dt = \left(-e^{-t}\right]_0^{\infty} = \lim_{t\to\infty} -e^{-t} + e^0 = 0 + 1 = 1$$

One interesting property of the gamma function is that it generalizes the factorial function. In order to see this we use the following property of $\Gamma(s)$.

Proposition 1. *For* $\mathrm{Re}(s) > 0$*, we have*

$$\Gamma(s+1) = s\Gamma(s) \tag{1}$$

Proof. We start by defining the left hand side using the definition of the gamma function.

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{(s+1)-1} dt = \int_0^\infty e^{-t} t^s dt$$

We can evaluate this integral using integration by parts, letting $u = t^s$ and $dv = e^{-t}dt$. Then we have $du = st^{s-1}dt$ and $v = -e^{-t}$. This integral becomes

$$\int_0^\infty e^{-t} t^s dt = \left(-e^{-t}t^s\right]_0^\infty - \int_0^\infty -e^{-t}(st^{s-1})dt.$$

Simplifying the expression gives us

$$\left(-e^{-t}t^s\right]_0^\infty - \int_0^\infty -e^{-t}(st^{s-1})dt = \lim_{x\to\infty}\left(-\frac{x^s}{e^x} + \frac{0^s}{e^0}\right) + s\int_0^\infty e^{-t}t^{s-1}dt.$$

The first part of the expression evaluates to 0 and we notice that the second part has the definition of $\Gamma(s)$ in it. Thus we get

$$\lim_{x\to\infty}\left(-\frac{x^s}{e^x} + \frac{0^s}{e^0}\right) + s\int_0^\infty e^{-t}t^{s-1}dt = 0 + s\Gamma(s) = s\Gamma(s).$$

$\square$

Knowing this fact, we start seeing the connection to the factorial function. Let n be a nonnegative integer. We know from the proposition that

$$\Gamma(n) = (n-1)\Gamma(n-1).$$

Using the proposition again for $\Gamma(n-1)$, we get

$$(n-1)\Gamma(n-1) = n-1\,[(n-2)\Gamma(n-2)]$$

We can repeat this process until we $1\Gamma(1)$, which we know equals 1. Thus we have

$$\Gamma(n) = (n-1)(n-2)\cdots 1 = (n-1)!$$

We can say that $\Gamma(n) = (n-1)!$ for nonnegative integers n. This is an example of **analytic continuation** since we can use the gamma function to define more values of the factorial function. The idea of analytic continuation will appear later on.

2.3. **PRIME COUNTING FUNCTIONS.** This section covers functions that help count primes and find prime numbers as well. To start this section, we look at the most important function that counts prime numbers.

Definition 3. Let $\pi(x)$ be the **prime-counting function**, which counts the number of prime numbers less than or equal to x, where $x \in \mathbb{R}^+$. In other words,

$$\pi(x) = \#\{p \leq x \colon p \text{ is prime}\}$$

One important theorem deals with approximating $\pi(x)$. In the 18th century, Gauss and Legendre conjectured a way to approximate it for any value of x, which is

$$\frac{x}{\ln x}.$$

Using this, we have the theorem below.

