Abstract

$u_1, \ldots, u_r$ are in $k[[x_1, \ldots, x_s]]$ with $k$ and $\deg(u_1, \ldots, u_r)$ finite. Intending applications to Hilbert-Kunz theory, we code the numbers $\deg(u_1^{a_1}, \ldots, u_r^{a_r})$ into a function $\varphi_u$, which empirically satisfies many functional equations related to “magnification by $p$”, where $p = \text{char} k$. $p$-fractals, introduced here, formalize these ideas.

In the first interesting case ($r = 3$, $s = 2$), the $\varphi_u$ are $p$-fractals. Our proof uses functions $\varphi_I$ attached to ideals $I$ and square-free elements $h$ of $A = k[[x, y]]$. The finiteness of the set of ideal classes in $A/(h)$ and the existence of “magnification maps” on this set show the $\varphi_I$ to be $p$-fractals.

We describe further functional equations coming from a theory of reflection maps on ideal classes, and the paper concludes with examples and open questions.

1 Introduction

Let $A = k[[x_1, \ldots, x_s]]$ be the $s$-variable power series ring over a field $k$ of characteristic $p > 0$. If $I$ is an ideal of $A$, the degree of $I$, denoted by $\deg I$, is the $k$-dimension of $A/I$. We shall reserve the letter $q$ for powers of $p$; $I^{[q]}$ is the ideal generated by all $z^q$, $z \in I$. Note that $\deg I^{[q]} = q^s \deg I$.

The following two difficult questions are closely related to the theory of the Hilbert-Kunz function [4]:

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(1) Suppose \( \deg I < \infty \). How does \( \deg \left( I^{[q]}, \prod_{i=1}^{r} h_i^{a_i} \right) \), \( 0 \leq a_i \leq q \), depend on \( a_1, \ldots, a_r \)?

(2) Suppose \( \deg(u_1, \ldots, u_r) < \infty \). How does \( \deg(u_1^{a_1}, \ldots, u_r^{a_r}) \) depend on \( a_1, \ldots, a_r \)?

Question (2) has an easy answer when \( r = s \), in which case \( \deg(u_1^{a_1}, \ldots, u_s^{a_s}) = \deg(u_1, \ldots, u_s) \cdot \prod_{i=1}^{s} a_i \). Also (1) and (2) have easy answers when \( s = 1 \). To understand the general case it’s useful to encode the degrees appearing in (1) and (2) into functions \( \varphi_I, \varphi_u : \mathcal{I}^r \to \mathbb{Q} \), where \( \mathcal{I} \) consists of all rational numbers in \([0, 1]\) whose denominators are powers of \( p \). Explicitly:

(1) \( \varphi_I \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) = q^{-s} \deg \left( I^{[q]}, \prod_{i=1}^{r} h_i^{a_i} \right) \);

(2) \( \varphi_u \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) = q^{-s} \deg \left( u_1^{a_1}, \ldots, u_r^{a_r} \right) \).

Note that the right hand sides of (1) and (2) are unchanged when \( q \) is replaced by \( pq \) and \( a_i \) by \( pa_i \); this allows us to view \( \varphi_I \) and \( \varphi_u \) as functions on \( \mathcal{I}^r \).

Computer experiments indicate that these functions have remarkable self-similarity properties under “magnification” by a power of \( p \) which completely characterize them. We shall make this notion of self-similarity precise in the next section, through the introduction of \( p \)-fractals. There is experimental and theoretical evidence that the \( \varphi_I \) and \( \varphi_u \) are \( p \)-fractals. (A consequence of this would be that each function can be described by a finite set of functional equations of a simple type, and a finite set of initial values.) One result is implicit in [2] and [3]—if \( r = s + 1 \), \( u_i = x_i \) and \( u_{s+1} = \sum_{i=1}^{s} x_i \), then \( \varphi_u \) is a \( p \)-fractal.

In this paper we answer (1) and (2) in the simplest non-trivial cases, proving:

**Theorem 1** Suppose \( A = \mathbb{k}[x, y] \), with \( \mathbb{k} \) finite, and \( \deg I < \infty \). Then the function \( \varphi_I : \mathcal{I}^r \to \mathbb{Q} \), \( \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) \mapsto q^{-2} \deg \left( I^{[q]}, \prod_{i=1}^{r} h_i^{a_i} \right) \), is a \( p \)-fractal.

