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# Obstruction criteria for modular deformation problems

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For a cuspidal newform $f = \sum a_n q^n$ of weight $k \geq 3$ and a prime $\lambda$ of $\mathbf{Q}(a_n)$, the deformation problem for its associated mod $\lambda$ Galois representation is unobstructed for all primes outside some finite set. Previous results gave an explicit bound on this finite set for $f$ of squarefree level; we modify this bound and remove the squarefree hypothesis. We also show that if the $\lambda$-adic deformation problem for $f$ is unobstructed, then $f$ is not congruent mod $\lambda$ to a newform of lower level.

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## 1. Introduction

Let $f = \sum a_n q^n$ be a cuspidal newform of weight $k \geq 2$ and level $\Gamma_1(N)$. Write $K = \mathbf{Q}(a_n)$ for the number field generated by its Fourier coefficients. Let $\lambda$ be a prime of $K$, and let $\ell$ be the characteristic of its residue field $k_\lambda$. For any finite set $S$ of places which contains the primes dividing $N\infty$, let $\mathbf{Q}_{S\cup\{\ell\}}$ be the maximal extension of $\mathbf{Q}$ unramified outside $S \cup \{\ell\}$, and let $G_{\mathbf{Q},S\cup\{\ell\}}$ be its Galois group over $\mathbf{Q}$. By work of Deligne, there is an associated semisimple residual Galois representation

$$\bar{\rho}_{f,\lambda} : G_{\mathbf{Q},S\cup\{\ell\}} \to \mathrm{GL}_2(k_\lambda)$$

and this representation is absolutely irreducible for almost all primes $\lambda$.

Given such a representation $\bar{\rho}$, it is interesting to study its lifts to other coefficient rings. If $A$ is a local ring with residue field $k_\lambda$, we say $\rho$ is a *lift* of $\bar{\rho}$ if the following diagram commutes:

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\rho} & \mathrm{GL}_2(A) \\ & \searrow^{\bar{\rho}_{f,\lambda}} & \downarrow \\ & & \mathrm{GL}_2(k_\lambda) \end{array}$$

The vertical arrow is induced by the reduction map $A \to k_\lambda$; we consider two lifts equivalent if they are conjugate to one another by a matrix in the kernel of this induced map. An equivalence class of lifts is called a *deformation* of $\bar{\rho}$.

The study of the deformation theory of such Galois representations, which began with Mazur's seminal paper [5], has been the subject of much important research in number theory; in particular, it featured prominently in the proof of the Taniyama–Shimura conjecture, and more recently, in the proof of Serre's Conjecture. See Sec. 2 for a brief introduction to deformation theory and the terms used below.

In a pair of papers [7, 8], Weston proved that for any newform $f$ of weight $k \geq 2$, the deformation problem for $\bar{\rho}_{f,\lambda}$ is unobstructed for infinitely many primes $\lambda$, and when the level of $f$ is squarefree, he gave an explicit description of the obstructed primes. In fact, when $k \geq 3$, there are only finitely many obstructed primes, while for $k = 2$ the obstructed primes are a set of density zero. The first main result of this paper is the removal of the squarefree hypothesis from Weston's result; only a minor modification of the bound given in [8] is necessary. See Theorem 3.6 in Sec. 3 for the full statement.

While Theorem 3.6 gives sufficient conditions for a deformation problem to be unobstructed, the second main result of this paper focuses on a necessary condition. For any modular Galois representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_\ell)$, there is an optimal (least) level $N$ coprime to $\ell$ such that $\bar{\rho}$ arises from a newform of level $N$. Call a deformation problem *minimal* if the set $S$ of primes (as in the first paragraph) contains only those places dividing $N\infty$. We show in Theorem 4.1 that minimal deformation problems are only unobstructed when they arise from modular forms of optimal level. This is analogous to a similar phenomenon which occurs in Hida Hecke algebras.

**Notation.** We fix an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$, and for each rational prime $\ell$, we fix an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_\ell$. Let $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and let $G_\ell = \mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$. Whenever $S$ is a finite set of primes, $G_{\mathbf{Q},S}$ denotes the Galois group (over $\mathbf{Q}$) of the maximal extension of $\mathbf{Q}$ which is unramified outside of $S$.

We write $\epsilon_\ell$ for the $\ell$-adic cyclotomic character. For any character $\psi$ we denote its reduction mod $\lambda$ by $\bar{\psi}$, where $\lambda$ is made clear in context.

If $\rho : G \to V$ is a representation, the adjoint representation $\mathrm{ad}\,\rho : G \to \mathrm{End}(V)$ is defined by letting $g \in G$ act on $\mathrm{End}(V)$ via conjugation by $\rho(g)$; we write $\mathrm{ad}^0\rho$ for the trace-zero component of the adjoint.