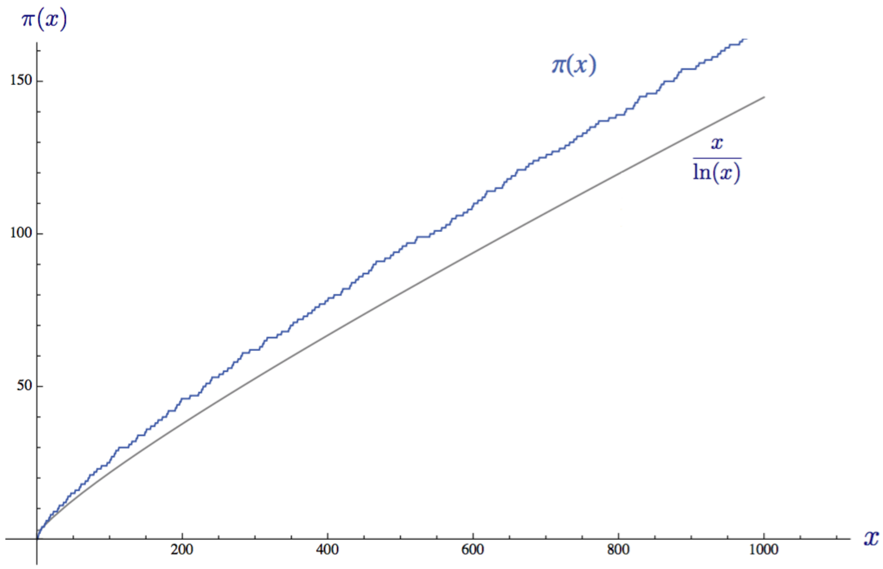

FIGURE 2.1. The graph of the first 1000 integers for the two functions in the Prime Number Theorem [5]

Theorem 1. *The* ***Prime Number Theorem*** *states that*

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1.$$

Another way to think of this theorem is that the error between the two functions approaches 0 as x increases towards infinity. The graph in Figure 2.1 shows the two different functions for the first 1000 positive integers.

We now take a look at a functions that help us study primes.

Definition 4. The **von Mangoldt function** Λ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k\text{, where } p \text{ is a prime number} \\ 0 & \text{otherwise} \end{cases}$$

While $\Lambda(n)$ is useful at looking at individual primes, we want to use it to count the number of primes up to a certain number. We make use of the function below.

Definition 5. The **second Chebyshev function** ψ is defined on $\mathbb{R}^+$ by

$$\psi(x) = \sum_{n \leq x} \Lambda(x)$$

When looking at the second Chebyshev function and the prime counting function, it is clear that the two are related in counting up to a certain number of primes. However, one would notice that $\psi(x)$ also includes values that are powers of primes. This brings up a concern that these prime powers would contribute greatly to the value of $\psi(x)$, not making it useful to compare to $\pi(x)$. In this case, we can see that most of the contribution for $\psi(x)$ comes from the primes, not the prime powers. Table 1 shows how much these prime powers contribute to the second Chebyshev function.

x	$\pi(x)$	$\pi(x) + \chi(x)$	$\frac{\pi(x)}{\pi(x)+\chi(x)}$
10	4	7	0.571
25	9	14	0.643
100	25	35	0.714
1000	168	193	0.870
10000	1229	1280	0.960
50000	5133	5217	0.984
100000	9592	9700	0.989

Table 1. Values of x and the contribution of prime powers. Note that $\chi(x)$ is the function that counts the number of prime powers up to x.

Looking at the table, we see that there are not that many prime powers up to a certain number. By looking at the ratio between $\pi(x)$ and $\pi(x) + \chi(x)$, we see that the value gets closer to 1 as x gets bigger and bigger. Since the ratio behaves this way, it shows that prime powers do not contribute much to $\psi(x)$. More explicitly, the contribution from prime powers p^k, with $k \geq 2$, is $O(x^{\frac{1}{2}} \log(x))$ [4]. From this deduction, it is reasonable to use the von Mangoldt function and the second Chebyshev function when studying prime numbers.

3. PROPERTIES OF THE RIEMANN ZETA FUNCTION

The Riemann zeta function is a particular example of a more general class of objects called Dirichlet series, which is a series of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where $s \in \mathbb{C}$ and $a(n)$ is a sequence of complex numbers. The zeta function is defined by setting $a(n) = 1$ for all $n \in \mathbb{N}$. When studying a series, one thing to determine is where a series converges or diverges. In this case, we want to see what values of s converge for $\zeta(s)$. We determine this in the following lemma.

