Abstract

Following [5], a relational variable set on a category $B$ is a lax functor $B \rightarrow \text{Rel}$, where $\text{Rel}$ is the category of sets and relations. Change-of-base functors and their adjoints are considered for certain categories of relational variable sets and applied to construct the simplification of a dynamic set (in the sense of [11]).

Key Words: relational variable set, specification structure, dynamic set, relational presheaf, change of base, exponentiable

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1 Introduction

A relational variable set on a category $B$ is a lax functor $B \rightarrow \text{Rel}$, where $\text{Rel}$ is the locally partially-ordered 2-category of sets and relations. Also called relational presheaves [8], specification structures [1], and dynamic sets [11], these lax functors have played a role in the study of modal and basic predicate logic [4, 5], concurrency [1], automata [8], and spatio-temporal databases [11].

For a category $\text{Rel}^B$ of relational variable sets and a functor $p: E \rightarrow B$, we consider adjoints to the change-of-base functor $p^*: \text{Rel}^B \rightarrow \text{Rel}^E$. We will see that $p^*$ always has a left adjoint $\Sigma_p$, while the existence of a right adjoint $\Pi_p$ is related to the exponentiability of $p$ in the category $\text{Cat}$ of small categories and functors.
In “Powerful functors” [12], Street describes exponentiability in $\text{Cat}$ via an equivalence (due to Bénabou) between the slice 2-category $\text{Cat}/B$ and a 2-category of normal lax functors $m: B^{\text{op}} \to \text{Mod}$, where $\text{Mod}$ denotes the bicategory whose objects are small categories and the hom category $\text{Mod}(A, B)$ is the functor category $\text{Set}^{B^{\text{op}} \times A}$, and a normal lax functor strictly preserves identities. In particular, $p: E \to B$ is exponentiable if and only if the corresponding $m_E: B^{\text{op}} \to \text{Mod}$ is a pseudofunctor, and the latter readily translates into the Giraud-Conduché [6, 3] factorization lifting condition for exponentiability in $\text{Cat}$.

A modification of Bénabou’s result yields an equivalence between $\text{Rel}^B$ and the category $\text{Cat}_f/B$ of faithful functors over $B$. It turns out that a faithful functor $p: E \to B$ is exponentiable in $\text{Cat}_f/B$ if and only if the corresponding relational variable set $B \to \text{Rel}$ is a non-unitary (i.e., not necessarily identity preserving) functor, or equivalently a certain weak factorization lifting property (WFLP) holds. Thus, when $p: E \to B$ is faithful, using the general relationship between exponentiability of $p$ and the existence of $\Pi_p$, it follows that $\Pi_p: \text{Rel}^E \to \text{Rel}^B$ exists if and only if $p$ satisfies WFLP, and this generalizes to the case when $p$ is not assumed to be faithful.

We begin with the introduction of the category $\text{Rel}^B$ of relational variable sets and morphisms, and its equivalence to $\text{Cat}_f/B$. In §4, we consider adjoints to the change-of-base functor $p^*: \text{Rel}^B \to \text{Rel}^E$, and their relationship to the exponentiability of $p$. We conclude, in §5, with an application to the construction of the simplification of a dynamic set with respect to a change in time domain (in the sense of [11]).

## 2 Relational Variable Sets

Let $\text{Rel}$ denote the locally partially-ordered 2-category of sets and relations, i.e., $\text{Rel}(X, Y)$ is the poset of relations $R \subseteq X \times Y$, with the identity morphism on $X$ given by the diagonal $\Delta \subseteq X \times X$ and composition by the usual relation composites. To distinguish relations from functions, elements of $\text{Rel}(X, Y)$ will be denoted by $R: X \rightarrowtail Y$. The composite of $R: X \rightarrowtail Y$ and $S: Y \rightarrowtail Z$ will be written $S \circ R$, and abbreviated as $SR$.

A relational variable set or $\text{Rel}$-set on a category $B$ consists of a set $X_b$, for every object $b$ of $B$, and a relation $X_\beta: X_b \rightarrowtail X_Y$, for every morphism $\beta: b \to b'$ of $B$, satisfying
\[ \Delta_{X_b} \subseteq X_{i_b} \]
\[ X_{\beta'} X_{\beta} \subseteq X_{\beta' \beta} \]

for all objects \( b \) and for all morphisms \( \beta: b \to b' \) and \( \beta': b' \to b'' \) of \( B \). Writing \( x \xrightarrow{\beta} x' \) for the infix form of \( (x, x') \in X_\beta \), and \( x \xrightarrow{b} x' \) when \( \beta \) is the identity morphism on \( b \), these conditions become

\[ \text{(RS1*) } x \xrightarrow{b} x, \text{ for all } x \in X_b \]
\[ \text{(RS2*) } x \xrightarrow{\beta} x', x' \xrightarrow{\beta'} x'' \Rightarrow x \xrightarrow{\beta' \beta} x'' \]

Note that a \( \text{Rel}-\text{set} \) is just a \emph{lax functor} or \emph{morphism of bicategories}, in the sense of [7] or [2], respectively. Of course, \( X \) is a functor if and only if the containments in (RS1) and (RS2) are equalities.