## 2. Deformation Theory

Consider an odd, continuous Galois representation $\bar{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{F})$, where $\mathbf{F}$ is some finite field and $S$ is a finite set of primes containing the characteristic of $\mathbf{F}$ and the infinite place. Let $\mathcal{C}$ be the category whose objects are local rings which are inverse limits of artinian local rings with residue field $\mathbf{F}$, and whose morphisms $A \to B$ are continuous local homomorphisms inducing the identity map on residue fields. As explained in the introduction, if $A \in \mathcal{C}$, then we say $\rho : G_{\mathbf{Q},S} \to \mathrm{GL}_2(A)$

is a lift of $\bar{\rho}$ if the composition

$$G_{\mathbf{Q},S} \xrightarrow{\rho} \mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbf{F})$$

is equal to $\bar{\rho}$. Two lifts $\rho_1, \rho_2$ of $\bar{\rho}$ to $A$ are considered equivalent if they are conjugate to one another by a matrix in the kernel of the map $\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbf{F})$, and a *deformation* of $\bar{\rho}$ to $A$ is an equivalence class of lifts of $\bar{\rho}$ to $A$. There is an associated *deformation functor*

$$D^S_{\bar{\rho}} : \mathcal{C} \to \mathrm{Sets}$$

which sends a ring $A$ to the set of deformations of $\bar{\rho}$ to $A$. When $\bar{\rho}$ is absolutely irreducible, this functor is representable by a ring $R_{\bar{\rho}} \in \mathcal{C}$ [5, 2].

For $i = 1, 2$, let $d_i$ be the $\mathbf{F}$-dimension of the Galois cohomology group $H^i(G_{\mathbf{Q},S}, \mathrm{ad}\,\bar{\rho})$. Mazur showed that $d_1 - d_2 \geq 3$ and

$$R_{\bar{\rho}} \simeq W(\mathbf{F})[\![T_1, \ldots, T_{d_1}]\!]/(r_1, \ldots, r_{d_2}), \ldots, r_{d_2}),$$

where $W(\mathbf{F})$ is the ring of Witt vectors of $\mathbf{F}$. When $d_2 = 0$, it can be shown that $d_1 = 3$, so $R_{\bar{\rho}}$ is simply a power series ring in three variables. In this case, the deformation problem for $\bar{\rho}$ is said to be *unobstructed.*

Let $\ell$ be the characteristic of $\mathbf{F}$. As in [8, Lemma 2.5], an application of the Poitou–Tate exact sequence allows one to show that

$$\begin{aligned}\dim_{\mathbf{F}} H^2(G_{\mathbf{Q},S}, \mathrm{ad}\bar{\rho}) \leq{}& \dim_{\mathbf{F}} Ш^1(G_{\mathbf{Q},S}, \bar{\epsilon}_\ell \otimes \mathrm{ad}^0\bar{\rho}) \\ &+ \sum_{p \in S} \dim_{\mathbf{F}} H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho})\end{aligned} \tag{2.1}$$

with equality if $\ell \neq 3$. Here $Ш^1(G_{\mathbf{Q},S}, \bar{\epsilon}_\ell \otimes \mathrm{ad}^0\bar{\rho})$ is a sort of Selmer group; when $\bar{\rho} = \bar{\rho}_{f,\lambda}$ for some newform $f$, this term can be controlled by the set $\mathrm{Cong}(f)$ of congruence primes for $f$, as described in [8, Sec. 4]. Our focus will instead be on the local invariants $H^0(G_{\mathbf{Q},S}, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho})$ for $p \in S$, which we refer to as *obstructions at* $p$.

## 3. Removing the Squarefree Hypothesis

We fix some notation to be used throughout Sec. 3. Let $f = \sum a_n q^n$ be a newform of level $N$ and weight $k \geq 2$. Let $\omega$ be its nebentypus character, and let $M$ be the conductor of $\omega$. Let $K$ be its associated number field, and fix a prime $\lambda$ in $K$ with residue field $k_\lambda$ of characteristic $\ell$ such that $(N, \ell) = 1$ and $\bar{\rho}_{f,\lambda}$ is absolutely irreducible. Let $S$ be a finite set of places containing the primes dividing $N\infty$. We wish to study the conditions under which the deformation problem for

$$\bar{\rho}_{f,\lambda} : G_{\mathbf{Q},S\cup\{\ell\}} \to \mathrm{GL}_2(k_\lambda)$$

is unobstructed, and as described in Sec. 2, as long as $\lambda \notin \mathrm{Cong}(f)$, then this amounts to determining when $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}) \neq 0$ for $p \in S$.

Let $\pi$ be the automorphic representation associated to $f$, and write $\pi = \bigotimes' \pi_p$ for its decomposition into admissible complex representations $\pi_p$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. By the local Langlands correspondence, the classification of each $\pi_p$ allows us to study $\bar{\rho}_{f,\lambda}|_{G_p}$ in an explicit fashion. In [8], the assumption that $N$ be squarefree aided in the determination of $\pi_p$ for each $p \in S$; in particular, in this case it is easy to determine when $\pi_p$ is an unramified principal series, a principal series with one ramified character and one: Unramified character, or a special (twist of Steinberg) representation, and these are the only possibilities. When $p^2 \mid N$, it is not so easy to determine the structure of $\pi_p$. However, determining the exact structure of $\pi_p$ turns out to be unnecessary.

### 3.1. *Twists and p-primitive newforms*

Recall that for any primitive Dirichlet character $\chi$ of conductor $M$, we may twist the newform $f$ to obtain a newform $f \otimes \chi = \sum b_n q^n$, where $b_n = \chi(n)a_n$ for almost all $n$. The level of $f \otimes \chi$ is at most $NM^2$, but it may be smaller. For any newform $f$ and any prime $p$, one says that $f$ is *p-primitive* if the $p$-part of its level is minimal among all its twists by Dirichlet characters. We have the following lemma.

