Geometric Group Theory and Hyperbolic Groups

By

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ABSTRACT

ALLEGRA DAWES Geometric Group Theory and Hyperbolic Groups.

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This paper provides a concise and accessible introduction to the field of geometric group theory and hyperbolic groups. Geometric group theory is a relatively new field of mathematics that seeks to understand the connections between the algebraic properties of groups and the geometric or topological properties of the spaces on which they act. It is a field with opportunities for algebraic, geometric, topological, and combinatorial analysis. We first provide the background necessary for a discussion of geometric group theory including group presentations, group actions, and Cayley graphs in section 2. Next, in section 3 we explore one of the geometric properties that some groups exhibit, that is hyperbolicity. We include a brief discussion of the hyperbolic plane to establish an understanding of hyperbolicity and then move into characterizing hyperbolic metric spaces, δ-hyperbolicity, and hyperbolic groups. Finally, we develop one of the combinatorial aspects of geometric group theory by examining the word problem in hyperbolic groups in section 4.

Contents

1. INTRODUCTION

1.1. **An Introduction to Geometric Group Theory.** Geometric group theory is a relatively new field of mathematics that seeks to understand the connections between the algebraic properties of groups and the geometric or topological properties of the spaces on which they act. Groups are commonly understood to encode the symmetries of a certain object. This is clearly exhibited by the dihedral groups of symmetries of regular polygons. The dihedral group includes rotations and reflections of geometric shapes. As we will see, all groups can be understood in this way. In some sense, the elements of a group can be understood as performing an action on a geometric object or space. Groups not only act on geometric objects; they can, in fact, be understood as geometric objects themselves. By realizing a group through its Cayley Graph and endowing it with a metric, we can view groups as metric spaces. Geometric group theory is then concerned with how groups relate to the spaces they act on. If a group acts on a space, what does it say about the group? What does it say about the space? How can we think about a group as metric space itself?

These are the broad questions that underpin geometric group theory. Geometric group theory grew out of combinatorial group theory which studies groups through their presentations. Group presentations allow us to equip groups with a metric and consider them as geometric objects directly. Understanding a group as a metric space gives rise to a variety of interesting geometric properties. The focus of this thesis is hyperbolic groups, that is, groups whose Cayley graphs can be understood as negatively curved in some sense. As we will see, the 'negative curvature' is exhibited through δ-hyperbolicity when a Cayley graph satisfies certain requirements in relation to some non-negative constant δ.

Section 2 of this paper will provide background on groups, spaces, and the word problem. Section 3 will briefly introduce hyperbolicity in the hyperbolic plane and then hyperbolic groups. Section 4 will explore the word problem in hyperbolic groups.

2. BACKGROUND

2.1. **Groups and Free Groups.** We will develop some of the background definitions and theorems to build the basis for geometric group theory. The ultimate goal of this section is to show that every group is identified with the symmetries of some geometric object. We first define a group.

Definition 2.1. A **group** is a set G with an operation $*$ that satisfies the following properties:

(1) Closure: For all $a, b \in G$, $a * b \in G$.
(2) Identity: There exists an element $e \in G$ such that for all $a \in G$, $e * a = a * e = a$.
(3) Inverse: For all $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$.
(4) Associativity: For all $a, b, c \in G$, $a * (b * c) = (a * b) * c$.

We will think of groups as encoding the symmetries of objects. Some groups, such as the dihedral group D_n or the symmetric group S_n, can easily be seen as groups of symmetries. Other groups are less obviously symmetries, but, as we will see, by considering a Cayley graph (introduced in section 2.4) these too encode the symmetries of some object.

Free groups are important in developing our understanding of group presentations that allow us to generate the Cayley graphs for a group. Before we define and explore free groups, here are some useful definitions for our future discussion.

Definition 2.2. Let S be a set of letters (we call S an **alphabet**) and let S^{-1} be the set of symbols $S^{-1} = \{s^{-1} | s \in S\}$. We call s^{-1} the **inverse** of s for each $s \in S$. Then a **word** in S is a string of letters in $S \cup S^{-1}$.

The empty word is the string containing no letters. We can multiply two words together by concatenating them. Consider the example below of words in $S = \{a, b\}$ to see how concatenation works.

$$(aab^{-1}, a^{-1}bb^{-1}) \mapsto aab^{-1}a^{-1}bb^{-1}$$

The above example shows how strings of letters in $S \cup S^{-1}$ can be combined to form longer words. However, concatenation alone is not the operation of the free group – we will also require that words be reduced.

Definition 2.3. We say that a word is **reduced** if for all $s \in S$ and $s^{-1} \in S^{-1}$, s and s^{-1} never appear next to each other in the word.

With the above operation we see that s and s^{-1} are indeed inverses in the sense that their concatenation reduces to the identity, which is the empty word. Thus we can reduce words by simply deleting the offending terms as they naturally cancel to the identity. As an example, we can reduce our concatenated word from above as follows:

$$aab^{-1}a^{-1}bb^{-1} \mapsto aab^{-1}a^{-1}.$$

In some words, like $aa^{-1}bbb^{-1}aab$, there are multiple places where we could begin the reduction. We could first delete aa^{-1} and then bb^{-1}, or vice-a-versa leading to two different reduction processes. As we will show later, every word in $a, a^{-1}, b,$ and b^{-1} can be reduced to a unique reduced word. We can now define the free group on some set S.

Definition 2.4. ([CM17b]) Let S be some set. Then the **free group** on S, denoted F_S, is defined as follows. The elements of F_S are all reduced words in S. The group multiplication is concatenation followed by reduction. If S is finite and $|S| = n$, then the **rank** of F_S is n.

As the free group is essential in understanding the manner in which we present groups, we will consider the following examples concerning the free group of rank 2. The free group of rank two is generated by the set $S = \{a, b\}$. We will confirm that F_2 is a group, but first we will prove a lemma that will be used to show that the group multiplication is associative.

Theorem 2.5. *Every word has a unique reduction.*

Proof. Take a word w for which there are choices for how to reduce it: $w \rightarrow_{\tau_1} w_1$ or $w \rightarrow_{\tau_2} w_1'$. These two choices fall into the following cases.