An example (in the spirit of [11]) of a \( \text{Rel}-\text{sets} \) as a model of a data base changing over time is given by

Here, the objects of the category \( t_1 \to t_2 \to t_3 \) are time values (perhaps, certain years), the elements of \( X_t \) represent daily flights to three airports (say, Newark, Boston, and Washington) with subscripts used to indicate multiple flights.

Among the examples from \emph{Logics for Concurrency} [1] is the \( \text{Rel}-\text{set} \) on \( B = \text{Rel} \) with \( X_b = \mathcal{P}(b) = \text{Rel}(1, b) \), and \( S \to_T T \) whenever \( RS \subseteq T \). Note that this example can be generalized to any bicategory \( B \), where \( 1 \) is replaced by any fixed object of \( B \).

Also, \( \text{Rel}-\text{sets} \) on the power set \( \mathcal{P}(M) \) of a monoid \( M \) arise in a categorical approach to automata theory. For details of this and other applications, see [8].

A \emph{morphism} \( f: X \to Y \) of \( \text{Rel}-\text{sets} \) on \( B \) is an op-lax natural transformation. Thus, \( f \) consists of a function \( f_b: X_b \to Y_b \), for every object \( b \), such that for every morphism \( \beta: b \to b' \) there is a diagram
\[
\begin{array}{c}
X_b \xrightarrow{f_b} Y_b \\
\downarrow x_\beta \quad \subseteq \quad \downarrow y_\beta \\
X_{b'} \xrightarrow{f_{b'}} Y_{b'}
\end{array}
\]

in \(\text{Rel}\), i.e., \(x \rightarrow_\beta x' \Rightarrow f_b x \rightarrow_\beta f_{b'} x'\).

Note that this is the notion of morphism given by Ghilardi and Meloni [4, 5] and Rosenthal [8]. A more general definition of morphism, in which the functions \(f_b\) are replaced by relations, is also given in [8]. Abramsky, Gay, and Nagarajan do not consider morphisms of specification structures in [1].

When \(B\) is a small category, \(\text{Rel}\)-sets and their morphisms form a locally small locally preordered 2-category \(\text{Rel}^B\) with \(f \rightarrow g\), if \(f_b x \rightarrow_b g_b x\), for all \(b \in B\) and \(x \in X_b\), and consequently, \(x \rightarrow_\beta x' \Rightarrow f_b x \rightarrow_\beta g_{b'} x'\), for all \(\beta: b \rightarrow b', x \in X_b\), and \(x' \in X_{b'}\). The symbol \(\rightarrow\) is used here for a preorder to distinguish it from a partial order \(\leq\) since both arise on the same set in §5.

### 3 Rel-Sets and Faithful Functors

In this section, we assume \(B\) is a small category and consider a 2-adjunction

\[
\text{Rel}^B \xleftarrow{\Gamma} \text{Cat}/B
\]

which gives rise to an equivalence between \(\text{Rel}^B\) and the category \(\text{Cat}_f/B\) of faithful functors over \(B\). In particular, \(\Gamma\) is given by the lax fibration for the Grothendieck construction on a \(\text{Rel}\)-set.

Recall that \(\text{Cat}/B\) is the 2-slice category whose objects are functors \(p: E \rightarrow B\), morphisms are commutative triangles

\[
\begin{array}{ccc}
E & \xrightarrow{f} & F \\
\downarrow p & & \downarrow q \\
B & & B
\end{array}
\]

in \(\text{Cat}\), and 2-cells are natural transformations \(\theta: f \rightarrow g\) such that \(q\theta = id_p\).

For \(p: E \rightarrow B\) and an object \(b\) of \(B\), the fiber \(E_b\) of \(E\) over \(b\) is the subcategory of \(E\) consisting of objects over \(b\) and morphisms over \(id_b\). Let
$Lp$ denote the $Rel$-set with $(Lp)_b = E_b$ and $e \to_\beta e'$ if there exists $\xi: e \to e'$ such that $p\xi = \beta$. As is customary for slice categories, we will often suppress the explicit reference to $p$ and write $LE$ instead of $Lp$. If $f: E \to F$ is a functor over $B$, then $f_b: E_b \to F_b$ defines a morphism $f: LE \to LF$, since $e \to_\beta e'$ when there exists $\xi: e \to e'$ over $\beta: b \to b'$, and hence $f\xi: fe \to f'e'$ over $\beta$, showing that $f_b e \to_\beta f_b e'$. Moreover, $f \to g$ for every 2-cell $\theta: f \to g$, since the morphism $\theta_e: fe \to ge$ satisfy $q\theta_e = id_{pe} = id_b$, and so $f_b e \to_b g_b e$, for all $e$ in $(LE)_b$. Thus, $L: Cat/B \to Rel^B$ is a 2-functor.

Note that the lax functor $LE: B \to Rel$ need not be a functor. In fact, it is unitary (i.e., identity preserving) precisely when $p: E \to B$ has discrete fibers, and it preserves composition when $p$ satisfies the weak factorization lifting property (WFLP)

\[
\begin{array}{c}
E \\
p \\
B
\end{array} \xymatrix{
\ar[r]^e & e' \\
\ar[r]_\xi & e'' \\
\ar[r]_p & p e'' \\
\ar[r]_\beta & p e''}
\]

i.e., for every morphism $\xi''$ of $E$ and every factorization $p\xi'' = \beta'/\beta$ in $B$, there exists a factorization $\xi'' = \xi' \xi$ in $E$ such that $p\xi = \beta$ and $p\xi' = \beta'$.