**Lemma 3.1.** *Let $f$ be a newform and let $f_p$ be a p-primitive twist. Then*

$$H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\, \bar{\rho}_{f,\lambda}) = H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\ \bar{\rho}_{f_p,\lambda}).$$

*In particular, $f$ has local obstructions at $p$ if and only if $f_p$ has local obstructions at $p$.*

**Proof.** For some Dirichlet character $\chi$ we have $f_p = f \otimes \chi$. It follows that $\bar{\rho}_{f_p,\lambda} \simeq \chi \otimes \bar{\rho}_{f,\lambda}$, and a straightforward matrix calculation then shows that $\mathrm{ad}(\bar{\rho}_{f_p,\lambda}) \simeq \mathrm{ad}(\bar{\rho}_{f,\lambda})$. The lemma follows. □

By Lemma 3.1, when studying local obstructions at $p$ for a newform $f$, we may assume that $f$ is $p$-primitive. The utility of considering $p$-primitive newforms is given by the following result, which comes from [4, Proposition 2.8].

**Proposition 3.2.** *Let $\pi_p$ be the local component of a p-primitive newform $f \in S_k(\Gamma_1(Np^r))$ with $p \nmid N$ and $r \geq 1$. Then one of the following conditions holds.*

(1) $\pi_p \simeq \pi(\chi_1, \chi_2)$ *is principal series, where $\chi_1$ is unramified and $\chi_2$ is ramified*;
(2) $\pi_p \simeq \mathrm{St} \otimes \chi$ *is special* (*twist of Steinberg*) *with $\chi$ unramified*;
(3) $\pi_p$ *is supercuspidal.*

**Proof.** See [4, Proposition 2.8] for the proof. □

**Remark 3.3.** If the level of a newform $f$ is divisible by $p^2$, it may be difficult to explicitly determine its $p$-minimal twist. Loeffler and Weinstein have made this computationally feasible in many cases; see [4]. We will avoid this extra difficulty and simply determine where obstructions might occur in all three cases of the above proposition.

### 3.2. *Supercuspidal obstruction conditions*

The arguments used by Weston in [8] are robust enough to carry over into the non-squarefree setting when we are cases (1) and (2) of Proposition 3.2. We instead focus on case (3), where $\pi_p$ is supercuspidal. We will frequently make use of the fact that

$$\dim_k H^0(G_p, \bar{\epsilon}_\ell \otimes (\mathrm{ad}^0 \bar{\rho}_{f,\lambda})^{\mathrm{ss}}) \leq \dim_k H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}^0\, \bar{\rho}_{f,\lambda}).$$

When $p > 2$, a supercuspidal $\pi_p$ is always induced from a quadratic extension of $\mathbf{Q}_p$, and these will be the focus of Proposition 3.5 below. When $p = 2$, there are additional supercuspidal representations, called extraordinary representations, and we consider these first. The case where $\pi_p$ is extraordinary was actually already dealt with in [7, Proposition 3.2] and are not a problem if $\ell \geq 5$. We reproduce the proof here.

**Proposition 3.4.** *Suppose $\pi_p$ is extraordinary, so $p = 2$. Then $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\, \bar{\rho}_{f,\lambda}) = 0$ if $\lambda$ has residue characteristic at least* 5.

**Proof.** Let $\rho : G_2 \to \mathrm{GL}_2(\bar{\mathbf{Q}}_\ell)$ be the representation of $G_2$ which is in Langlands correspondence with $\pi_2$. In this case, the projective image of inertia, proj $\rho(I_2)$, in $\mathrm{PGL}_2(\bar{\mathbf{Q}}_\ell)$ is isomorphic to either $A_4$ or $S_4$, and the composition

$$\mathrm{proj}\ \rho(I_2) \hookrightarrow \mathrm{PGL}_2(\bar{\mathbf{Q}}_\ell) \xrightarrow{\mathrm{ad}^0} \mathrm{GL}_3(\bar{\mathbf{Q}}_\ell)$$

is an irreducible representation of proj $\rho$. Since $\mathrm{proj}\, \rho(I_2)$ has order 12 or 24, it follows that $\mathrm{ad}^0 \bar{\rho}_{f,\lambda}$ is an irreducible $\bar{\mathbf{F}}_\ell$-representation of $I_2$ since $\mathrm{char}(\lambda) \geq 5$, thus $H^0(I_2, \bar{\epsilon}_\ell \otimes \mathrm{ad}^0\, \bar{\rho}_{f,\lambda}) = 0$ and the proposition follows. □

We will henceforth assume $\ell \geq 5$, so by this proposition there are no obstructions in the extraordinary case.

Now we deal with the final remaining possibility for $\pi_p$, which is the supercuspidal, non-extraordinary case. Since extraordinary supercuspidal representations only occur in the case $p = 2$, this is a very weak hypothesis. Recall that for any character $\psi$, we write $\bar{\psi}$ for its reduction mod $\lambda$.