Case 1: Adjacent Reductions

In this case we have our word $w = ...baa^{-1}ab....$ Let one reduction delete aa^{-1} and the other delete $a^{-1}a$. Then we have $w_1 = ...bab...$ and $w_1^{'} = ...bab....$ So we find the the reductions result in the same word.

Case 2: Nonadjacent Reductions

In this case, we have non-adjacent subwords as in $w = baa^{-1}bab^{-1}ba...$ Let τ_1 delete aa^{-1} and τ_2 delete $b^{-1}b$. Then we have

$$w \rightarrow_{\tau_1} ...bbab^{-1}ba... \rightarrow_{\tau_2} ...bbaa... = w_0$$
$$w \rightarrow_{\tau_2} ...baa^{-1}baa... \rightarrow_{\tau_1} ...bbaa... = w_0$$

So we find a common reduction of w_1 and w_1'. Thus no matter which reduction (τ_1 or τ_2) we choose, we obtain the same word.

Hence w has a unique reduction.

□

We can now check the construction of the free group of rank 2.

Proposition 2.6. *The free group of rank 2 is a group.*

Proof. It is clear that the empty word (the string with no letters) is the identity. The inverse of a word is obtained by reversing the word and then replacing each a with a^{-1}, a^{-1} with a, b with b^{-1}, and b^{-1} with b. For example, the inverse of $bab^{-1}a^{-1}$ is $aba^{-1}b^{-1}$.

We will now show that the group multiplication is associative. Let w_1, w_2, and w_3 be three words. We will show that $(w_1w_2)w_3 = w_1(w_2w_3)$. We will write $w_1w_2 = w_{12}$ to signify that the concatenation of w_1w_2 is reduced to w_{12}. Then consider the two following reductions of $w_1w_2w_3$:

$$w_1w_2w_3 \rightarrow w_1(w_{23}) \rightarrow w_{1(23)}$$
$$w_1w_2w_3 \rightarrow (w_{12})w_3 \rightarrow w_{(12)3}.$$

By Lemma 1, we know that the two reductions result in the same reduce word, so $w_{1(23)} = w_{(12)3}$. Thus the group multiplication is associative. □

While we have focused on the free group of rank 2, the above examples can easily be applied to free groups of other ranks. Additionally, note that a free group can also be defined on alphabet sets of infinite rank. We will now see how free groups are used in group presentations.

2.2. **Group Presentations.** Group presentations allow us to fully describe a group without needing to list every element. Our ultimate goal is to show that every group can be given by a group presentation. In other words, we want to show that every group is a quotient of a free group. To prove this statement, we will first provide the necessary definitions to understand group presentations and their relationship to free groups. These definitions can be found in [CM17b], but are basic concepts in abstract algebra.

Definition 2.7. A **homomorphism** from a group G to a group H is a function $f : G \to H$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Definition 2.8. A **normal subgroup** of a group G is a group $N \subseteq G$ such that $gng^{-1} \in N$ for all $n \in N$ and for all $g \in G$.

Definition 2.9. Let G be a group and R be a subset of G. Then the **normal closure** of R in G is the smallest normal subgroup that contains R.

Definition 2.10. The **kernel** of a group homomorphism $f : G \to H$ is the set of $g \in G$ such that $f(g)$ is the identity in H.

Definition 2.11. Let N be a normal subgroup of G. Then we define the **quotient group** G/N as the set of all left cosets of N in G. In other words, $G/N = \{gN \mid g \in G\}$. The group multiplication is $(g_1N)(g_2N) = g_1g_2N$.

We can understand G/N as the set of equivalence classes of elements in G where two elements $g_1, g_2 \in G$ are equivalent if $g_1g_2^{-1} \in N$. Then it is clear that there is a homomorphism $G \to G/N$ that maps g to its equivalence class in G/N. The kernel of this homomorphism is N.

Group presentations allow us to describe a group without listing all of the elements (tedious at best and impossible in infinite groups). These presentations involve two parts: a set of generators, and a set of defining equalities such that the entire group multiplication table follows from these equalities. We can now define a group presentation.

Definition 2.12. A **group presentation** is a pair (S, R) where S is a set and R is a set of words in S, so R is a subset of the free group on S, F_S. Then a group G has a presentation $\langle S \mid R \rangle$ if G is isomorphic to the quotient of F_S by the normal closure of R. We write $G \cong \langle S \mid R \rangle$.

We can understand the group G as a group of reduced words in $S \cup S^{-1}$ where all words in R are equivalent to the empty word. Then the elements in S are the generators of G and the elements in R are defining relators. If we are given a defining relation such as $abba = bab$, then we can turn it into a relator by moving all the terms to one side, which in this example results in the relator $abbab^{-1}a^{-1}b^{-1}$.

Examples. To clarify group presentations, we will consider some simple examples. We will provide presentations for the groups F_2, D_n, and S_n.

(1) The Free Group on S (F_S): $F_S = \langle S | \emptyset \rangle$
(2) The Dihedral group of order $2n$ (D_n): $D_n = \langle \sigma, \tau | \sigma^n, \tau^2, (\sigma\tau)^2 \rangle$. Interpreting ρ as a rotation and τ as a reflection, D_n is the group of symmetries for a regular $n-$gon.

(3) The Symmetric group of order n (S_n): S_n is defined over a set of n elements and is the group that consists of permutation operations on the n elements. Our generator set is $S = \{\sigma_1, ..., \sigma_{n-1}\}$, where σ_i is the transposition that swaps the i^{th} and $(i+1)^{th}$ position. The relators are $\sigma_i^2 = 1$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $j \neq i \pm 1$, and $(\sigma_i\sigma_{i+1})^3 = 1$. So a presentation of S_n is

$$S_n = \langle \sigma_1, ..., \sigma_{n-1} | \sigma_i^2 = 1\,,\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } j \neq i \pm 1,\ (\sigma_i\sigma_{i+1})^3 = 1 \rangle.$$

For any given group we can always find a way to present it simply by listing all the elements in the group and every relation that is equivalent to the empty word. This is an incredibly inefficient presentation, but it is still important to recognize that every group has a group presentation. Additionally, given two different presentations, there is no known finite procedure to determine whether they represent the same group.