This condition is a weakening of the usual Giraud-Conduché [6, 3] factorization lifting property characterizing exponentiable objects of $Cat/B$. In particular, it does not require the usual “zigzag” relating any two liftings of the same factorization. However, restricting to posets, $p: E \to B$ satisfies WFLP if and only if it is exponentiable in the category $Pos$ of posets and order-preserving maps [10]. The fact that WFLP arises (instead of FLP) should be clear from the discussion of exponentiability in the next section.

To define $\Gamma: Rel^B \to Cat/B$, let $X$ be a $Rel$-set on $B$, and consider the category $E_X$ whose objects are pairs $(x, b)$ where $b$ is an object of $B$ and $x \in X_b$, and morphisms $\beta: (x, b) \to (x', b')$ are morphisms $\beta: b \to b'$ of $B$ such that $x \to_\beta x'$ in $X$. Then $E_X$ is a category over $B$ via the projection $\Gamma X: E_X \to B$ which is, in fact, a faithful functor. A morphism $f: X \to Y$ of $Rel$-sets gives rise to a functor $\Gamma f: E_X \to E_Y$ over $B$ defined by $\Gamma f(x, b) = (f_b x, b)$ and $\Gamma f(\beta) = \beta$. Note that every functor $g: E_X \to E_Y$ over $B$ is of this form, for given such a $g$, define $f_b: X_b \to Y_b$ by $f_b(x) = \pi_1 g(x, b)$ and $f(\beta) = \beta$. Then $f$ is a morphism of $Rel$-sets, since

$x \to_\beta x' \Rightarrow \beta: (x, b) \to (x', b') \Rightarrow g\beta: (f_b x, b) \to (f_b x', b') \Rightarrow f_b x \to_\beta f_b x'$

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and $\Gamma f = g$. A 2-cell $f \to g$ induces a natural transformation $\theta: \Gamma f \to \Gamma g$ over $B$ given by $id_b: (f_b x, b) \to (g_b x, b)$ since $f_b x \to g_b x$, for all $b$. Thus, $\Gamma: Rel^B \to Cat/B$ is a 2-functor whose image is the full subcategory of $Cat_f/B$ of faithful functors over $B$.

To see that $L$ is left adjoint to $\Gamma$, define $\varepsilon: L\Gamma \to id$ and $\eta: id \to \Gamma L$, as follows. Given a $Rel$-set $X$, note that $(L\Gamma X)_b = (EX)_b = \{(x, b) | x \in X_b\}$ Then the projections $(\varepsilon_X)_b: (L\Gamma X)_b \to X_b$ define a morphism $\varepsilon_X: L\Gamma X \to X$, which is clearly a 2-natural isomorphism. Likewise, given $p: E \to B$ in $Cat/B$, $\Gamma LE$ is the category whose objects are pairs $(e, b)$ where $e \in LE_b$, i.e., $pe = b$, and morphisms $\beta: (e, b) \to (e', b')$ are morphisms $\beta: b \to b'$ such that $e \to \beta e'$ in $LE$, i.e., there exists $\xi: e \to e'$ in $E$ with $p\xi = \beta$. Then $\eta_E: E \to \Gamma LE$, given by $\eta_E(e) = (e, pe)$ and $\eta_E(\xi) = p\xi$, gives rise to a 2-natural transformation $\eta: p \to Lp$, such that $\eta_E$ is full and surjective on objects, in any case, and faithful if and only if $p$ is faithful. Moreover, the adjunction identities easily follow. Therefore, the following has been established:

**Theorem 3.1** There is a 2-adjunction

$$Rel^B \xleftarrow{\Gamma} Cat/B$$

which induces an equivalence between the locally preordered 2-category $Rel^B$ and the slice 2-category $Cat_f/B$ of faithful functors over $B$. As a result, $Cat_f/B$ is a reflective subcategory of $Cat/B$ with reflection $\Gamma L$.

Note that the equivalence $Rel^B \simeq Cat_f/B$ is essentially that of Bénabou mentioned in the introduction and described by Street in [12]. But, the “op” does not appear here since the morphisms of $Rel$ are opposite those of $Mod$. Also, normal (i.e., strictly identity-preserving) lax functors $B^{op} \to Mod$ are considered there in order to obtain all of $Cat/B$. Normality does not arise here since the identity morphisms of $Rel$ are the diagonals. In fact, normal lax functors $B \to Rel$ correspond to faithful functors $E \to B$ whose fiber $E_b$ are discrete.

Since $Cat_f/B$ is a reflective subcategory of $Cat/B$, it is closed under limits. Using the equivalence with $Rel^B$, we get:
Corollary 3.2 Limits exist and are computed point-wise in $\text{Rel}^B$.

In the case where $B$ a preordered set (in particular, a poset), $\text{Cat}_f/B$ is the category $\text{Pr}/B$ of preordered sets over $B$. Thus, Theorem 3.1 gives rise to the following equivalence which was used by Ghilardi and Meloni [4] in their relational semantics.