**Proposition 3.5.** *Suppose $f$ is a newform of weight $k \geq 2$ such that $\pi_p$ is supercuspidal but not extraordinary. Suppose also that $\ell > 5$. If $p^4 \not\equiv 1 \,(\mathrm{mod}\ \ell)$, then $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\, \bar{\rho}_{f,\lambda}) = 0$.*

**Proof.** The Langlands correspondence (cf. [7, Proposition 3.2] or [4, Remark 3.11]) implies that there is a quadratic extension $E/\mathbf{Q}_p$ such that in characteristic zero we have

$$\rho_{f,\lambda}|_{G_p} \simeq \mathrm{Ind}_{G_E}^{G_p} \chi,$$

where $G_E = \mathrm{Gal}(\bar{E}/E)$ is the absolute Galois group of $E$ and $\chi : G_E \to \bar{\mathbf{Q}}_\ell$ is a continuous character. Let $\chi_E : \mathrm{Gal}(E/\mathbf{Q}_p) \to \{\pm 1\}$ be the non-trivial character for

$E/\mathbf{Q}_p$. Let $\chi^c$ be the Galois conjugate character of $\chi$, and let $\psi = \chi \cdot (\chi^c)^{-1}$. We have

$$\bar{\epsilon}_\ell \otimes \mathrm{ad}^0(\bar{\rho}_{f,\lambda})|_{G_p}^{\mathrm{ss}} \simeq \bar{\epsilon}_\ell \chi_E \oplus (\bar{\epsilon}_\ell \otimes \mathrm{Ind}_{G_E}^{G_p} \bar{\psi}).$$

Since $\ell > 3$, the first summand has no $G_p$-invariants, so we may focus on the second summand. By Mackey's criterion, the induced representation $\mathrm{Ind}_{G_E}^{G_p} \bar{\psi}$ is irreducible if and only if $\bar{\psi} \neq \bar{\psi}^c$. If it is irreducible, then so is its twist and we are done.

So suppose that $\bar{\psi} = \bar{\psi}^c$. We first note that, since $\bar{\psi} = \bar{\chi}(\bar{\chi}^c)^{-1}$, we have $\bar{\psi}^c = \bar{\chi}^c \bar{\chi}^{-1}$, hence

$$\bar{\psi}^2 = \bar{\psi}\bar{\psi}^c = [\bar{\chi}(\bar{\chi}^c)^{-1}] \cdot [\bar{\chi}^c(\bar{\chi})^{-1}] = 1.$$

Thus, $\bar{\psi}$ is a quadratic character on $G_E$.

Restricting the induced representation to $G_E$ we have

$$(\mathrm{Ind}_{G_E}^{G_p} \bar{\psi})|_{G_E} \simeq \bar{\psi} \oplus \bar{\psi}^c = \bar{\psi} \oplus \bar{\psi}$$

where the first equality is a generality about induced representations and the second comes from our assumption that $\bar{\psi} = \bar{\psi}^c$. So we already have $H^0(G_E, \bar{\epsilon}_\ell \otimes \mathrm{Ind}_{G_E}^{G_p} \bar{\psi}) = 0$ unless $\bar{\psi} = \bar{\epsilon}_\ell|_{G_E}^{-1}$, in which case $\bar{\epsilon}_\ell|_{G_E}$ is quadratic. Since $G_E$ has index 2 in $G_p$, this would imply that on $G_p$ the cyclotomic character $\bar{\epsilon}_\ell$ has order at most 4. Evaluating at $\mathrm{Frob}_p$, this implies that $p^4 \equiv 1 \pmod{\ell}$. So if $p^4 \not\equiv 1 \pmod{\ell}$, then the representation has no $G_E$-invariants and hence it has no $G_p$ invariants, completing the proof. □

We are now ready to prove the first main theorem, which removes the squarefree hypothesis from [8, Theorem 4.3]. For a newform $f$ of level $N$, $\mathrm{Cong}(f)$ is the set of congruence primes for $f$, i.e. the primes $\lambda$ such that there exists a newform $g$ (which is not a Galois conjugate of $f$) of level dividing $N$ with $f \equiv g \pmod{\lambda}$.

**Theorem 3.6.** *Assume that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible and $\ell > 3$. If $H^2(G_{\mathbf{Q},S\cup\{\ell\}}, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda}) \neq 0$ then one of the following conditions holds*:

(1) $\ell \leq k$;
(2) $\ell \mid N$;
(3) $\ell \mid \phi(N_S)$, *where $N_S$ is the product of the primes in $S$;*
(4) $\ell \mid (p+1)$ *for some* $p \mid N$;
(5) $a_p^2 \equiv (p+1)^2 p^{k-2}\omega(p) (\mathrm{mod}\,\lambda)$ *for some* $p \in S, p \nmid N, p \neq \ell$;
(6) $\ell = k+1$ *and $f$ is ordinary at* $\lambda$;
(7) $k = 2$ *and* $a_\ell^2 \equiv \omega(\ell) (\mathrm{mod}\,\lambda)$;
(8) $N = 1$ *and* $\ell \mid (2k-3)(2k-1)$;
(9) $\lambda \in \mathrm{Cong}(f)$;
(10) $p^4 \equiv 1 \,(\mathrm{mod}\,\ell)$ *for some $p$ such that* $p^2 \mid N$.

**Remark 3.7.** We note that conditions (1)–(9) are essentially the same conditions from [8, Theorem 4.3]; these conditions deal with the non-supercuspidal primes in $S$, while condition (10) deals with the (potentially) supercuspidal primes.

**Proof.** By Eq. (2.1), if $H^2(G_{\mathbf{Q},S\cup\{\ell\}}, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda}) \neq 0$ then either $\dim_k \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_\ell \otimes \mathrm{ad}^0\bar{\rho}) \neq 0$ or $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda}) \neq 0$ for some $p \in S$. By [8, Lemma 17], the former is only possible if $\lambda \in \mathrm{Cong}(f)$. This is accounted for in condition (9).

Now, let $p \in S$. While determining whether $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda}) = 0$, Lemma 3.1 allows us to replace $f$ by its $p$-minimal twist. In this case, by Lemma 3.2, there are only three possibilities for the local representation $\pi_p$.