2.3. **Group Actions.** To understand groups as a collection of symmetries of some geometric object, we will introduce the concept of group actions. We will first define a group action and the orbit of an element.

Definition 2.13. [CM17a] An **action** of a group G on a set X is a function $G \times X \to X$ where the image of (g, x) is denoted $g \cdot x$ and the following properties are satisfied:

(1) For all $x \in X$, $1 \cdot x = x$, where 1 is the identity element in G, and
(2) For all $g, h \in G$ and $x \in X$, $g \cdot (h \cdot x) = (gh) \cdot x$.

Definition 2.14. Let G be a group on a set X. The **orbit** of $x \in X$ under G is $G \cdot x = \{g \cdot x \,|\, g \in G\}$.

If we are given a permutation group G on a set X, then the orbits of elements partition X into equivalence classes. In order to familiarize ourselves with group actions and orbits, we will include an example of orbits for a set X and a permutation group G.

Example. Let $X = \{1, 2, 3, 4, 5\}$ and $G = \{(id, (12), (34), (12)(34)\}$. The equivalence classes that G induces on X are as follows:

$$G \cdot 1 = G \cdot 2 = \{1, 2\}$$
$$G \cdot 3 = G \cdot 4 = \{3, 4\}$$
$$G \cdot 5 = \{5\}$$

Thus the orbits of each element of X partition X.

We can also see that a group G always acts on itself. There are at least two ways for G to act on itself. The first is through left multiplication $(g \cdot h = gh)$ and the second is by conjugation $(g \cdot h = ghg^{-1})$. The following proposition will confirm that these are in fact actions.

Proposition 2.15. *Left multiplication and conjugation of elements in G are actions of G on itself.*

Proof. Let 1 be the identity element in G. Then we know $1g = g$ for all $g \in G$. Since G is a group, its multiplication is associate, so we find that $g \cdot (h \cdot x) = g \cdot hx = ghx = (gh) \cdot x$. Thus left multiplication of elements in G is a group action on G.

Now we will consider conjugation. Clearly $1 \cdot g = g$ for all $g \in G$. Now let g, h, and x be elements in G. Then we have

$$\begin{aligned} g \cdot (h \cdot x) &= g \cdot (hxh^{-1}) \\ &= g(hxh^{-1})g^{-1} \\ &= (gh)x(h^{-1}g^{-1}) \\ &= (gh)x(gh)^{-1} \\ &= (gh) \cdot x. \end{aligned}$$

Hence conjugation of elements in G is a group action on G. $\square$

Definition 2.16. Let X be a set. Then we define the **group of permutations of** X to be the symmetric group of X, denoted S_X.

We can understand S_X as the set of symmetries on X. Then the group action of G on X can be understood as a homomorphism $G \to S_X$. If we consider the fact that $G \to S_G$ is injective then we have the following theorem.

Theorem 2.17. *Every group is isomorphic to a subgroup of some symmetric group.*

Group actions enable us to understand groups as collections of symmetries for some geometric object. The next section will introduce the two geometric objects that we will focus on.

2.4. **Cayley Graphs and Metric Spaces.** In this section we will establish that every group can be thought of as the symmetries of some geometric object. The first (and most important) of these objects is the Cayley graph of a group. We will first briefly define a graph and then examine how groups can be represented as graphs. We will then see how graphs can be understood as metric spaces themselves.

Definition 2.18. A **graph** is an ordered pair $\Gamma = (V, E)$ where V is a set of vertices and E is a set of edges. The endpoint function $f : E \to V \times V$ associates endpoints with edges between them.

One type of graph that will be of particular interest for our discussion of hyperbolic groups is the tree graph.

Definition 2.19. A **tree** is an undirected graph in which any two vertices are connected by exactly one path.

In other words, trees are acyclic, meaning they contain no cycles. We are particularly interested in infinite trees, that is, trees with an infinite number of vertices.

We can also discuss isomorphisms and automorphisms of graphs. An isomorphism between two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ is a pair of bijections, one from $V \to V'$, and one from $E \to E'$ that preserves the endpoint function. An automorphism on a graph is then an isomorphism from a graph to itself. We will denote the set of automorphisms of a graph Γ by $\mathrm{Aut}(\Gamma)$. This set can be thought of as the collection of symmetries of Γ. We can now define a group action on a graph.

Definition 2.20. An action of a group G on a graph $\Gamma = (V, E)$ is a pair of group actions on V and E respectively that preserve the endpoint function. So we have:

(1) For all x in either V or E, $1 \cdot x = x$.

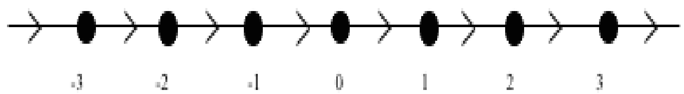

FIGURE 2.1.

(2) For all $g, h \in G$ and x in V or E, $g \cdot (h \cdot x) = (gh) \cdot x$.

To say that the group action must preserve the endpoint function means that for every $e \in E$ with endpoints v and w, the endpoints of $g \cdot e$ are $g \cdot v$ and $g \cdot w$.

We can now define the Cayley graph of a group G as follows.

Definition 2.21. Let G be a group and S be a generating set for G. Then the **Cayley graph** of G with respect to S is a directed, labeled graph $\Gamma = (V, E)$ where $V = G$ and for all $g \in G$ and $s \in S$ there is a directed edge from g to gs. We label the edge from g to gs with $s \in G$.

As an illustrative example of Cayley graphs, we will consider the group $(\mathbb{Z}, +)$, that is, the integers under the operation of addition. We will first consider the generating set $S = \{1\}$. Then the Cayley graph is simply a vertex for every integer and a directed line between each pair of adjacent points. Figure 2.1 shows the Cayley graph for $(\mathbb{Z}, +)$ and the generating set $\{1\}$.