Corollary 3.3 If $B$ is a preordered set, then $\text{Rel}^B \simeq \text{Pr}/B$.

4 Change of Base and Exponentiability

A functor $p: E \to B$ induces a 2-functor $p^*: \text{Rel}^B \to \text{Rel}^E$, given by $(p^* X)_e = X_{pe}$ with $x \to x'$ whenever $x \to_p x'$ in $X$, and $(p^* f)_e = f_{pe}$, since $f \to f'$ implies $p^* f \to p^* f'$. A straightforward calculation shows that $p^*$ has a left adjoint $\Sigma_p$ given by $(\Sigma_p Y)_b = \prod_{pe=b} Y_e$ with $y \to y'$ whenever $y \to \xi y'$ for some $\xi$ such that $p\xi = \beta$, and $(\Sigma_p g)_b = \prod_{pe=b} g_e$.

In this section, we will show that $p^*$ has a right adjoint if and only if $p$ satisfies the weak factorization lifting property (WFLP) introduced in the previous section, and then establish a connection to exponentiability in $\text{Cat}_f/B$.

To begin, one easily shows that pulling back along $p$ preserves faithful functors, i.e., if $q: F \to B$ is faithful, then so is the projection $E \times_B F \to E$ in the pullback diagram

$$
\begin{array}{ccc}
E \times_B F & \longrightarrow & F \\
\downarrow & & \downarrow q \\
E & \longrightarrow & B \\
\downarrow p & & \\
B & & \\
\end{array}
$$

and that $p^*: \text{Rel}^B \to \text{Rel}^E$ corresponds (via the equivalence of Theorem 3.1) to the pullback functor $\text{Cat}_f/B \to \text{Cat}_f/E$, also denoted by $p^*$. Moreover, by uniqueness of adjoints, $\Sigma_p: \text{Rel}^E \to \text{Rel}^B$ corresponds to the functor obtained by first composing with $p$ and then reflecting, i.e.,

$$\text{Cat}_f/E \xrightarrow{\Sigma_p} \text{Cat}/B \xrightarrow{\text{hat}} \text{Cat}_f/B$$

Theorem 4.1 The following are equivalent for a functor $p: E \to B$.

(a) $p^*: \text{Rel}^B \to \text{Rel}^E$ has a right 2-adjoint
(b) \(p^*: \text{Cat}_f/B \to \text{Cat}_f/E\) has a right 2-adjoint

(c) \(p\) satisfies the weak factorization lifting property (WFLP)

(d) The lax functor \(Lp: B \to \text{Rel}\) is a non-unitary functor

**Proof.** Since (a) \(\iff\) (b) and (c) \(\iff\) (d) in any case, it suffices to prove (c) \(\implies\) (a) and (b) \(\implies\) (c).

For (c) \(\implies\) (a), suppose \(p\) satisfies WFLP, and define a 2-functor

\[ \Pi_p: \text{Rel}^E \to \text{Rel}^B \]

as follows. Given \(Y\) in \(\text{Rel}^E\), let \((\Pi_p Y)_b\) denote the set of functions

\[ \sigma: E_b \to \prod_{pe=b} Y_e \]

such that \(\sigma e \in Y_e\) and \(\sigma e_1 \underset{i}{\to} \sigma e_2\) for all \(\nu: e_1 \to e_2\) over \(id_b\), and define \(\sigma \to \beta\ \sigma'\) if \(\sigma e \to \xi \ \sigma' e'\) for all \(\xi: e \to e'\) over \(\beta: b \to b'\).

Then (RS\(^1_\star\)) holds, since \(\sigma \to b\ \sigma\) for all \(\sigma \in (\Pi_p Y)_b\), by definition. For (RS\(^2_\star\)), suppose \(\sigma \to \beta\ \sigma'\) and \(\sigma' \to \beta'\ \sigma''\), where \(\beta: b \to b'\) and \(\beta': b' \to b''\).

To see that \(\sigma \to \beta\ \sigma''\) for \(\beta'' = \beta' \beta\), suppose \(\xi'': e \to e''\) over \(\beta''\). Applying WFLP, there exists a factorization \(\xi'' = \xi' \xi\) such that \(p\xi = \beta\) and \(p\xi' = \beta'\). Then \(\sigma e \to \xi' \ \sigma' e'\) and \(\sigma' e' \to \xi'' \ \sigma'' e''\), and so \(\sigma e \to \xi'' \ \sigma'' e''\), since \(Y\) is a \(\text{Rel}\)-set over \(E\). Therefore, \(\sigma \to \beta''\ \sigma''\), and it follows that \(\Pi_p Y\) is a \(\text{Rel}\)-set over \(B\).

A morphism \(g: Y \to Z\) of \(\text{Rel}^E\) induces a function

\[ (\Pi_p g)_b: (\Pi_p Y)_b \to (\Pi_p Z)_b \]

for each \(b\), which takes \(\sigma\) to the composite

\[ E_b \xrightarrow{\sigma} \prod_{pe=b} Y_e \xrightarrow{\prod_{pe=b} g_e} \prod_{pe=b} Z_e \]

A straightforward calculation shows that this is a morphism of \(\text{Rel}^B\).