If $\pi_p$ is principal series or special as in cases (1) and (2) of Lemma 3.2, then the local Galois representation has exactly the same form as the cases handled by Weston (see [8, Theorem 4.3]). This accounts for conditions (1)–(9). The only difference occurs in condition (4). In Weston's original condition, it is only necessary to avoid $\ell \mid (p+1)$ for primes $p$ dividing $N/M$, where $M$ is the conductor of the nebentypus character of $f$. Since we have replaced $f$ by its $p$-minimal twist $f_p$, and we do not know the conductor of the character of $f_p$, we replace Weston's original condition with our coarser condition.

If $\pi_{f,p}$ is supercuspidal, then Proposition 3.5 yields condition (10). This covers all the possibilities for $\pi_{f,p}$, thus completing the proof. □

## 4. Minimal Deformation Problems and Optimal Levels

Given a modular form $f$, a prime $\lambda$ of $\bar{K}$, and a finite set of places $S$, let us write $\mathbf{D}(f, S)$ for the corresponding deformation problem. (We suppress $\lambda$ from the notation, as it will always be clear from context.) If $S$ contains only the primes dividing the level of $f$ and the infinite place, then we may simply write $\mathbf{D}(f)$, and we call this the *minimal deformation problem* for $f$.

For any odd, continuous, absolutely irreducible representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k_\lambda)$, with $k_\lambda$ a finite field of characteristic $\ell$, let $\mathcal{H}(\bar{\rho})$ be the set of newforms of level prime to $\ell$ giving rise to this representation. Among all such newforms, there is a least level appearing, which we call the *optimal* level for $\mathcal{H}(\bar{\rho})$. In fact, this optimal level is the prime-to-$\ell$ Artin conductor of $\bar{\rho}$ (see [1]).

Let $f$ and $g$ be newforms in $\mathcal{H}(\bar{\rho})$ with associated minimal sets of primes $S$ and $S'$, respectively. We have an isomorphism of residual Galois representations $\rho_{f,\lambda} \simeq \rho_{g,\lambda}$, and if $S \subset S'$ then we have an equality of deformation problems $\mathbf{D}(f, S') = \mathbf{D}(g)$. Furthermore, since $S \subset S'$, if $\mathbf{D}(f)$ is obstructed then so is $\mathbf{D}(g)$. In fact, we prove the following theorem.

**Theorem 4.1.** *If $\mathbf{D}(f)$ is unobstructed, then $f$ is of optimal level for $\mathcal{H}(\bar{\rho})$.*

In Sec. 4.2 we present the proof of this theorem; our strategy is to prove the contrapositive. By Proposition 4.3 below, we know the factorization of any non-optimal level. If $g$ is a newform of non-optimal level, we compare it to an optimal level newform $f$. Since, as discussed above, $\mathbf{D}(g)$ inherits any obstructions that $\mathbf{D}(f)$ might have, we may assume that $\mathbf{D}(f)$ is unobstructed, and we show that even in this case, $\mathbf{D}(g)$ is necessarily obstructed.

This theorem is motivated by the following heuristic: If $\bar{\rho}$ is $\ell$-ordinary and $\ell$-distinguished, and if $\mathcal{H}(\bar{\rho})$ is the set of of all $\ell$-ordinary, $\ell$-stabilized newforms with mod $\ell$ Galois representation isomorphic to $\bar{\rho}$, then $\mathcal{H}(\bar{\rho})$ is a dense set of classical points in a *Hida family* **H**. This object has a geometric interpretation in which its irreducible components have associated integers, called levels, which correspond to levels of modular forms. The components of **H** of non-minimal level have associated (full) Hecke algebras of higher $\Lambda = \mathbf{Z}_\ell[\![T]\!]$-rank than the minimal-level component (see [3, Sec. 2.4] for more details). Thus, if a general enough $\mathcal{R} = \mathbf{T}$ theorem is known (or believed), then this forces the deformation ring to grow as well. Our theorem shows that this sort of behavior is not a special property of Hida families, and that it actually occurs independent of any geometric structure.

**Remark 4.2.** It is worth pointing out two things about this theorem. The first is that it does *not* follow immediately from Theorem 3.6, because condition (9) of that theorem is not a sharp obstruction criterion, i.e. it does not *guarantee* the existence of obstructions. The other noteworthy aspect is that in [8], congruence primes are primarily shown to (potentially) give rise to *global* obstruction classes, whereas our proof uses the existence of a newform congruence to produce *local* obstruction classes.

### 4.1. *Preliminaries*

In this section we record the results which we will use to prove the theorem. Let us first set some notation to be used throughout Sec. 4.

Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level $N$ (coprime to $\ell$), and nebentypus $\omega$, and let $M$ be the conductor of $\omega$. Let $S$ be a finite set of places containing the primes which divide $N\infty$. Let $K = \mathbf{Q}(a_n)$, and fix a prime $\lambda$ of $\bar{K}$ which lies over $\ell$. We have $f \in \mathcal{H}(\bar{\rho})$, where $\bar{\rho}_{f,\lambda} \simeq \bar{\rho}$.

Suppose $f$ is of optimal level for $\mathcal{H}(\bar{\rho})$. If $g \in \mathcal{H}(\bar{\rho})$ is of non-optimal level, we will want to know what form its level can have. The following is a result of Carayol (see the introduction of [1]).