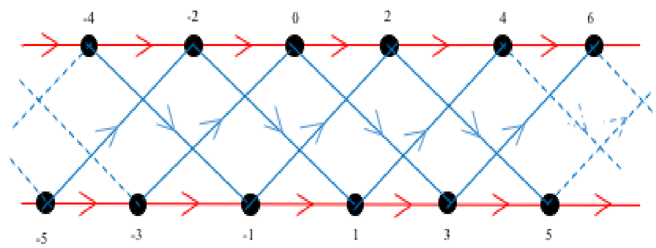

Figure 2.2.

We know that $\mathbb{Z}$ is also generated by the set $S' = \{2, 3\}$. How does this change the Cayley graph? Again the integers are the vertices of the graph. The generator 2 connects any two vertices that differ by 2 while the generator 3 connects any two vertices that differ by 3. Figure 2.2 shows the Cayley graph for the integers and the generating set S'. The red edges correspond to the generator 2 and the blue edges correspond to the generator 3.

As previously mentioned, the group G acts on itself by left multiplication. We can now view this as an action of G on its Cayley graph. We can view all groups as encoding the symmetries of an object; the Cayley graph is just this object.

A metric space is quite simply a set that is equipped with a metric function (or distance function) that measures the distance between any two points in the set. We will first make this description more precise with the following definition.

Definition 2.22. A **metric space** is a a set X and a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

(1) For all $x, y \in X$, $d(x, y) \geq 0$.

(2) For all $x, y \in X$, $d(x, y) = 0 \iff x = y$.

(3) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(4) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

There are many familiar metrics including the Euclidean metric on $\mathbb{R}^n$. A particularly useful metric is the path metric on graphs as we understand all groups as the collection of symmetries of their Cayley graph. The path metric on a group's Cayley graph enables us to view a group as a metric space itself. We define the path metric as follows.

Definition 2.23. Let Γ be a connected graph (there is an edge between any two vertices). Then the **word metric** or **path metric on** Γ is the metric that defines the distance between two vertices as the number of edges in the shortest edge path (sequence of edges) connecting them. The metric ignores the direction of the edges in order to satisfy the symmetry requirement of metrics.

As the path metric is essential for our ability to view groups as geometric objects, let us prove that it satisfies all four properties of a metric as listed above.

Theorem 2.24. *The path metric satisfies the four properties of a metric.*

Proof. Let Γ be a connected graph in which each edge between any two vertices has length 1; then it is clear that the length of an edge path between any two vertices x and y must be non-negative. In other words, $d(x, y) \geq 0$. Now assume that $d(x, y) = 0$. Since Γ is connected there must be an edge path between the vertices x and y. The shortest possible nontrivial path consists of one edge and has a length of 1. Since $d(x, y) = 0$ and Γ is connected, we see that $x = y$. And clearly if $x = y$, then $d(x, y) = 0$. Symmetry is satisfied by the path metric since, in the definition of the path metric, we ignore the directions of the edges.

□

The path metric can clearly be applied to Cayley graphs, and when discussing the path metric on a Cayley graph we usually call it the word metric. Given a group G with a finite set of generators S, the word metric (applied to its corresponding Cayley graph) thus defines the distance between $g, g' \in G$ to be the minimum length of a path among all paths connecting these vertices, where we consider each edge of the graph to have length 1. The word metric derives its name from the fact that the distance between g and g' is the same as the length of the shortest word w in letters from S such that $gw = g'$. Put another way, the distance between $g, h \in G$ is the word length of $h^{-1}g$ with respect to the generating set S.

Thus, the word metric shows us that a group with a generating set is itself a metric space. It is natural, then, to ask about the geometry of these groups.

As we will later discuss, some groups can be seen as negatively curved or, in other words, hyperbolic.

2.5. **The Word Problem.** Before we move on to hyperbolic groups we will introduce Dehn's word problem. This is an important area of study in combinatorial group theory. The word problem for groups concerns our ability to determine from some presentation whether two words represent the same element of a group. Equivalently, the word problem concerns our ability to determine whether an element is the identity of the group or not.

We will now formalize the relationship between the word problem and group presentations. Let S be an alphabet set, let R be a set of words on $S \cup S^{-1}$, and suppose that G is a group that is presented by $\langle S \,|\, R \rangle$. Then a word on $S \cup S^{-1}$ represents the identity when it can be converted to the empty word via a null sequence.

Definition 2.25. A **null sequence** for a word w is a finite series of reductions ($uss^{-1}v \to uv$), expansions ($uv \to uss^{-1}v$), and applications of defining relators ($wuw' \to wvw'$, where uv^{-1} or vu^{-1} is a permutation of an element of R), that convert w to the empty word.

Solving the word problem requires finding an algorithm that, given some word, can determine whether or not the word represents the empty word.

Example. As an example of how trivial words can be non-obvious, we will consider the Dihedral group D_4. We have the following presentation $D_4 = \langle \sigma \,, \tau \,|\, \sigma^4, \tau^2, (\sigma\tau)^2 \rangle$. Then consider the word $w = \sigma^5\tau\sigma\tau$. At first glance, w does not appear to be the trivial word, yet the following null sequence shows that it in fact is:

$$w = \sigma^5\tau\sigma\tau = \sigma^4\sigma\tau\sigma\tau = \sigma^4(\sigma\tau)^2 = 1.$$

Thus we see that w is the trivial word of D_4.

We are not directly concerned with solving the word problem, but as we will discuss later, some specific types of groups ensure that the word problem always has a solution.

3. HYPERBOLICITY AND HYPERBOLIC GROUPS

As previously mentioned, by equipping a Cayley graph with the word metric we are able to consider a group itself as a metric space. The overarching idea of geometric group theory that finitely generated groups can be thought of as geometric objects through the study of their Cayley graphs gives rise to a number of large scale geometric phenomena. Hyperbolic groups are a class of groups with Cayley graphs that are, in some sense, negatively curved.

There are three main curvature classifications: positive, flat, and negative. We think of the sphere as exhibiting positive curvature, the Euclidean plane as a flat surface, and the saddle surface as exhibiting negative curvature. Our study of hyperbolic groups allows us to make sense of curvature using the distance function of the space.