**Theorem 2** Suppose \( A \) is as above, and \( \deg(u_1, u_2, u_3) < \infty \). Then the function \( \varphi_u : \mathcal{I}^3 \to \mathbb{Q} \), \( \left( \frac{a_1}{q}, \frac{a_2}{q}, \frac{a_3}{q} \right) \mapsto q^{-2} \deg \left( u_1^{a_1}, u_2^{a_2}, u_3^{a_3} \right) \), is a \( p \)-fractal.

The second author proved closely related results in his thesis [5], though the context there was of polynomial rings and homogeneous polynomials and ideals. He also gave applications to proving the rationality of various Hilbert-Kunz series—we’ll do the same in sequels to this paper.

The main idea behind the proof of Theorem 1 when the \( h_i \) are pairwise prime and irreducible is the use of ideal classes in \( B = \mathbb{k}[x, y]/(\prod_{i=1}^{r} h_i) \). The fact that the \( \varphi_I \) are \( p \)-fractals is due to the finiteness of the set of ideal classes in \( B \) and the existence of certain “magnification operators” on this set. The theory of these magnification operators is developed in section 5. There are
also “reflection operators” on ideal classes. Such maps are studied briefly in section 5, and in more detail in an appendix, written by the first author. We conclude with several examples and open questions, in sections 6 and 7.

2 \textit{p-Fractals and a Finiteness Lemma}

Fix a positive integer \( r \), and let \( \mathcal{F}^r \) be the \( \mathbb{Q} \)-algebra of functions \( \varphi : \mathcal{I}^r \to \mathbb{Q} \). Let \( \varphi \in \mathcal{F}^r \), and let \( b = (b_1, \ldots, b_r) \) be an integer vector with \( 0 \leq b_i < q \). Then \( (t_1, \ldots, t_r) \mapsto \varphi \left( \frac{t_1+b_1}{q}, \ldots, \frac{t_r+b_r}{q} \right) \) is again an element of \( \mathcal{F}^r \), which will be denoted by \( T_{q,b} \varphi \). Roughly speaking, we’re breaking up \( \mathcal{I}^r \) into \( q^r \) smaller pieces, and \( T_{q,b} \varphi \) describes the restriction of \( \varphi \) to one of these pieces.

**Definition 2.1** \( \varphi \) is a \textit{p-fractal} (of dimension \( r \)) if all the \( T_{q,b} \varphi \), \( q \) a power of \( p \), \( 0 \leq b_i < q \), span a finite dimensional \( \mathbb{Q} \)-subspace \( V \) of \( \mathcal{F}^r \).

**Remark 2.2** Equivalently, \( \varphi \) is a \( p \)-fractal if it lies in a finite dimensional \( \mathbb{Q} \)-subspace of \( \mathcal{F}^r \) stable under the operators \( T_{p,b} \), \( 0 \leq b_i < p \). We call such a subspace \textit{p-stable}. One sees easily from this criterion that the \( p \)-fractals of dimension \( r \) form a \( \mathbb{Q} \)-subalgebra of \( \mathcal{F}^r \). The coordinate functions \( t_i \) are clearly \( p \)-fractals, so the same is true for polynomial functions.

**Remark 2.3** Suppose \( \varphi \) is a \( p \)-fractal, and \( V \) is as in definition 2.1. Choose \( q^* \) so that \( V \) is spanned by the \( T_{q,b} \varphi \) with \( q \leq q^* \), \( 0 \leq b_i < q \). Then each \( T_{pq^*c} \varphi \), \( 0 \leq c_i < pq^* \), can be written as a linear combination of these generators. This gives us \( (pq^*)^r \) functional equations for \( \varphi \)—each has the form

\[
\varphi \left( \frac{t_1+c_1}{pq^*}, \ldots, \frac{t_r+c_r}{pq^*} \right) = \sum_{0 \leq b_i < q \leq q^*} k(q,b,c) \varphi \left( \frac{t_1+b_1}{q}, \ldots, \frac{t_r+b_r}{q} \right),
\]

with \( k(q,b,c) \in \mathbb{Q} \). Evidently \( \varphi \) can be recovered from its values at the finitely many points \( \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) \), \( 0 \leq a_i \leq q^* \), by using these functional equations.

The rest of this section will be devoted to the treatment of a special case of Theorem 1. We shall assume that \( A = \mathbb{k}[x,y] \), with \( \mathbb{k} \) finite, that \( r = 1 \) and that \( h_1 \), which we’ll simply denote by \( h \), is irreducible in \( A \).

**Definition 2.4** If \( I \) is an ideal of \( A \) with \( \deg(I,h) < \infty \), \( \varphi_I : \mathcal{I} \to \mathbb{Q} \) is the function with \( \varphi_I \left( \frac{a}{q} \right) = q^{-2} \deg \left( I[q], h^a \right) \), \( 0 \leq a \leq q \).