To see that a 2-cell \(g \to g'\) gives rise to \(\Pi_p g \to \Pi_p g'\), we must show that \((\Pi_p g)_b \sigma \to (\Pi_p g')_b \sigma\), for all \(\sigma \in (\Pi_p Y)_b\), i.e., \(g_{e_2} \sigma e_1 \to g'_{e_2} \sigma e_2\), for all \(\nu: e_1 \to e_2\) over \(id_b\). Now, \(\sigma e_1 \to \sigma e_2\) since \(\sigma \in (\Pi_p Y)_b\), and so \(g_{e_1} \sigma e_1 \to g_{e_2} \sigma e_2\) since \(g\) is a morphism. Also, \(g_{e_2} \sigma e_2 \to g'_{e_2} \sigma e_2\), since \(g \to g'\), and so \(g_{e_1} \sigma e_1 \to g'_{e_2} \sigma e_2\) follows from transitivity in \(Z\).
It remains to show that $p^*$ is left adjoint to $\Pi_p$. For the counit $\varepsilon: p^*\Pi_p \to id$, note that $(p^*\Pi_p Y)_e = (\Pi_p Y)_{pe}$, where elements are functions
\[
\sigma: E_{pe} \to \prod_{pe'=pe} Y_{e'}
\]
such that $\sigma e' \in Y_{e'}$ and $\sigma e_1 \to e' \sigma e_2$ for all $e': e'_1 \to e'_2$ over $id_{pe}$. Then one can show that the evaluation map $(\varepsilon_Y)_e: (\Pi_p Y)_{pe} \to Y_e$ given by $(\varepsilon_Y)_e \sigma = \sigma e$, defines a morphism of $Rel^E$. To define the unit $\eta: id \to \Pi_p p^*$, note that $(\Pi_p p^* X)_b$ is the set of functions
\[
\sigma: E_b \to \prod_{pe=b} X_{pe}
\]
such that $\sigma e \in X_{pe}$ and $\sigma e_1 \to e \sigma e_2$ for all $e: e_1 \to e_2$ over $id_b$. Then the function $(\eta_X)_b: X_b \to (\Pi_p p^* X)_b$, which takes $x$ to the constant function at $x$
\[
[x]: E_b \to \prod_{pe=b} X_{pe}
\]
defines a morphism of $Rel^B$. One checks that the adjunction identity holds, to complete the proof of $(c) \Rightarrow (a)$.

For $(b) \Rightarrow (c)$, suppose $p^*: Cat_f/B \to Cat_f/E$ has a right adjoint. To see that $p$ satisfies WFLP, suppose $\xi''': e \to e''$ in $E$ and $p\xi'''' = \beta''\beta'$, where $\beta: pe \to b'$ and $\beta': b' \to pe''$. Then the composite $\beta''\beta'$ gives rise to a pushout in $Cat_f/B$ of the form

\[
\begin{array}{ccc}
1 & \xrightarrow{1} & 2 \\
\downarrow & & \downarrow \\
2 & \xrightarrow{\beta'} & 3 \\
\downarrow \beta & & \downarrow \beta'' \beta' \\
& B & \\
\end{array}
\]

where 2 and 3 are the categories $0 \to 1$ and $0 \to 1 \to 2$, respectively.

Since $p^*$ preserves pushouts (being a left adjoint), we get a corresponding pushout in $Cat_f/E$

\[
\begin{array}{ccc}
E \times_B 1 & \xrightarrow{1} & E \times_B 2 \\
\downarrow & & \downarrow \\
E \times_B 2 & \xrightarrow{p^* \beta'} & E \times_B 3 \\
\downarrow p^* \beta & & \downarrow \\
& E & \\
\end{array}
\]

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Since $\text{Cat}_f/E$ is a reflective subcategory of $\text{Cat}/E$, pushouts are formed in $\text{Cat}/E$ and reflected to $\text{Cat}_f/E$. Thus, the pushout $P \to E$ of this diagram can be constructed as follows. Let $E_\beta$ and $E_{\beta'}$ denote the subcategories of $E$ obtained by identifying $p^*\beta$ and $p^*\beta'$ with their images in $E$. Then the objects of $P$ are the union of those of $E_\beta$ and $E_{\beta'}$, and the morphisms are those of $E_\beta$ and $E_{\beta'}$ together with pairs $(\xi, \xi'): e \to e''$ such that $\xi: e \to e'$ in $E_\beta$ and $\xi': e' \to e''$ in $E_{\beta'}$, subject to an appropriate equivalence relation. Since $\xi'': e \to e''$ corresponds to a morphism of $E \times_B 3$, and hence one of $P$ over $\beta\beta'$, the desired factorization of $\xi''$ follows, to complete the proof.

Recall that, for a category $\mathcal{A}$ with binary products, an object $X$ is called exponentiable if the functor $X \times -: \mathcal{A} \to \mathcal{A}$ has a right adjoint, and $\mathcal{A}$ is called cartesian closed if every object is exponentiable. By Corollary 3.2, $\text{Rel}^B$ has products and $X \times Y$ is given by $(X \times Y)_b = X_b \times Y_b$ with $(x, y) \to (x', y')$ whenever $x \to \beta x'$ and $y \to \beta y'$. It turns out that $\text{Rel}^B$ is not cartesian closed, and so exponentiable objects are of interest there.