In the following section we will examine the hyperbolic plane to attain an understanding of the geometry and negative curvature that is meant by hyperbolicity. Next, we will provide the definitions of hyperbolicity that are applied to groups.

3.1. **The Hyperbolic Plane.** Unsurprisingly, hyperbolic groups are so named because they exhibit the behavior that is typically associated with hyperbolic space. There are many models for hyperbolic space, but we will focus on the hyperbolic plane, denoted $\mathbb{H}^2$, which is obtained by putting a non-Euclidean metric on the upper half plane $\mathbb{U}^2$. Every point in the hyperbolic plane exhibits saddle point behavior. Thus hyperbolicity captures the idea of the negative curvature of surfaces. The following definition is of use throughout this section and the following sections of Chapter 3.

Definition 3.1. A **geodesic** in a metric space is a path of minimal length between two points.

Thus geodesics in Euclidean space are straight lines. We will see that in $\mathbb{H}^2$ geodesics are quite different.

3.1.1. *The Upper Half Plane Model.* The upper half plane in $\mathbb{R}^2$ is simply defined as the set

$$\mathbb{U}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}.$$

We know that in $\mathbb{R}^2$ equipped with the standard geometry, the lengths of parameterized curves can be computed using the Euclidean metric

$$ds_{\mathbb{E}}^2 = dx^2 + dy^2$$

Given a curve $f(t) = (x(t), y(t))$ and $t_1 = \alpha$, $t_2 = \beta$ we find that the arc length is given by the integral

$$\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

It follows that in the Euclidean plane, geodesics are given by straight line segments.

In the upper half plane model of hyperbolic space we instead use the Riemannian metric defined by

$$ds_{\mathbb{H}}^2 = \frac{dx^2+dy^2}{y^2} = \frac{ds_{\mathbb{E}}^2}{y^2}.$$

Definition 3.2. We define the **hyperbolic plane**, $\mathbb{H}^2$, as the upper half plane $\mathbb{U}^2$ equipped with the Riemannian metric $ds^2_{\mathbb{H}}$.

Similar to before, we can find the length of a parameterized curve $f(t) = (x(t), y(t))$ in $\mathbb{H}^2$ by computing the integral

$$\int_{\alpha}^{\beta} \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Due to the presence of $y(t)$ in the denominator, a horizontal line drawn close to the x-axis in $\mathbb{H}^2$ will have a larger length than the Euclidean length, while a horizontal line far above the x-axis in $\mathbb{H}^2$ will have a shorter length than the Euclidean length.

Thus, the geometry of the hyperbolic plane is very different from that of Euclidean space. In particular, geodesics in $\mathbb{H}^2$ are not simply straight lines connecting points, since along a horizontal line does not result in the shortest distance between points. Simply put, distances are smaller when they are farther from the x-axis, thus, in the hyperbolic plane, an arc between two points is shorter than a straight line. In fact, geodesics in $\mathbb{H}^2$ are vertical lines and arcs of semicircles that are orthogonal to the x-axis.

Definition 3.3. A **geodesic triangle** consists of three points that are connected by geodesic segments.

Figure 3.1 shows a geodesic triangle between three points that are very close to the x-axis.

The key property of triangles in hyperbolic spaces that will be used to study hyperbolic groups is the idea of δ-thinness. The following theorem formalizes this property.

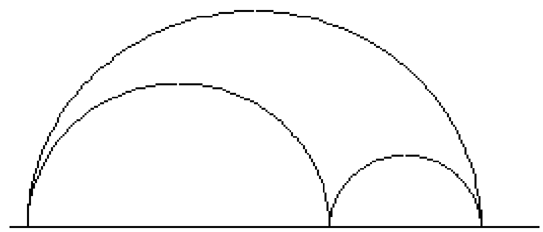

FIGURE 3.1.

Theorem 3.4. [Ger99] *There is a number $\delta > 0$ so that for all geodesic triangles in $\mathbb{H}^2$ with vertices A, B, and C and all points P on side AB, there exists a point Q on at least one of the sides AC or CB so that the distance $d_{\mathbb{H}}(P, Q) \leq \delta$.*

Gersten provides a proof of this property, but we will not include it here. As we will see, this thin triangle property leads directly to our discussion of δ-hyperbolic metric spaces and thus of hyperbolic groups. Hyperbolic groups can be studied as metric spaces. They are groups that to some degree resemble the hyperbolic plane.

3.2. **Definitions of δ-hyperbolicity.** Before unpacking the definitions of δ-hyperbolicity we will define a geodesic metric space. First we must consider a way to compare metric spaces using the following definition.

Definition 3.5 ([MT17] Def.)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Then a function $f : X \to Y$ is an **isometric embedding** if f preserves distances, that is, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

This definition is weaker than that of an isometry. An isometric embedding is only guaranteed to be injective. If it is also surjective then it is an isometry. We can now define a geodesic metric space.

Definition 3.6. [Ger99] A metric space (X, d) is called a **geodesic metric space** if for all pairs of points x_1 and x_2 in X there is an isometric embedding $f : [0, d(x_1, x_2)] \to X$ taking the endpoints of the interval to x_1 and x_2.

The image of f is clearly a geodesic in X between these two points. Thus, a metric space is geodesic if for all pairs $x_1, x_2 \in X$ there is a geodesic between them.

We are now ready to discuss δ-hyperbolicity. We will explore three equivalent definitions of δ-hyperbolicity, all of which can be used to characterize hyperbolic groups.

3.2.1. *Definition 1: δ-thin triangles.* Our first definition of δ-hyperbolicity is that geodesic triangles are δ-thin. A geodesic triangle consists of three points pairwise connected by geodesic segments, say α, β, and γ. We write $\mathcal{N}_\delta(x)$ to mean the δ neighborhood around x.

Definition 3.7. Consider a geodesic triangle consisting of three points connected by geodesic segments α, β, and γ. Then this triangle is **δ-thin** if the following inclusions hold: $\gamma \subset \mathcal{N}_\delta(\alpha \cup \beta)$, $\alpha \subset \mathcal{N}_\delta(\gamma \cup \beta)$, and $\beta \subset \mathcal{N}_\delta(\alpha \cup \gamma)$.

