Energy Stored in a Capacitor

We saw in the demo with a large capacitor that charge can be stored for long periods of time on an isolated capacitor. Let’s calculate the amount of stored energy by imagining that the capacitor is charged by moving bits of charge \( dq \) from one plate of the capacitor to the other, starting with an uncharged capacitor. The very first bit of charge requires no work to move because there is no potential difference between the starting plates. But for every bit of charge moved thereafter, it must be moved through a potential difference that is constantly increasing as more charge is separated and the capacitor charge increases. Once there is some potential difference \( \Delta V \) between the two plates, the amount of work required to move a bit of charge \( dq \) from one plate to the other is

\[
dW = (\Delta V)dq = (q/C)dq,
\]

where \( q \) is the charge that has already been moved from one plate to the other. To be explicit, we are moving positive charge \( dq \) from the negative plate already having charge \( -q \) to the positive plate with charge \( +q \). This will clearly require positive work since the charge \( dq \) is repelled by the positive charge on the plate it is being moved toward. If this process is continued until a total charge \( Q \) has been moved, so that the final charges on the plates are \( +Q \) and \( -Q \), the total work required will be given by

\[
W = \int_0^Q \frac{q}{C}dq.
\]

Evaluating both sides we have that the total work required to charge the capacitor is

\[
W = \frac{Q^2}{2C}.
\]

This outside work to charge a capacitor is then equal to the potential energy \( U \) it stores and we can write this in several equivalent forms using \( Q = C(\Delta V) \) to eliminate each of the variables one at a time:

\[
U = \frac{Q^2}{2C} = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2.
\]

For a parallel-plate capacitor (of area \( A \) and plate separation \( d \)), one further expression that we can obtain for the potential energy is to replace \( C = \varepsilon_oA/d \), and to replace \( \Delta V = Ed \), since the \( E \) field is uniform between the plates of \( C \), to obtain

\[
U = \frac{1}{2}\varepsilon_oA(Ed)^2 = \frac{1}{2}\varepsilon_oE^2(Ad).
\]

But since \( (Ad) = V \), the volume of the space between the capacitor plates, we can find a simple expression for the energy density, or potential energy per unit volume, of the capacitor, \( u_E \),

\[
u_E = \frac{1}{2}\varepsilon_oE^2.
\]

This turns out to be a very general result, although we derived it here using the parallel-plate capacitor formula. We conclude that the energy density at a point in space is proportional to the square of the electric field at that point.

Example: Two capacitors \( C_1 \) and \( C_2 \) (where \( C_1 > C_2 \)) are charged to the same potential difference \( V_0 \), but with opposite polarity. The charged capacitors are removed from the battery and their places are connected as shown. The switches S1 and S2 are then closed as shown in (b)
(a) Find the final potential difference between a and b and (b) find the total energy stored in the capacitors before and after the switches are closed.