Now, it is well-known that if $\mathcal{A}$ is has pullbacks and $p: E \to B$ is a morphism of $\mathcal{A}$, then the pullback functor $p^*: \mathcal{A}/B \to \mathcal{A}/E$ has a left adjoint (denoted by $\Sigma p$) defined by composition with $p$. Moreover, $p^*$ has a right adjoint (denoted $\Pi p$) if and only if $p: E \to B$ is exponentiable in $\mathcal{A}/B$ (e.g., see [9]).

Thus, when $p: E \to B$ is faithful, it is an object of $\text{Cat}_f/B$, and so Theorem 4.1 yields:

**Corollary 4.2** The following are equivalent for a faithful functor $p: E \to B$.

(a) $p^*: \text{Rel}^B \to \text{Rel}^E$ has a right 2-adjoint

(b) $Lp: B \to \text{Rel}$ is 2-exponentiable in $\text{Rel}^B$

(c) $p^*: \text{Cat}_f/B \to \text{Cat}_f/E$ has a right 2-adjoint

(d) $p: E \to B$ is 2-exponentiable in $\text{Cat}_f/B$

(e) $p$ satisfies WFLP

(f) $Lp: B \to \text{Rel}$ is a non-unitary functor

Using the equivalence $\text{Cat}_f/B \simeq \text{Rel}^B$, one obtains:

**Corollary 4.3** The following are equivalent for a $\text{Rel}$-set $X$ on $B$. 
(a) $X$ is 2-exponentiable in $\text{Rel}^B$

(b) Given $x \rightarrow_{\beta''} x''$ and a factorization $\beta'' = \beta' \beta$, there exists $x'$ such that $x \rightarrow_{\beta} x'$ and $x' \rightarrow_{\beta'} x''$

(c) The lax functor $X : B \rightarrow \text{Rel}$ is a non-unitary functor

Also, the equivalence $\text{Rel}^B \simeq \text{Pr}/B$ of Corollary 3.3 yields:

**Corollary 4.4** The following are equivalent for $p : E \rightarrow B$ in $\text{Pr}/B$.

(a) $p^* : \text{Rel}^B \rightarrow \text{Rel}^E$ has a right 2-adjoint

(b) $Lp : B \rightarrow \text{Rel}$ is 2-exponentiable in $\text{Rel}^B$

(c) $p^* : \text{Pr}/B \rightarrow \text{Pr}/E$ has a right 2-adjoint

(d) $p : E \rightarrow B$ is 2-exponentiable in $\text{Pr}/B$

(e) $p$ satisfies WFLP, i.e., if $e \leq e''$ in $E$ and $pe \leq b' \leq pe''$ in $B$, there exists $e' \in E$ such that $e \leq e' \leq e''$ and $pe' = b'$

(f) The lax functor $Lp : B \rightarrow \text{Rel}$ is a non-unitary functor

5 **Application to Granularity**

In [11], Stell uses unitary lax functors $X : T \rightarrow \text{Rel}$, on a (finite) poset $T$, to model data varying over time. In this context, $T$ is called a *time domain* and $X$ is a *dynamic set*. Thus, a dynamic set is just a unitary $\text{Rel}$-set. The laxity is intended to account for data such as “countries or states which have had multiple episodes of existence through history, such as Austria [11].” A classification structure is then added so that objects of the data base can be further identified, for example, as roads, railways, houses, or by other features.

A classification structure is a preordered set $(\Phi, \rightarrow)$ together with a partial order $\leq$ (denoted by $\sqsubseteq$ in [11]) on $\Phi$. In practice, $\phi \leq \phi'$ indicates that $\phi'$ is a more general class than $\phi$, e.g., a building is more general than a house. And, $\rightarrow$ signifies that one class can evolve into another, e.g., a child can become an adult. The structure is assumed to satisfy

(C1) For all $\phi \rightarrow \phi'$, there exists $\psi$ with $\phi \leq \psi$ and $\phi' \leq \psi$.  

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(C2) Any set of elements in the same connected component of \((\Phi, \leq)\) has a least upper bound.

(C3) These least upper bounds preserve \(\rightarrow\), in the sense that, if some element of one such set \(A\) can evolve into an element of another set \(B\), then \(\text{lub} \ A \rightarrow \text{lub} \ B\).

A classification of a \(\text{Rel}\)-set \(X\) is given by a family of functions \(\lambda_t: X_t \rightarrow \Phi, \ t \in T\), such that

\[
\lambda_t x \rightarrow \lambda_{t'} x', \quad \text{for all } x \rightarrow x' x \in X_t, x' \in X_{t'},
\]

where the subscript has been omitted on \(x \rightarrow x'\) since there is at most one morphism \(t \rightarrow t'\) in \(T\). Thus, a classification on \(X\) is just a morphism \(\lambda: X \rightarrow T^* \Phi\) in \(\text{Rel}^T\), where \(T^* \Phi\) is the image of \(\Phi\) under the functor

\[
\text{Pr} \simeq \text{Rel}^1 \xrightarrow{T^*} \text{Rel}^T
\]

and \(T\) denotes the unique morphism \(T \rightarrow 1\). Then a classified dynamic set (in the sense of \([11]\)) is just a unitary classified \(\text{Rel}\)-set.