Note that replacing \( I \) by \( (I,h) \) doesn’t change this function. We’ll show that the \( \varphi_I \) are all \( p \)-fractals. This is accomplished by showing that the \( \mathbb{Q} \)-vector space spanned by the constant function 1 and all the \( \varphi_I \) is \( p \)-stable and finite dimensional. (The finiteness of \( \mathbb{k} \) and the irreducibility of \( h \) are needed in the proof of finite dimensionality.)
Definition 2.5 Let $B$ be a domain, and $I$ and $J$ be nonzero ideals of $B$. We say that $I$ and $J$ are equivalent if there are nonzero elements $u$ and $v$ of $B$ with $uI = vJ$.

Note that we don’t assume that $I$ and $J$ are invertible—so we don’t have a natural group structure on the ideal classes. The following lemma is crucial to us:

Lemma 2.6 Let $(B, \mathfrak{m})$ be a complete local Noetherian domain of dimension one, with finite residue field. Then there are only finitely many equivalence classes of ideals in $B$.

Proof. Let $D$ be the integral closure of $B$ in its field of fractions. Since $B$ is complete local, $D$ is complete local and a finite $B$-module. So the conductor $J = (B : D)$ is a nonzero ideal of $B$. Since $B/\mathfrak{m}$ is finite, the finitely generated torsion $B$-module $D/J$ is finite.

The definition of equivalence of ideals extends naturally to nonzero $B$-submodules $M$ of $D$. Now since $B$ is one-dimensional, $D$ is a discrete valuation ring, and therefore is a principal ideal domain. Let $I$ be a nonzero ideal of $B$. Then $ID = dD$, for some $d \neq 0$. Set $M = d^{-1}I$. Then $M$ is a $B$-submodule of $D$ with $MD = D$. Since $M \supseteq M(JD) = JD = J$ and $D/J$ is finite, there are only finitely many possibilities for $M$. But $I$ and $M$ are equivalent, giving the lemma. $\Box$

Note that for any ideal $J$ of $A$ and $f \in A$ there is an exact sequence

$$0 \to A/(J : f) \overset{J}{\to} A/J \to A/(J, f) \to 0,$$  \hspace{1cm} (2)

which gives

$$\deg(J) = \deg(J, f) + \deg(J : f).$$  \hspace{1cm} (3)

Replacing $J$ by $(J, fg)$, Eq. (3) becomes

$$\deg(J, fg) = \deg(J, f) + \deg((J : f), g).$$  \hspace{1cm} (4)

Lemma 2.7 If $u$ and $g \in A$ have no common factor, and $I$ is an ideal of $A$ with $\deg(I, g) < \infty$, then $\deg(uI, g) = \deg(I, g) + \deg(u, g)$.

Proof. This follows from Eq. (3), with $J = (uI, g)$ and $f = u$, noticing that $((uI, g) : u) = (I, g)$, since $u$ and $g$ have no common factor. $\Box$

Lemma 2.8 Suppose $\deg I < \infty$ and $u \in A$ is not divisible by $h$. Set $J = (uI, h)$. Then $\varphi_J(t) = \varphi_I(t) + ct$, where $c = \deg(u, h)$. 
Proof. Lemma 2.7 shows that

\[
\deg \left( J^{[q]} : h^a \right) = \deg \left( u^q I^{[q]} : h^a \right) = \deg \left( I^{[q]} : h^a \right) + \deg(u, h^a) = \deg \left( I^{[q]} : h^a \right) + aq \deg(u, h), \tag{5}
\]

for all \( q \) and \( 0 \leq a \leq q \). Dividing by \( q^2 \) we find that

\[
\varphi_J \left( \frac{a}{q} \right) = \varphi_I \left( \frac{a}{q} \right) + c \cdot \frac{a}{q}. \tag{6}
\]

\( \square \)

**Proposition 2.9** The \( \mathbb{Q} \)-vector space spanned by the \( \varphi_I \) is finite dimensional.

Proof. Since \( \varphi_I = \varphi(I, h) \), we may restrict our attention to ideals \( I \) containing \( h \). These are in one-to-one correspondence with nonzero ideals in \( B = A/(h) \), a complete one dimensional local Noetherian domain, with finite residue field. Lemmas 2.6 and 2.8 show that, modulo integer multiples of \( t \), there are only finitely many distinct functions \( \varphi_I \), giving the proposition. \( \square \)