Lemma 1. *The Dirichlet series which defines the Riemann zeta function converges when* $\mathrm{Re}(s) > 1$*. If* $\mathrm{Re}(s) \leq 1$*, then the function diverges.*

Proof. This can be shown with the Integral Test using the function $f(x) = \frac{1}{x^s}$. For this proof we only use values of s that only have a real part and no imaginary part, since the imaginary portion does not effect magnitude. In the case that $s = 1$, the integral is evaluated as

$$\lim_{b\to\infty} \int_1^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} (\ln x]_1^b = \lim_{b\to\infty} \ln b - \ln 1 = \infty.$$

This means that the function diverges when $s = 1$. When $s \neq 1$, the integral evaluates as

$$\begin{aligned}
\int_1^\infty \frac{1}{x^s} dx &= \lim_{b\to\infty} \int_1^\infty \frac{1}{x^s} dx \\
&= \lim_{b\to\infty} \left(\frac{x^{1-s}}{1-s} \right]_1^b \\
&= \lim_{b\to\infty} \frac{b^{1-s}}{1-s} - \frac{1}{1-s}
\end{aligned}$$

If $s > 1$, then we have $1 - s < 0$, making $\lim_{b\to\infty} b^{1-s} = 0$. Thus we have

$$\lim_{b\to\infty} \frac{b^{1-s}}{1-s} - \frac{1}{1-s} = -\frac{1}{1-s}$$

which converges, making $\zeta(s)$ converge for all values with $\mathrm{Re}(s) > 1$. If we have $s < 1$, then $1 - s > 0$, which makes $\lim_{b\to\infty} b^{1-s} = \infty$. Then we have

$$\lim_{b\to\infty} \frac{b^{1-s}}{1-s} - \frac{1}{1-s} = \infty,$$

which diverges. □

Knowing that the zeta function converges when $\mathrm{Re}(s) > 1$, we can rewrite it as an Euler Product.

Theorem 2. *For $\zeta(s)$ with $Re(s) > 1$, we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where the product runs over all prime numbers. This is the product expansion of $\zeta(s)$, which is known as the Euler Product.

Proof. We first write out the first few terms of $\zeta(s)$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

Multiplying $\zeta(s)$ by $\frac{1}{2^s}$ gives us

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} \dots \tag{2}$$

If we subtract (2) from $\zeta(s)$, we get

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots \tag{3}$$

Now multiplying (3) by the second term that appears in it, $\frac{1}{3^s}$, we get

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \dots \tag{4}$$

Subtracting (4) from (3) results in

$$\begin{aligned}\left(1 - \frac{1}{2^s}\right)\zeta(s) - \left(\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s)\right) &= \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s) \\ &= 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \dots \qquad (5)\end{aligned}$$

If we repeat this process, multiplying the second term to the new equation and subtracting from the previous one, this results in

$$\cdots\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 \tag{6}$$

Dividing both sides by these new terms gives us

$$\zeta(s) = \left(\frac{1}{1 - \frac{1}{2^s}}\right)\left(\frac{1}{1 - \frac{1}{3^s}}\right)\left(\frac{1}{1 - \frac{1}{5^s}}\right)\left(\frac{1}{1 - \frac{1}{7^s}}\right)\cdots \tag{7}$$

We see that the terms that we get are prime numbers. To explain this better, we can generalize the product on the right hand side of (7) as $\left(\frac{1}{1-\frac{1}{p^s}}\right)$, where p is a prime number. We see that this is a geometric series since $\left|\frac{1}{p^s}\right| < 1$ for all primes p. We can expand this to get

$$\left(\frac{1}{1-\frac{1}{p^s}}\right) = 1 + \frac{1}{p^s} + \left(\frac{1}{p^s}\right)^2 + \left(\frac{1}{p^s}\right)^3 + \cdots \tag{8}$$

Since this is for all primes, we expand all the prime numbers as such and distribute, which results in all natural numbers. This is guaranteed by the Fundamental Theorem of Arithmetic. Thus (7) can be rewritten as

$$\zeta(s) = \prod_p \left(\frac{1}{1-\frac{1}{p^s}}\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{9}$$

□

We can apply this theorem to prove the following result.