Loss of detail in the time domain is represented in \([11]\) via a simplification from \(T\) to \(S\), which is a span

\[
\begin{array}{c}
T \\
p \downarrow \\
U \\
\downarrow q \\
S
\end{array}
\]

of order-preserving maps, where \(p\) is injective and \(q\) is surjective. In \([11]\), the author constructs the simplification of a classified dynamic set \(X\) over \(T\) to one over \(S\). Properties (C1)–(C3) play a crucial role in obtaining the classification in the unitary case.

In what follows, we obtain this construction in stages. For general \(\text{Rel}\)-sets, we get a simplification from \(T\) to \(S\) via the functor

\[
\text{Rel}^T \xrightarrow{p^*} \text{Rel}^U \xrightarrow{\Sigma_q} \text{Rel}^S
\]

To obtain the simplification of a dynamic set (in the sense of \([11]\)), we first introduce a unitary reflection, and then we adapt the construction to the classified case.
Let $\text{Dyn}^T$ denote the full subcategory of $\text{Rel}^T$ consisting of dynamic sets, and define $(\bar{\ }) : \text{Rel}^T \to \text{Dyn}^T$ as follows. Given a $\text{Rel}$-set $X$, let $\sim_t$ (or simply $\sim$) denote the equivalence relation on $X_t$ generated by $\to$, and let $\bar{X}_t = X_t / \sim_t$. For $t \leq t'$, $x \in \bar{X}_t$, and $x' \in \bar{X}_{t'}$, define $x \to x'$ if 

\[ x \to x_1 \sim_{t_1} x'_1 \to \cdots \to x_n \sim_{t_n} x'_n \to x' \]

for some $x_1, x'_1, \ldots, x_n, x'_n$. Then $\to$ is well-defined and makes $\bar{X}$ into a dynamic set on $T$. Note that $\bar{X}$ would not necessarily be unitary if $T$ were merely a preordered set. If $f : X \to Y$ is a morphism of $\text{Rel}$-sets on $T$, then $\bar{f}_t : \bar{X}_t \to \bar{Y}_t$, given by $\bar{f}_t(x) = \bar{f}(\bar{x})$, is well-defined and provides a $\text{Rel}$-set morphism $\bar{f} : \bar{X} \to \bar{Y}$, since $f$ is order-preserving.

**Proposition 5.1** The functor $(\bar{\ }) : \text{Rel}^T \to \text{Dyn}^T$ is left 2-adjoint to the inclusion.

**Proof.** With the unit $\eta_X : X \to \bar{X}$ given by $\eta_X(x) = \bar{x}$ and the counit by the “identity” functor, the adjunction identities easily follow.

Now, suppose $T \xleftarrow{p} U \xrightarrow{q} S$ is any span of posets. Then

\[ \text{Dyn}^T \xrightarrow{p^*} \text{Rel}^U \xrightarrow{\Sigma q} \text{Rel}^S \xrightarrow{(\bar{\ })} \text{Dyn}^S \]

gives a simplification functor for (non-classified) dynamic sets. If $p$ is injective, then $p^*$ preserves dynamic sets, since unitary $\text{Rel}$-sets on $T$ correspond to posets with discrete fibers over $T$, and so this simplification functor becomes

\[ \text{Dyn}^T \xrightarrow{p^*} \text{Dyn}^U \xrightarrow{\Sigma q} \text{Rel}^S \xrightarrow{(\bar{\ })} \text{Dyn}^S \]

Next, we consider classified $\text{Rel}$-sets. Let $\text{Rel}^T/\Phi$ denote the category whose objects are classified $\text{Rel}$-sets on $T$, i.e., $\lambda : X \to T^*\Phi$ in $\text{Rel}^T$, with morphisms given by triangles

\[ \xymatrix{ X \ar[r]^f \ar[dr]_{\lambda} & Y \\
& T^*\Phi \ar[ur]_{\mu} & } \]

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i.e., morphisms \( f: X \to Y \) in \( \text{Rel}^T \) such that \( \lambda_t x \leq \mu_t f_t x \), for all \( x \in X_t \).

Given a morphism \( p: U \to T \) of posets, it is easy to show that the adjunction \( \Sigma_p \dashv p^* \) restricts to

\[
\begin{array}{c}
\text{Rel}^U \sslash \Phi & \xrightarrow{\Sigma_p} & \text{Rel}^T \sslash \Phi \\
\xleftarrow{p^*} & & \\
\end{array}
\]

Thus, for a span \( T \leftarrow U \xrightarrow{q} S \), we get a simplification functor for classified \( \text{Rel} \)-sets

\[
\text{Rel}^T \sslash \Phi \xrightarrow{p^*} \text{Rel}^U \sslash \Phi \xrightarrow{\Sigma_q} \text{Rel}^S \sslash \Phi
\]

Note that (C1)–(C3) were not needed for this general (not necessarily unitary) case which was not considered in [11].