**Proposition 4.3.** *Suppose* $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_\ell)$ *is modular of weight* $k \geq 2$ *and level* $N'$ *coprime to* $\ell$. *Then*

$$N' = N \cdot \prod p^{\alpha(p)},$$

*where* $N$ *is the conductor of* $\bar{\rho}$, *and for each* $p$ *with* $\alpha(p) > 0$, *one of the following conditions holds*:

(1) $p \nmid N\ell$, $p(\mathrm{tr}\rho(\mathrm{Frob}_p)^2) = (1+p)^2 \det\rho(\mathrm{Frob}_p)$ *in* $\bar{\mathbf{F}}_\ell$ *and* $\alpha(p) = 1$;
(2) $p \equiv -1 \bmod \ell$ *and one of the following conditions holds*:
   (a) $p \nmid N, \mathrm{tr}(\rho(\mathrm{Frob}_p) = 0$ *in* $\bar{\mathbf{F}}_\ell$ *and* $\alpha(p) = 2$, *or*
   (b) $p \mid N$, $\det \rho$ *is unramified at* $p$, *and* $\alpha(p) = 1$;

(3) $p \equiv 1 \bmod \ell$ *and one of the following conditions holds*:

  (a) $p \nmid N$ *and* $\alpha(p) = 2$, *or*

  (b) $p^2 \nmid N$, *or the power of* $p$ *dividing* $N$ *is the same as the power dividing the conductor of* $\det \rho$, *and* $\alpha(p) = 1$.

Our goal, then, is to show that each of the possible supplementary primes appearing in Proposition 4.3 gives rise to an obstruction. We collect some lemmas in this direction.

The first two lemmas come from [8].

**Lemma 4.4.** *If* $\ell \,|\, (p-1)$ *for some* $p \in S$, *then* $\mathbf{D}(f, S)$ *is obstructed.*

**Proof.** This is proved in the discussion at the beginning of [8, Section 3]. Since $\ell \,|\, p-1$, we have $H^0(G_p, \bar{\epsilon}_\ell) \neq 0$. Since $\bar{\epsilon}_\ell \otimes \mathrm{ad}\ \bar{\rho}_{g,\lambda} \simeq \bar{\epsilon}_\ell \oplus (\bar{\epsilon}_\ell \otimes \mathrm{ad}^0 \bar{\rho}_{g,\lambda})$ this shows that $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\bar{\rho}_{g,\lambda}) \neq 0$ and so $\mathbf{D}(f, S)$ is obstructed. □

The previous lemma gives us a tool we can use when $p \equiv 1 \pmod{\ell}$. The next lemma deals with the case when $p \not\equiv 1 \pmod{\ell}$ and $p \neq \ell$.

**Lemma 4.5.** *Assume* $p \nmid N\ell$ *and* $p \not\equiv 1 \pmod{\ell}$. *Then* $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\ \bar{\rho}_{f,\lambda}) \neq 0$ *if and only if* $a_p^2 \equiv (p+1)^2 p^{k-2} \omega(p) (\bmod \lambda)$.

**Proof.** This is Lemma 3.1 of [8]. □

We record one final lemma before proving our theorem.

**Lemma 4.6.** *If* $p \,\|\, N, p \nmid M$, *and* $p^2 \equiv 1 (\bmod \ell)$, *then* $\mathbf{D}(f, S)$ *is obstructed.*

**Proof.** As explained in [7, Sec. 5.2], in this case $\pi_p$ is special, which translates on the Galois side to the existence of an unramified character $\chi : G_p \to \bar{K}_\lambda^\times$ (where $K$ is the field of Fourier coefficients of $f$ and $K_\lambda$ is its completion at $\lambda$) such that

$$\rho_{f,\lambda}|_{G_p} \otimes \bar{K}_\lambda \simeq \begin{pmatrix} \epsilon_\ell \chi & * \\ 0 & \chi \end{pmatrix},$$

with the upper right corner ramified. Upon reduction this matrix becomes either

$$A = \begin{pmatrix} \bar{\epsilon}_\ell \bar{\chi} & \nu \\ 0 & \bar{\chi} \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} \bar{\chi} & \nu \\ 0 & \bar{\epsilon}_\ell \bar{\chi} \end{pmatrix}$$

for some $\nu : G_p \to \bar{k}_\lambda$. We note that by Lemma 5.1 of [7], possibility $B$ can only occur if $p^2 \equiv 1 \pmod{\ell}$.

Let $C = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. One computes

$$ACA^{-1} = \begin{pmatrix} 0 & \bar{\epsilon}_\ell \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BCB^{-1} = \begin{pmatrix} 0 & (\bar{\epsilon}_\ell)^{-1} \\ 0 & 0 \end{pmatrix},$$

and so

$$(\bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda}) \cdot C = \begin{pmatrix} 0 & \bar{\epsilon}_\ell^j \\ 0 & 0 \end{pmatrix}$$

for $j = 0$ or 2, so $C \in H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda})$. If $j = 0$, this is obvious; if $j = 2$, this follows from the facts that, since $\epsilon_\ell$ is ramified only at $\ell$, it factors through a group which is topologically generated by $\mathrm{Frob}_p$, but $\epsilon_\ell(\mathrm{Frob}_p) = p$ and $p^2 \equiv 1 \pmod{\ell}$. So in either case $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\,\bar{\rho}_{f,\lambda}) \neq 0$, hence $\mathbf{D}(f, S)$ is obstructed. □

## 4.2. *Optimal level deformation problems*

Let $f \in S_k(\Gamma_1(N), \omega)$ and $g \in S_k(\Gamma_1(N'))$ be newforms in $\mathcal{H}(\bar{\rho})$ with $f$ of optimal level and $N' > N$; by Proposition 4.3, $N \mid N'$. Let $S$ (respectively, $S'$) be the set of places of $\mathbf{Q}$ dividing $N\infty$ (respectively, $N'\infty$), so $S \subset S'$.