**Lemma 2.10** Suppose \( \deg I < \infty \), \( 0 \leq b < p \), and \( J = (I^{[p]} : h^b) \). Then

\[
\varphi_J = p^2 T_{p|b} \varphi_I - c, \text{ where } c = \deg \left( I^{[p]} : h^b \right).
\]

Proof. Applying Eq. (4) we find that

\[
\deg \left( J^{[q]} : h^a \right) = \deg \left( (I^{[pq]} : h^{bq}) : h^a \right) = \deg \left( I^{[pq]} : h^{a+bq} \right) \tag{6}
\]

for all \( q \) and \( 0 \leq a \leq q \). Dividing by \( q^2 \) we get

\[
\varphi_J \left( \frac{a}{q} \right) = p^2 \varphi_I \left( \frac{a+bq}{pq} \right) - c = p^2 T_{p|b} \varphi_I \left( \frac{a}{q} \right) - c. \tag{7}
\]

\( \square \)

**Proposition 2.11** Suppose \( A = k[x, y] \), with \( k \) finite. Let \( I \) be an ideal of \( A \) with \( \deg I < \infty \), and \( h \) an irreducible element of \( A \). Then the function

\[
\varphi_I : \mathcal{I} \to \mathbb{Q}, \ \frac{a}{q} \mapsto q^{-2} \deg \left( I^{[q]} : h^a \right), \text{ is a } p\text{-fractal.}
\]
PROOF. In view of proposition 2.9, it suffices to show that the \( \mathbb{Q} \)-vector space spanned by the constant function 1 and all the \( \varphi_I \) is \( p \)-stable. But this is immediate from lemma 2.10. \( \square \)

3 Theorem 1

In this section we prove Theorem 1. Once again \( A = \mathbb{k}[x,y] \), with \( \mathbb{k} \) finite, but now \( r \) is arbitrary. We assume at first that the \( h_i \) are pairwise prime irreducible elements of \( A \), and later we remove that restriction. Set \( h = \prod_{i=1}^r h_i \).

Definition 3.1 If \( I \) is an ideal of \( A \) with \( \deg(I,h) < \infty \), \( \varphi_I : \mathfrak{X}^r \to \mathbb{Q} \) is the function with \( \varphi_I \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) = q^{-2} \deg \left( I[i], \prod_{i=1}^r h_i^{a_i} \right) \).

Note that replacing \( I \) by \( (I,h) \) doesn’t change the function defined above. A slight modification of the argument of the last section will show that the \( \varphi_I \) are all \( p \)-fractals. Set \( B = A/(h) \). \( B \) is a one dimensional reduced complete Noetherian local ring, with minimal primes \( (h_i) \) and finite residue field. Consider ideals \( I \) of \( B \) not contained in any \( (h_i) \). (These are just the ideals containing a non zero-divisor; alternatively, they are the ideals \( I \) with \( B/I \) finite.) We say that two such ideals \( I \) and \( J \) are equivalent if \( uI = vJ \), with \( u \) and \( v \) non zero-divisors in \( B \).

Lemma 3.2 There are only finitely many equivalence classes of ideals in \( B \).

Proof. Let \( S \) be the set of non zero-divisors in \( B \), and \( D \) be the integral closure of \( B \) in \( S^{-1}B \). It’s not hard to see that \( D/B \) is finite. In fact, let \( L_i \) be the field of quotients of \( A/(h_i) \), and \( D_i \) be the integral closure of \( A/(h_i) \) in \( D_i \). Then \( S^{-1}B \) identifies with \( \prod_{i=1}^r L_i \) and \( D \) identifies with \( \prod_{i=1}^r D_i \). We have \( B \hookrightarrow \prod_{i=1}^r A/(h_i) \hookrightarrow \prod_{i=1}^r D_i = D \) and, since \( A/(h_i) \hookrightarrow D_i \) has finite cokernel (see the proof of lemma 2.6), it suffices to show that \( B \hookrightarrow \prod_{i=1}^r A/(h_i) \) has finite cokernel. But this is clear—the localization of this inclusion at each height zero prime of \( B \) is onto.