Now, let \( \text{Dyn}^T \sslash \Phi \) be the full subcategory of \( \text{Rel}^T \sslash \Phi \) consisting of classified dynamic sets. Then the reflection \( (\overline{\cdot}): \text{Rel}^T \sslash \Phi \to \text{Dyn}^T \sslash \Phi \) extends to the classified case, as follows. Given a classified \( \text{Rel} \)-set \( \lambda: X \to T^* \Phi \), using (C1) and (C2), define \( \overline{\lambda}: \overline{X} \to \Phi \) by \( \overline{\lambda} \overline{x} = \text{lub} \{ \lambda_t a | a \sim x \} \). Then, by property (C3), \( \overline{\lambda} \) is a classification on \( \overline{X} \), i.e.,

\[
\overline{x} \to \overline{x}' \Rightarrow \overline{\lambda} \overline{x} \to \overline{\lambda} \overline{x}'
\]

for all \( x \in X_t, x' \in X_{t'} \), and \( t \leq t' \). To see that \( (\overline{\cdot}): \text{Rel}^T \sslash \Phi \to \text{Dyn}^T \sslash \Phi \) is a functor, suppose \( f: X \to Y \) is a morphism of classified \( \text{Rel} \)-sets. Then so is \( \overline{f} \), that is,

\[
\overline{X} \xrightarrow{\overline{f}} \overline{Y}
\]

since \( \overline{\lambda} \overline{x} = \text{lub} \{ \lambda_t a | a \sim x \} \leq \text{lub} \{ \mu_t f_t a | a \sim f_t x \} \leq \text{lub} \{ \mu_t b | b \sim f_t x \} = \overline{\mu} \overline{f_t x} \).

**Lemma 5.2** The functor \( (\overline{\cdot}): \text{Rel}^T \sslash \Phi \to \text{Dyn}^T \sslash \Phi \) is left adjoint to the inclusion.

**Proof.** It suffices to show that the unit \( \eta_X: X \to \overline{X} \) (given by \( \eta_X(x) = \overline{x} \) in the proof of Proposition 5.1) is a morphism of \( \text{Dyn}^T \sslash \Phi \). But, this is clear, since \( \lambda_t x \leq \text{lub} \{ \lambda_t a | a \sim x \} = \overline{\lambda} \overline{x} = \overline{\lambda} \eta_x \).

Thus, we get:
Theorem 5.3 The simplification of classified dynamic sets relative to the span $T \vdash_p U \xrightarrow{q} S$ (in the sense of [11]) is given by

$$\text{Dyn}^T/\Phi \xrightarrow{p^*} \text{Rel}^U/\Phi \xrightarrow{\Sigma_q} \text{Rel}^S/\Phi \xrightarrow{(\cdot)} \text{Dyn}^S/\Phi$$

We conclude with a consideration of right adjoints to the simplification functors.

Theorem 5.4 If $T \vdash_p U \xrightarrow{q} S$ is a span of preordered sets, then the simplification functor $\text{Rel}^T \xrightarrow{p^*} \text{Rel}^U \xrightarrow{\Sigma_q} \text{Rel}^S$ has a right adjoint if and only if $p^*$ does, i.e., $p$ is a WFLP map.

Proof. Consider the corresponding composite $\text{Pr}/T \xrightarrow{p^*} \text{Pr}/U \xrightarrow{\Sigma_q} \text{Pr}/S$. By [9, Proposition 1.1], a functor $F: \text{Pr}/T \to \text{Pr}/S$ has a right adjoint if and only if

$$\text{Pr}/T \xrightarrow{F} \text{Pr}/S \xrightarrow{\Sigma_S} \text{Pr}$$

does, and so the desired result follows.

Theorem 5.5 If $T \vdash_p U \xrightarrow{q} S$ is a span of posets and $p$ is an injective WFLP map, then the simplification functor

$$\text{Dyn}^T \xrightarrow{p^*} \text{Dyn}^U \xrightarrow{\Sigma_q} \text{Rel}^S \xrightarrow{(\cdot)} \text{Dyn}^S$$

has a right 2-adjoint.

Proof. Since $\Sigma_q: \text{Rel}^U \to \text{Rel}^S$ and $(\cdot): \text{Rel}^S \to \text{Dyn}^S$ have right adjoints and $q^*$ preserves dynamic sets, it suffices to show that $p^*$ has a right adjoint. Now, $p^* : \text{Rel}^T \to \text{Rel}^U$ does by Corollary 4.4, since $p$ satisfies WFLP, and one can show using the description in the proof of Theorem 4.1, that $\Pi_p$ preserves dynamic sets, since $p$ is injective, and the desired result follows.

For the classified case, the situation is more complicated. In particular, $\Pi_p$ cannot be easily adapted unless we impose conditions on $\Phi$ which may not make sense for the intended interpretation [11] of the relations on $\Phi$. For example, taking $T = \{0, 1\}$ and $p$ to be the inclusion of $U = \{1\}$, one can show that $\Phi$ would need an element $\phi_0$ such that $\phi \leq \phi_0$ and $\phi_0 \to \phi$, for all $\phi \in \Phi$, i.e., a class that is more general than and could evolve into any
other class. This can be seen using $Y = T^*\Phi$ and the description of $\Pi_p Y$ in Theorem 4.1. On the other hand, using the inclusion of $U = \{0\}$ in $T$, one can show that $\Phi$ would need an element $\phi_1$ such that $\phi \leq \phi_1$ and $\phi \rightarrow \phi_1$, for all $\phi \in \Phi$, i.e., $\phi_1$ would be a class that is more general than any other class and such that any class could evolve into $\phi_1$.

References