Corollary 1. *For any value of s that has $\mathrm{Re}(s) > 1$, we have*

$$\log \zeta(s) = -\sum_p \log\left(1 - p^{-s}\right)$$

where p is a prime number.

Proof. Using Theorem 2, we can rewrite $\log \zeta(s)$ as

$$\log \zeta(s) = \log\left[\prod_p (1 - p^{-s})^{-1}\right].$$

Expanding the Euler Product and using properties of logarithms, we get

$$\begin{aligned}
\log\left[\prod_p (1-p^{-s})^{-1}\right] &= \log\left[(1-2^{-s})^{-1}(1-3^{-s})^{-1}\cdots\right] \\
&= \log(1-2^{-s})^{-1} + \log(1-3^{-s})^{-1} + \cdots \\
&= \sum_p \log(1-p^{-s})^{-1} \\
&= -\sum_p \log(1-p^{-s})
\end{aligned}$$

$\square$

4. FUNCTIONAL EQUATION

As state in Lemma 1, the Dirichlet series of $\zeta(s)$ is only defined for values of s that have $\text{Re}(s) > 1$. In other words, if we try to compute $\zeta(-1)$, it would be

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots .$$

This answer does not make sense and has no real use. For our purposes, we want to look at values of s that have $\text{Re}(s) < 1$ and can be computed in a nice way. This leads us to the equation below, which is

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{10}$$

This function in (10) is known as a **functional equation**, which gives us a way to define more values in the zeta function, specifically for values of s that have $\mathrm{Re}(s) < 1$. If we try $s = -1$ into the function in (10), we get

$$\zeta(-1) = 2^{-1}\pi^{-2}\sin\left(-\frac{\pi}{2}\right)\Gamma(2)\zeta(2) = -\frac{1}{12}.$$

Using this functional equation of $\zeta(s)$, we can actually make sense of it and compute $\zeta(-1)$ and get an answer of $-\frac{1}{12}$ instead of having $1+2+3+4+\cdots$. This equation provide more insight to study the values of the zeta function for $\mathrm{Re}(s) < 1$ by the symmetry described by the functional equation.

5. ZEROS OF THE ZETA FUNCTION

Making use of the functional equation of the zeta function, we can find the zeros of the Riemann Zeta function. The zeros of the zeta function fall into two different categories, which are listed in the proposition below.

Proposition 2. (a) *There are no zeros for any s that has* $\mathrm{Re}(s) > 1$.

(b) *The zeros of $\zeta(s)$ that have* $\mathrm{Re}(s) < 0$ *are at $s = -2m$ for all $m \in \mathbb{N}$. These are called the **trivial zeros** of the zeta function.*

(c) *For all of the zeros that have* $0 \leq \mathrm{Re}(s) \leq 1$, *they lie symmetrically around the line* $\mathrm{Re}(s) = \frac{1}{2}$. *These zeros are called the **non-trivial zeros**.*

Proof. For part (a), we can use the Euler Product of the zeta function to determine if there are any zeros when we have $\mathrm{Re}(s) > 1$. In this case to have $\zeta(s) = 0$, it must be true that

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \left(\frac{1}{1 - \frac{1}{p^s}}\right) = 0$$

There is no value of p that can make this statement true, meaning that there are no values of s that make $\zeta(s) = 0$ when $\mathrm{Re}(s) > 1$.

For part (b), we will use the functional equation from (10), which is

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

From part (a), we know that $\zeta(s) \neq 0$ when $\mathrm{Re}(s) > 1$. In this case we have $\mathrm{Re}(s) < 0$, which means that the only values that are used for $\zeta(1-s)$ have $\mathrm{Re}(s) > 1$, making $\zeta(1-s) \neq 0$. Moreover, $\Gamma(s)$ has no zeros in the complex plane meaning that we can say that $\Gamma(1-s) \neq 0$. This means that the only way to find the zeros of $\zeta(s)$, we must look at $\sin\left(\frac{\pi s}{2}\right)$ and see where it equals zero. We know that only values of sin that make it equal to 0 are at $\theta = 0$ and $\theta = \pi$. This means that the only integers that make $\sin\left(\frac{\pi s}{2}\right) = 0$ true are the even integers. However, we are only looking at values that have $\mathrm{Re}(s) < 0$. Thus the only possible values that make $\sin\left(\frac{\pi s}{2}\right) = 0$ and have $\mathrm{Re}(s) < 0$ are $s = -2, -4, -6, \ldots$, i.e., the negative even integers.