Write $f = \sum a_n q^n$. Let $K$ be a field containing the Fourier coefficients of both $f$ and $g$, and let $\lambda$ be a prime of $K$ over $\ell$ such that $f \equiv g \pmod{\lambda}$ and hence $\bar{\rho} \simeq \bar{\rho}_{f,\lambda} \simeq \bar{\rho}_{g,\lambda}$. Write $k_\lambda$ for the residue field of $\lambda$.

Using this notation, we are now ready to prove Theorem 4.1. For the reader's convenience, we restate the theorem.

**Theorem 4.1.** *If* $\mathbf{D}(f)$ *is unobstructed, then* $f$ *is of optimal level for* $\mathcal{H}(\bar{\rho})$.

**Proof.** We will prove the contrapositive. Keeping the notation from the beginning of Sec. 4.2, let $f$ and $g$ be newforms in $\mathcal{H}(\bar{\rho})$, with $f$ of optimal level and $g$ of non-optimal level. We will show that $\mathbf{D}(g)$ is obstructed.

If $\mathbf{D}(f)$ is obstructed, then as noted earlier, this implies $\mathbf{D}(g)$ is also obstructed. So in proving the theorem, we may assume that $\mathbf{D}(f)$ is unobstructed.

We consider separately the primes $p \in S'$ which appear in cases (1), (2), and (3) of Proposition 4.3. Note that we have an equivalence of deformation problems $\mathbf{D}(g, S') = \mathbf{D}(f, S')$. We write $\mathbf{D}_\ell$ for these equivalent deformation problems.

First, suppose $p \mid N'$ is as in case (3), so in particular $p \equiv 1 \pmod{\ell}$. Then by Lemma 4.4 we see that $\mathbf{D}_\ell$ is obstructed.

Next, suppose $p \mid N'$ is as in case (1) of the proposition, so $p$ is a prime such that $p \nmid N\ell$, $\alpha(p) = 1$, and $pa_p^2 \equiv (1+p)^2 \omega(p) p^{k-1} \pmod{\lambda}$, or equivalently (since $p$ is invertible in $\bar{\mathbf{F}}_\ell$),

$$a_p^2 \equiv (p+1)^2 p^{k-2} \omega(p) \quad (\mathrm{mod} \lambda).$$

Then by Lemma 4.5 we see that $\mathbf{D}_\ell$ is obstructed.

Finally, suppose $p \mid N'$ is as in case (2), so $p \equiv -1 \pmod{\ell}$ and one of the following holds:

(a) $p \nmid N$, $a_p \equiv 0 \pmod{\lambda}$, and $\alpha(p) = 2$; or
(b) $p \parallel N$, $\det \rho$ is unramified at $p$, and $\alpha(p) = 1$.

If $p$ were as in case (b), then actually $p \in S$, and Lemma 4.6 shows that $\mathbf{D}(f, S)$ is obstructed. This contradicts our hypothesis on $\mathbf{D}(f)$, so we can ignore this case.

Finally, we must consider case (a), so that $p \equiv -1 \pmod{\ell}$, $p \nmid N$, $a_p \equiv 0 \pmod{\lambda}$, and $\alpha(p) = 2$. Recalling that $\mathbf{D} = \mathbf{D}_\ell(f, S')$, Lemma 4.5 gives the obstruction since $a_p \equiv (p+1) \equiv 0 \pmod{\lambda}$. □

**Remark 4.7.** It is not the case that every minimal, optimal level deformation problem is unobstructed. Indeed, for any prime $p$ we have

$$\bar{\epsilon}_\ell \otimes \operatorname{ad} \bar{\rho}_{f,\lambda} \simeq \bar{\epsilon}_\ell \oplus (\bar{\epsilon}_\ell \otimes \operatorname{ad}^0 \bar{\rho}_{f,\lambda})$$

and

$$H^0(G_p, \bar{\epsilon}_\ell) \neq 0 \Leftrightarrow p \equiv 1 \pmod{\ell},$$

so condition (3) of Theorem 4.1 is sharp. Let $\ell = 5$, $p = 11$, and $k = 3$. The space $S_3(\Gamma_1(11), 3)$ contains one newform defined over $\mathbf{Q}$ and four newforms which are Galois conjugates defined over $\mathbf{Q}(\alpha)$, where $\alpha$ is a root of $x^4 + 5x^3 + 15x^2 + 15x + 5$. The minimal set $S$ for any of these newforms is $S = \{11, \infty\}$. Since $S_3(\Gamma_1(1))$ is empty, all of these newforms are of optimal level for their respective mod $\ell$ representations, but since $p \equiv 1 \bmod \ell$ their minimal deformation problems are obstructed.

**Remark 4.8.** The techniques in this paper cannot rule out the possibility that two (or more) congruent modular forms of optimal level can exist for an unobstructed modular deformation problem.

Combining this result with Weston's result (Theorem 1 of [7]), we have the following corollary.