Thus, a (geodesic) triangle is δ-thin if each geodesic segment is contained in the δ-neighborhood of the union of the other two segments.

Figure 3.2 shows a δ-thin geodesic triangle. The relationship to triangles in the hyperbolic plane is clear, since by Theorem 3.4, triangles in $\mathbb{H}^2$ also satisfy

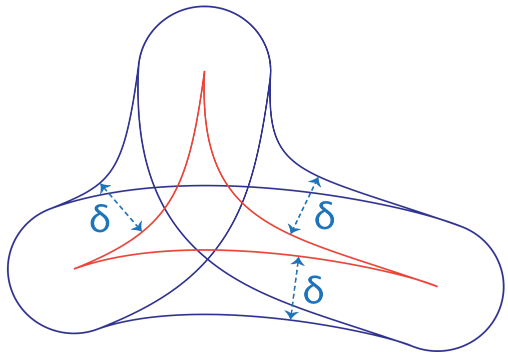

Figure 3.2.

this thinness condition. This leads to our first definition of a δ-hyperbolic space.

Definition 3.8. A geodesic metric space (X, d) is **δ-hyperbolic** if every geodesic triangle is δ-thin.

3.2.2. *Example and Non-Example of Thin Triangles.* Tree graphs (graphs with no cycles) exhibit an extreme form of hyperbolicity. In a tree, any triangle consisting of three points and the geodesic paths between these points has one path is completely contained in the union of the other two paths.

This means that every triangle in a tree has some vertex that is common in all three sides. It follows that trees are δ-hyperbolic with $\delta = 0$. Figure 3.3 illustrates this property of triangles in trees.

We will now consider a non-example of a space that is not hyperbolic and thus fails the thin triangle requirement. Consider the two dimensional space of the surface of a sphere. Geodesics on a sphere are great circles, so a triangle consists of three points on the sphere connected by segments of great circles. It is easy to see that a triangle on a sphere is "fat"– there does not exist a

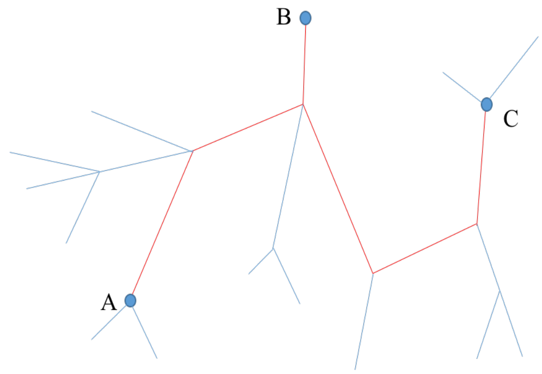

FIGURE 3.3. A triangle (in red) in which $\overline{AC} \subset \overline{AB} \cup \overline{BC}$

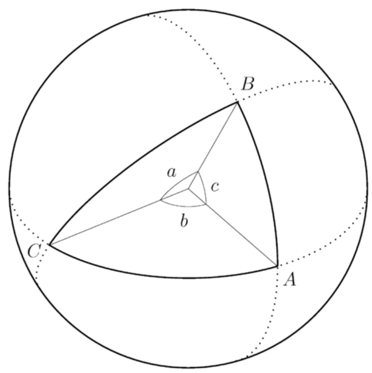

FIGURE 3.4. A "fat" triangle on the surface of a sphere.

single δ such that, for all triangles on the sphere, each side is contained within a δ neighborhood of the union of the other two side. Figure 3.4 shows such a fat triangle on the surface of a sphere.

3.2.3. *Definition 2: The Gromov Product and four point condition.* We can also define δ-hyperbolicity using the Gromov product. This product is essentially a measure of the gap in the triangle inequality that prevents it from being an equality. Recall that the triangle inequality for points x, y, and z is

$$d(x, y) + d(y, z) \geq d(x, z).$$

By subtracting $d(x, z)$ we have $d(x, y) + d(y, z) - d(x, z) \geq 0$. This inequality is reflective of the gap between equality. The Gromov product is simply half of this gap.

Definition 3.9. [Duc17] Let (X, d) be a metric space and $x, y, z \in X$. Then the **Gromov product** is

$$(x \cdot y)_z := \tfrac{1}{2}(d(x, z) + d(y, z) - d(x, y)).$$

The second definition of δ-hyperbolicity using the Gromov product is as follows.

Definition 3.10. Let (X, d) be a metric space. Then X is a **δ-hyperbolic space** if for all points x, y, z, and w in X

$$(x \cdot y)_w \geq \min((y \cdot z)_w, (x \cdot z)_w) - \delta.$$

How should we interpret this definition of δ-hyperbolicity? Kapovich and Benakli write that the Gromov product in δ-hyperbolic metric spaces measure how long two geodesics travel close to one another [KB02].

We can first illustrate this concept by again looking at the tree graphs – one of the clearest examples of a δ-hyperbolic metric space. First note that in a tree, the Gromov product $(x, y)_z$ is the distance from z to the geodesic segment $[x, y]$. Consider a tree in which points x, y, and z form a triangle and w is a fourth point. Since there are no cycles in a tree, there is only one path from w to some point on the geodesic of the triangle (a loop would be formed if there were more than one path to the same geodesic). Then we know from section 3.1.2 that x lies on at least two of the triangle sides. Thus there is always a tie between at least two geodesics for the minimum distance between w and the segments of the triangle. So for a tree we have

$$(x \cdot y)_w \geq \min((y \cdot z)_w, (x \cdot z)_w).$$

Like the δ-thin triangles condition above, the Gromov product version of δ-hyperbolicity in trees can be seen as the case where $\delta = 0$.

3.2.4. *Example.* As an illustrative example of the Gromov product's use in examining δ-hyperbolicity we will look at ultra-metric spaces. The following definitions are needed for this example.