To show that the zeros lie symmetrically around the line $\mathrm{Re}(s) = \frac{1}{2}$, we will make use of the functional equation of $\zeta(s)$ from (10). We notice that if we have $\zeta(\frac{1}{2})$, then the $\zeta(1-s)$ term is also equal to $\zeta(\frac{1}{2})$. Then if we have s such that $0 \leq \mathrm{Re}(s) \leq 1$, then we also know that $0 \leq \mathrm{Re}(1-s) \leq 1$, where s and $1-s$ are reflections across the line $\mathrm{Re}(s) = \frac{1}{2}$. Thus when $\zeta(s) = 0$, then we know that that $\zeta(1-s) = 0$ as well and that they are symmetric. $\square$

One of the most important topics of study is the observation of the non-trivial zeros. The area $0 \leq \mathrm{Re}(s) \leq 1$ is called the **critical strip**, and it is commonly believed that all of the non-trivial zeros are located here. These locations of the non-trivial zeros in the critical strip are the focus of the Riemann hypothesis. Recall that the Riemann hypothesis says that the zeros that lie between $0 \leq s \leq 1$ have $\mathrm{Re}(s) = \frac{1}{2}$. After looking at the zeros of the zeta function at a greater level, we can now redefine the Riemann hypothesis with more precision.

Riemann Hypothesis. *The* ***Riemann hypothesis*** *states that all of non-trivial zeros lie in the critical strip. In particular, these zeros lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$*, which is known as the* ***critical line****.*

These are the zeros that are crucial to understand the relationship between the zeta function and prime numbers. Knowing more about what the Riemann hypothesis is will help us see the idea between zeros of $\zeta(s)$ and prime numbers.

6. PRIMES AND ZEROS

To explore the connection between prime numbers and the Riemann zeta function further, we can use techniques with complex analysis and contour integration to discover more about this. After exploring more about the zeta function, we can now prove a major theorem that helps connect the prime numbers to the Riemann zeta function.

Theorem 3. *For* $\mathrm{Re}(s) > 1$*, we have*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n} \frac{\Lambda(n)}{n^s} \tag{11}$$

Proof. We first start on the left hand side, where we can rewrite it as the logarithmic derivative.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} \log(\zeta(s))$$

We can then rewrite it as

$$\frac{d}{ds}\log(\zeta(s)) = \frac{d}{ds}\left(-\sum_p \log(1-p^{-s})\right)$$

from Corollary 1. Since we are taking the derivative we can apply it to the term inside the summation and apply it, resulting in

$$\begin{aligned}\frac{d}{ds}\left(-\sum_p \log(1-p^{-s})\right) &= -\sum_p \left(\frac{d}{ds}\log(1-p^{-s})\right)\\ &= -\sum_p \left(\frac{1}{1-p^{-s}}\dot{\log}(p)p^{-s}\right)\end{aligned}$$

We see that the $|p^{-s}| < 1$, and knowing this we can rewrite the term $\frac{1}{1-p^{-s}}$ as a geometric series, which is

$$-\sum_p \left(\frac{1}{1-p^{-s}}\dot{\log}(p)p^{-s}\right) = -\sum_p \left[\log{(p)}p^{-s}\left(\sum_{k=0}^{\infty}(p^{-s})^k\right)\right]$$