**Corollary 4.9.** *Let $f$ be a newform of level $N$ and weight $k \geq 2$. For infinitely many primes $\ell$, $f$ represents an optimal modular realization of a* mod $\ell$ *representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_\ell)$.*

**Proof.** For infinitely many such $\ell$, $\mathbf{D}(f)$ is unobstructed (by Weston), and by Theorem 4.1, this implies that $f$ is of optimal level among modular forms realizing $\bar{\rho}$. □

**Remark 4.10.** Actually, there is a much simpler proof of this fact: If $f$ is of non-optimal level for its mod $\ell$ representation, then there is a modular form $g$ of lower level such that $f \equiv g$. But such a congruence can occur for only finitely many primes $\ell$, which follows from the $q$-expansion principle and the fact that these spaces of modular forms are finite dimensional.

We also get another corollary. For any integer $N$, let $d(N)$ be the number of prime divisors of $N$, i.e. $d(N) = \sum_{p \mid N} 1$.

**Corollary 4.11.** *Fix a prime $\lambda$ with residue field $k$ of characteristic $\ell > 3$, suppose $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ is a modular* mod $\lambda$ *representation of prime-to-$\ell$ conductor $N$,* *and let $f$ be a newform of level $N$ such that $\bar{\rho} \simeq \bar{\rho}_{f,\lambda}$. If $g$ is a newform of level $N'$ such that $f \equiv g \pmod{\lambda}$, then*

$$\dim_k H^2(G_{\mathbf{Q},S}, \operatorname{ad}\bar{\rho}) \geq d(N'/N)$$

*where $S$ is a finite set of places containing the primes which divide $N'\infty$.*

**Proof.** The proof of Theorem 4.1 shows that if $g$ is of non-optimal level $N'$, then for every prime $p$ dividing $N'/N$, we have $H^0(G_p, \bar{\epsilon}_\ell \otimes \operatorname{ad}\bar{\rho}_{f,\lambda}) \neq 0$. By Eq. (2.1) we have

$$\dim_k H^2(G_{\mathbf{Q},S}, \operatorname{ad}\bar{\rho}) \geq \sum_{p\in S} \dim_k H^0(G_p, \bar{\epsilon}_\ell \otimes \operatorname{ad}\bar{\rho}_{f,\lambda})$$

and the corollary follows. □

## 5. Explicit Computations in the Supercuspidal Case

**Example 5.1.** Let $k = 2$, $\ell = 11$, and $p = 7$. Consider the CM elliptic curve $E$ with Cremona label $49a1$; it is given by

$$E : y^2 + xy = x^3 - x^2 - 2x - 1,$$

and its associated modular form $f \in S_2(\Gamma_0(49))$ has $q$-expansion

$$f = q + q^2 - q^4 - 3q^8 - 3q^9 + \cdots.$$

The mod $\ell$ Galois representation $\bar{\rho}_{f,\ell}$ is irreducible, and one checks that none of conditions (1)–(8) of Theorem 3.6 are satisfied. Using Sage [6], one also verifies that $\ell \notin \operatorname{Cong}(f)$ and so $f$ is of optimal level for this representation.

Loeffler and Weinstein have incorporated their results from [4] into the Local Components package of [6]. Using this, one discovers that $\pi_p$ is supercuspidal. However, since $p^4 \equiv 3 \pmod{\ell}$, condition (10) is also not satisfied. Thus $\mathbf{D}(f)$ is unobstructed.

**Example 5.2.** This example shows that condition (10) of Theorem 3.6 is necessary but not sufficient for producing local obstructions at supercuspidal primes. Let us fix $k = 3$, $\ell = 5$, and $p = 7$. Note that $p^2 \equiv -1 \bmod \ell$ and $p^4 \equiv 1 \bmod \ell$.

Using [6], one finds a newform $f$ in $S_3(\Gamma_1(49))$ with a $q$-expansion that begins

$$f = q + \left(-\frac{1}{92}\alpha^3 + \frac{5}{92}\alpha^2 - \frac{41}{91}\alpha + \frac{229}{92}\right) q^2 + \left(-\frac{1}{184}\alpha^3 + \frac{5}{184}\alpha^2 - \frac{133}{184}\alpha + \frac{229}{184}\right) q^3 + \cdots.$$

Using the Local Components package of [6], one discovers that $\pi_p$ is supercuspidal. Let $E = \mathbf{Q}_p(s)$ be the unramified quadratic extension of $\mathbf{Q}_p$. Let $L = K(\beta)$

where $\beta$ satisfies the polynomial $x^2 + (\frac{3}{1288}\alpha^3 + \frac{11}{184}\alpha^2 - \frac{153}{1288}\alpha + \frac{467}{184})x - 1$. Then the character $\chi$ associated to $\pi_p$ is characterized by

$$\chi : E \to L$$
$$s \mapsto \beta, \quad 7 \mapsto 7.$$

(Here we are viewing $\chi$ as a character of $E^\times$ instead of $G_E$ via local class field theory.) Let $\lambda$ be either of the two primes of $L$ which lies over $\ell$. Then using [6], one verifies that $\chi$ and its conjugate $\chi^c$ are equivalent mod $\lambda$ by checking that $\beta - \beta^c$ has positive $\lambda$-valuation. In the notation of Proposition 3.5, this shows that $\bar{\psi} = 1$; the induction of this character is a symmetric representation, and so $\bar{\epsilon}_\ell \otimes \mathrm{Ind}_{G_E}^{G_p} \bar{\psi}$ is an invariant $G_p$-representation, hence $H^0(G_p, \bar{\epsilon}_\ell \otimes \mathrm{ad}\, \bar{\rho}_{f,\lambda}) = 0$.

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