Definition 3.11. A space is called **ultrametric** if all triples satisfy

$$d(x, y) \leq \max(d(y, z), d(x, z)).$$

Theorem 3.12. *Every triangle in an ultrametric space is isosceles.*

Proof. Let (X, d) be an ultrametric space. Let x, y, and z be points in X. Without loss of generality, assume that $d(x, z) \leq d(y, z)$ and $d(x, y) < \max(d(x, z), d(y, z))$. Then if $d(x, z) < d(y, z)$ we have $d(z, y) > \max(d(x, y), d(x, z))$. This contradicts the fact that (X, d) is an ultrametric space. Thus we find that $d(x, z) = d(y, z)$ so the triangle formed by x, y, and z is isosceles. $\square$

It follows from this property that ultrametric spaces are 0-hyperbolic. For instance, the p-adic numbers $\mathbb{Q}_p$, which are of particular interest in number theory, form a complete ultrametric space, and so the Gromov product can be used to show the hyperbolicity of the p-adic numbers.

3.2.5. *Definition 3: Insize.* Our final definition of δ-hyperbolicity concerns the insize of a triangle. The inpoints of a triangle are the three points that divide the sides of a triangle into three pairs of equal length. Figure 3.5 shows a triangle with its insize points marked.

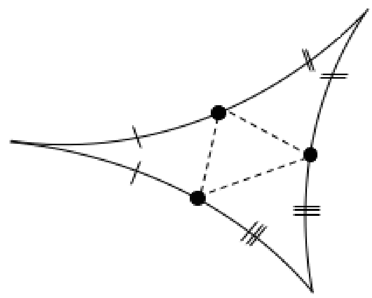

FIGURE 3.5.

Definition 3.13. The **insize** of a geodesic triangle is the largest of the three pairwise distances between the inpoints. We call a space **δ-hyperbolic** if the insize of any triangle in that space is less than or equal to δ.

Once again, the Euclidean plane $\mathbb{R}^2$ provides a clear example of a space that does *not* satisfy any of these three notions δ-hyperbolicity, since any triangle in $\mathbb{R}^2$ can be dilated with a constant multiplication. Thus for any value of δ, we can dilate both the insize and the thinness of the triangle without bound. Hence $\mathbb{R}^2$ is not δ-hyperbolic.

3.3. **The Invariance of Hyperbolicity.** We have now considered three different definitions of δ-hyperbolicity, but what does it mean for a space (or group) to be hyperbolic?

Definition 3.14. A space is **hyperbolic** if it is δ-hyperbolic for any of the three definitions of δ-hyperbolicity for any $\delta \geq 0$.

Thus, a space that satisfies one of the requirements of hyperbolicity satisfies all of the three definitions. However, it is not guaranteed that the same value of δ will work in all three cases. In fact the value of δ may change when one considers the same metric space but switches between definitions. In other words, if a space is hyperbolic for some value of δ_1 under one definition, it will be hyperbolic for some value δ_2 that may or may not be equal to δ_1, under a different definition.

While we will not go into a discussion of quasi-isometries here, note that there are isometry like functions that, given any two Cayley graphs of a group G, there is a quasi-isometry between them. As the Cayley graphs arise from presentations, we can select any two presentations and find a quasi-isometry between them. By the properties of these isometry-like functions, this guarantees that if one graph satisfies a definition of δ-hyperbolicity, then there is some δ' such that the other graph is δ'-hyperbolic. Thus, the hyperbolicity of a group is not dependant on the set of finite generators that is selected for the Cayley graph. We can now define a hyperbolic group.

Definition 3.15. A group is hyperbolic if any of its Cayley graphs generated by any finite set is δ-hyperbolic for some $\delta \geq 0$.

Thus the property of hyperbolicity carries over from one finite set of generators to another with different values of δ.

4. HYPERBOLIC GROUPS: THE WORD PROBLEM

In section 2.5 we introduced the word problem. We will briefly recall the word problem and then consider the word problem in hyperbolic groups.

Let G be a group with a presentation $\langle S|R\rangle$ where S is an alphabet and R is a set of words on $S \cup S^{-1}$. Then the word problem is the matter of determining whether or not two words represent the same group element or, equivalently, whether or not a word represents the group identity element. Determining whether or not a group has a solvable word problem depends on if the group possesses and algorithm to solve the problem.

In some groups the word problem is easily solved. For example in free abelian groups one can choose different generators and see is one is canceled. Other groups have undecidable word problems meaning there is no algorithm to determine in finite time whether or not a word is in fact the identity element. What is of interest to us is that every hyperbolic group has a solvable word problem. In fact, in a hyperbolic group, if we are given a word of length n, then we can determine whether or not it is the identity element in linear time with respect to n.

4.1. **Dehn Presentations.** Hyperbolic groups have solvable word problems because their geometry ensures the existence of a Dehn presentation that satisfies certain properties that lead to a fast algorithm.

Definition 4.1. Let G be a group with presentation $\langle a_1, ..., a_n \,|\, r_1, ..., r_m \rangle$. This is a **Dehn presentation** of G if it satisfies the following conditions.

(1) There is a set of strings $u_1, v_1, ..., u_m, v_m$ and each relator is of the form $r_i = u_i v_i^{-1}$.
(2) For each i, the word length of v_i is shorter than the word length of u_i.
(3) For any nonempty string w in $S = \{a_i\}$ that represents the identity element, if w has be reduced by canceling all occurrences of $a_i a_i^{-1}$, then at least one of the u_i or u_i^{-1} must appear as a substring.

First note from (1) that the set of relators encode the equivalence in the group of u_i and v_i. The third condition ensures that any string can be simplified using one of finitely many moves, each of which make the string strictly shorter. In a general presentation this is not guaranteed. A long word of generators does not necessarily have a clear simplification as you may have to use a relation that increases the length of the word before using another relation to shorten it. Thus the conditions that Dehn presentations satisfy ensure the existence of an algorithm that runs quickly.

Thus Dehn presentations ensure that the algorithm to solve the word problem exists and runs quickly. We will now examine how the geometry of hyperbolicity ensures the existence of a Dehn presentation for any hyperbolic group. Our goal is to prove the following theorem, which we learned from [Duc17, Theorem 7.1].