We can put p^{-s} into the geometric series and redefine the bounds, as well as rearrange the sums to get

$$\begin{aligned}-\sum_p \left[\log{(p)}p^{-s}\left(\sum_{k=0}(p^{-s})^k\right)\right] &= -\sum_p \left[\log{(p)}\left(\sum_{k\geq 1}(p^{-s})^k\right)\right]\\ &= -\sum_p \sum_{k\geq 1}\frac{\log{(p)}}{(p^s)^k}\\ &= -\sum_p \sum_{k\geq 1}\frac{\log{(p)}}{p^{ks}}\end{aligned}$$

Looking at this expression, we notice that it is similar to the von Mangoldt function since the sum of the different primes are counted. Moreover, we can actually simplify the p^{ks} term since the von Mangoldt function counts prime powers. Thus the function simplifies to

$$-\sum_{p}\sum_{k\geq 1}\frac{\log(p)}{p^{ks}} = -\sum_{n}\frac{\Lambda(n)}{n^s}$$

$\square$

Knowing the expression from Theorem 3, we can use it to show the connection between the primes and the zeros of the zeta function. To start off, we integrate each side of (11) over the perimeter of a box, where we choose a location such that all integrals in that location do converge. We will denote the perimeter of the box as B. This gives us

$$\oint_B \frac{\zeta'(s)}{\zeta(s)}ds = -\oint_B \sum_n \frac{\Lambda(n)}{n^s}ds \tag{12}$$

We multiply both sides of (12) by $\frac{x^s}{s}$, where we make x an arbitrary parameter. After multiplying the expression we can rearrange it to get

$$\begin{aligned}\oint_B \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds &= -\oint_B \sum_n \frac{\Lambda(n)}{n^s}\frac{x^s}{s}ds \\ &= -\oint_B \sum_n \Lambda(n)\left(\frac{x}{n}\right)^s\frac{1}{s}ds \\ &= -\sum_n \Lambda(n)\oint_B \frac{(x/n)^s}{s}ds \end{aligned}\tag{13}$$

On the right hand side of (13), we can evaluate $\oint \frac{(x/n)^s}{s}ds$ to be 1 if $n < x$ and 0 if $n \geq x$ [3]. We will let x go to infinity, meaning that the integral will evaluate to 1. This also changes the bounds for our summation, letting us look at all $n \leq x$. Thus, our expression from (13) becomes

$$\oint_B \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = -\sum_n \Lambda(n)(1)$$

$$\oint_B \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = -\sum_n \Lambda(n) \tag{14}$$

The right hand side of this equation deals with the prime numbers. The left hand side will tell us about the zeros and poles of $\zeta(s)$ in the region, which we will show now. To simplify the left hand side, we will examine the residues of $\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}$, which are at the zeros and poles of $\zeta(s)$. From observation, we see that $\zeta(s)$ has a pole at $s = 1$ with residue 1. Since there are many zeros of $\zeta(s)$, we let ρ range over all of the zeros. The residue at each zero is $\frac{x^\rho}{\rho}$, with the proof of this in [1]. Combining both of these sums, we evaluate the left hand side as

$$\sum_{\rho:\zeta(\rho)=0} \frac{x^\rho}{\rho} - x = -\sum_{n \leq x} \Lambda(n) \tag{15}$$

Note that the x term came from the residue of the pole at $s = 1$. Multiplying both sides of (15) by -1 gives us

$$x - \sum_{\rho:\zeta(\rho)=0} \frac{x^\rho}{\rho} = \sum_{n \leq x} \Lambda(n) \tag{16}$$

With the expression in (16), we begin to see the connection between primes and the zeros of the zeta function. If the Riemann Hypothesis is true, then we know that $\mathrm{Re}(\rho) = \frac{1}{2}$ for all zeros of the zeta function. Then the left hand side of (16) is approximately $x + O(x^{\frac{1}{2}})$ [4]. On the right hand side, we can make use of Theorem 1 and our analysis above to say that the sum evaluates to $\sum_{n \leq x} 1$

due to prime powers not contributing a lot to $\Lambda(n)$. The Euler Product gave us an important connection with the logarithmic derivative of $\zeta(s)$, relating to it as a sum over primes via contour integration. After using complex analysis, we then get information about the location of the zeros related to the number of primes.

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