Putnam Practice Problems for October 7, 2003

Those taking the Putnam Exam are generally expected to know some “basic” inequalities. Here are a few

(1) $\sqrt{ab} \leq \frac{a+b}{2}$, $0 < a, b$ and the generalization

(2) $(a_1a_2\cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$.

(3) $\frac{a+b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}}$, $0 < a, b$ and the generalization

(4) $\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}$

(5) $(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$

Warm-up Practice problems

(1) Prove formula (1)

(2) Now prove formula (2) (This is the arithmetic mean - geometric mean inequality.)

(3) Use (5) to deduce (4) (Inequality (5) is the Cauchy-Schwarz inequality, and (4) is arithmetic mean - quadratic mean inequality).

Now try the following:

(1) For $a, b, c \geq 0$, prove $(a + b)(b + c)(c + a) \geq 8abc$.

(2) If $a_i > 0$ for $i = 1, \ldots, n$, then

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) \geq 2^n.$$ 

(3) Prove the following inequality

$$n \left[(n+1)^{1/n} - 1\right] < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < n - (n-1)n^{-1/(n-1)}.$$