A recurrent topic on the Putnam exam is binomial coefficients. Binomial coefficients are the numbers \( \binom{n}{k} \) (read “n choose k”) defined by 
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]
(Recall: \( n! \), read “n factorial”, is the product \( 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \).) These numbers are sometimes denoted by \( nC_k \), and they correspond to “the number of ways to select \( k \) objects out of \( n \)” or, equivalently, the number of subsets of \( \{1, 2, \ldots, n\} \) with \( k \) elements. You’re expected to know a few facts about binomial coefficients, including:

(A) Pascal’s Recursion: \( \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \).

(B) Binomial Theorem: \( (1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \).

Warm-up problems:

1. Prove (A).

2. Use (A) to prove (B) by induction on \( n \).

3. Use (B) to show the following:
   - \( \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n. \)
   - \( \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = \binom{n}{n} = 0. \)
   - \( 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}. \) [One-word hint: derivative!]

4. Find \( \binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + 7\binom{n}{3} + \cdots + (2n+1)\binom{n}{n} \).

5. Find \( \binom{n}{0} + \binom{n}{1}/2 + \binom{n}{2}/3 + \cdots + \binom{n}{n}/(n+1) \).

6. Find \( \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} \).

Now try the following:

1. Show that the coefficient of \( x^k \) in the expansion of \( (1 + x + x^2 + x^3)^n \) is
   \[
   \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.
   \]
   (From 53\textsuperscript{rd} Putnam, 1992.)

2. Find \( \binom{n}{1}1^2 + \binom{n}{2}2^2 + \binom{n}{3}3^2 + \cdots + \binom{n}{n}n^2 \). (From 23\textsuperscript{rd} Putnam, 1962.)

3. Find \( F_1\binom{n}{0} + F_2\binom{n}{1} + F_3\binom{n}{2} + \cdots + F_{n+1}\binom{n}{n} \), where \( F_1, F_2, F_3, \ldots \) are the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, \ldots.