Theorem 4.2. *Hyperbolic groups admit Dehn presentations.*

Before proving this theorem we will introduce a key definition and prove a lemma to be used in the final proof.

Definition 4.3. A path γ is an **m-local geodesic** if every subpath of length m is a geodesic.

We will now prove the following lemma.

Lemma 4.4. *There do not exist 4δ-local geodesic loops of length $> 4\delta$ in any δ-hyperbolic space.*

Proof. We will show that any k-local geodesic segment must stay within 2δ of any true geodesic between its endpoints. So let γ be a k-local geodesic loop of length $> k$. Choose $p = \gamma(t)$ to be the farthest point on γ from the base-point $x = \gamma(0)$. Then we can find a segment of γ of length k with midpoint p, left endpoint p', and right endpoint p''. So we have $d(p', p) = \frac{k}{2}$ and $d(p, p'') = \frac{k}{2}$.

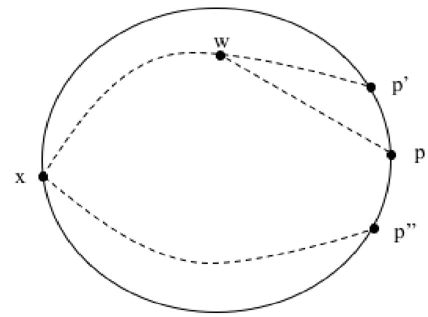

FIGURE 4.1.

Figure 4.1 illustrates this, where the dotted lines from x to p' and from x to p'' are geodesics.

Consider the geodesic triangle from x to p' to p''. By the insize property of triangles in hyperbolic space, we know that p must be within δ of some point, say w, on the geodesic segment from x to p'. First note that since p is the farthest point from x on γ we know that $d(p', x) \leq d(p, x)$. It follows that $d(p', w) \leq d(p, w)$. This is clear since if $d(p', w) > d(p, w)$, we would have a path from x through w to p that is shorter than the geodesic from x to p', contradicting our choice of p as the farthest point from x.

Now, we know that $d(p, w) \leq \delta$ and $d(p', w) \leq d(p, w)$, so it follows that $d(p', w) \leq \delta$. Using the triangle inequality we have the following

$$d(p, p') \leq d(p', w) + d(w, p) \leq 2\delta.$$

Since $d(p, p') = \frac{k}{2}$, we find that $k \leq 4\delta$.

$\square$

We are now ready to prove Theorem 4.2 using the proof provided by Duchin. Theorem 4.2 guarantees that since hyperbolic groups always admit Dehn presentations, they always have solvable word problems.

Proof of Theorem 4.2. We will show that any given hyperbolic group admits a Dehn presentation, that is, a presentation satisfying the three conditions in Definition 4.1. First fix some $K > 4\delta$. Let G be a δ-hyperbolic group and consider a finite generating set $S = \{a_i\}$ for G. Now form a list of all reduced words t_i with word length at most K, that is $|t_i| \leq K$ for all i. There are finitely many of these words.

We were given G by either a Cayley graph or a presentation, so we can build a finite amount of the Cayley graph in finite time. This allows us to identify and list which t_i represent the same word by simply following them in the graph. Let u_i be the non-geodesic words from this list, and for each u_i, let v_i be some geodesic word reaching the same point in the Cayley graph. Thus v_i is guaranteed to be strictly shorter than u_i. Let $R = \{r_i | r_i = u_i v_i^{-1}\}$. We claim that $G = \langle S | R \rangle$ is a Dehn presentation.

It is clear that the first two requirements of a Dehn presentation are satisfied by construction. For the last requirement, we know from Lemma 4.4 that in δ-hyperbolic space, there are no 4δ-local geodesic loops of length at least 4δ. So any loop of a length greater than or equal to 4δ has a sub-segment of length $\leq K$ that is non-geodesic. This sub-segment is one of our u_i. Thus the third requirement is satisfied and $G = \langle S | R \rangle$ is a Dehn presentation. $\square$

Dehn presentations ensure that there is an algorithm that solves the word problem in finite time. In fact, Dehn presentations give rise to the Dehn algorithm. The algorithm is as follows. We begin with a word that may or may not be the empty word. We then look for any u_i subwords. If we find any u_i subwords, then we can replace them with the corresponding v_i. This makes our word strictly shorter. If there are no u_i and no trivial cancellations of $a_i a_i^{-1}$ then the algorithm stops. At this point if there are still letters, then the original word was not the empty word in the group. Thus, hyperbolic groups have solvable word problems.

While we will not prove it here, it is important to note the following theorem.

Theorem 4.5. *A group is hyperbolic if and only if it has a Dehn presentation.*

Dehn presentations and the following algorithm that solves the word problem in linear time are characteristic of hyperbolic groups.

References

[CM17a] Matt Clay and Dan Margalit, ...and spaces, Office hours with a geometric group theorist, Princeton Univ. Press, Princeton, NJ, 2017, pp. 21–41. MR 3587215

[CM17b] ———, Groups, Office hours with a geometric group theorist, Princeton Univ. Press, Princeton, NJ, 2017, pp. 3–20. MR 3587214

[Duc17] Moon Duchin, Hyperbolic groups, Office hours with a geometric group theorist, Princeton Univ. Press, Princeton, NJ, 2017, pp. 176–202. MR 3587222

[Ger99] S. M. Gersten, Introduction to hyperbolic and automatic groups, Summer School in Group Theory in Banff, 1996, CRM Proc. Lecture Notes, vol. 17, Amer. Math. Soc., Providence, RI, 1999, pp. 45–70. MR 1653684

[KB02] Ilya Kapovich and Nadia Benakli, Boundaries of hyperbolic groups, Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math., vol. 296, Amer. Math. Soc., Providence, RI, 2002, pp. 39–93. MR 1921706

[MT17] Dan Margalit and Anne Thomas, Quasi-isometries, Office hours with a geometric group theorist, Princeton Univ. Press, Princeton, NJ, 2017, pp. 125–145. MR 3587220