The conductor \( J = (B : D) \) is an ideal of \( B \) containing a non zero-divisor, and \( D/J \) is finite. Extend the notion of equivalence to \( B \)-submodules \( M \) of \( D \) in the obvious way, and let \( I \) be an ideal of \( B \) containing a non zero-divisor. Since each \( D_i \) is a principal ideal domain, \( ID = dD \) for some non zero-divisor \( d \) of \( D \). Arguing as in the proof of lemma 2.6 we find that \( I \) is equivalent to \( M = d^{-1}I \) and that \( M \supseteq J \). Since there are only finitely many \( B \)-submodules between \( J \) and \( D \), the lemma follows. \( \square \)
Remark 3.3 When $k$ is infinite one can prove a weakened form of lemma 3.2—there is a non zero-divisor $u$ in $B$ such that every ideal class is represented by some ideal containing $u$. To see this we choose a non zero-divisor $w \in J = (B : D)$ and let $u = w^2$. Then since $MD = D$, $wMD$ is contained in $B$, and $wM$ is an ideal of $B$ in the class of $I$. Since $M$ contains $J$, $wM$ contains $w^2 = u$.

Lemma 3.4 Suppose $\deg I < \infty$ and $u \in A$ is prime to each $h_i$. Set $J = (uI, h)$. Then $\varphi_J(t_1, \ldots, t_r) = \varphi_I(t_1, \ldots, t_r) + \sum_{i=1}^r c_i t_i$, where $c_i = \deg(u, h_i)$.

Proof. By lemma 2.7 we have

\[
\deg \left( J^{[q]}_i, \prod_{i=1}^r h_i^{a_i} \right) = \deg \left( u^q I^{[q]}_i, \prod_{i=1}^r h_i^{a_i} \right) = \deg \left( I^{[q]}_i, \prod_{i=1}^r h_i^{a_i} \right) + \deg \left( u^q, \prod_{i=1}^r h_i^{a_i} \right) = \deg \left( I^{[q]}_i, \prod_{i=1}^r h_i^{a_i} \right) + \sum_{i=1}^r a_i q \deg(u, h_i),
\]

for all $q$ and $0 \leq a_i \leq q$, and dividing by $q^2$ we obtain the desired result. \(\square\)

Proposition 3.5 The $\mathbb{Q}$-vector space spanned by the $\varphi_I$ is finite dimensional.

Proof. This follows from lemmas 3.2 and 3.4—see the proof of proposition 2.9. \(\square\)

Lemma 3.6 Suppose $\deg I < \infty$, $0 \leq b_i < p$, and $J = (I^{[p]} : \prod_{i=1}^r h_i^{b_i})$. Then $\varphi_J = p^2 T_{p,b} \varphi_I - c$, where $c = \deg \left( I^{[p]}, \prod_{i=1}^r h_i^{b_i} \right)$.

Proof. Eq. (4) shows that

\[
\deg \left( J^{[q]}_i, \prod_{i=1}^r h_i^{a_i} \right) = \deg \left( I^{[pq]}_i, \prod_{i=1}^r h_i^{b_i+q} \right) - q \deg \left( I^{[pq]}_i, \prod_{i=1}^r h_i^{b_i} \right) = \deg \left( I^{[pq]}_i, \prod_{i=1}^r h_i^{a_i+b_i+q} \right) - q^2 c,
\]

for any $q$ and $0 \leq a_i \leq q$, and dividing by $q^2$ yields the desired formula. \(\square\)
Proposition 3.7 Suppose $A = \mathbb{k}[x, y]$, with $\mathbb{k}$ finite, that $I$ is an ideal of $A$ with $\deg I < \infty$, and that $h_1, \ldots, h_r$ are pairwise prime irreducible elements of $A$. Then the function $\varphi_I : I^r \rightarrow \mathbb{Q}$, $(\frac{a_1}{q}, \ldots, \frac{a_r}{q}) \mapsto q^{-2} \deg \left( I^{[q]} : \prod_{i=1}^{r} h_i^{a_i} \right)$, is a $p$-fractal.

**Proof.** The $\mathbb{Q}$-vector space spanned by the constant function 1 and all the $\varphi_i$ is finite dimensional, by proposition 3.5, and $p$-stable, by lemma 3.6. □

We now remove the restriction on the $h_i$—suppose $h_i$ are arbitrary non-zero elements of $A$, and write $h_i = (\text{unit}) \cdot \prod_{j=1}^{m} g_j^{c_{ij}}$, where $g_1, \ldots, g_m$ are pairwise prime irreducible elements of $A$. Let $\ell_1, \ldots, \ell_m$ be the linear forms $\sum_{i=1}^{r} c_{ij} T_i$, and $\ell = (\ell_1, \ldots, \ell_m)$. Then for each $t = \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) \in I^r$ we have

$$\varphi_I(t) = q^{-2} \deg \left( I^{[q]} : \prod_{i=1}^{r} h_i^{a_i} \right) = q^{-2} \deg \left( I^{[q]} : \prod_{j=1}^{m} g_j^{\ell_j(a)} \right). \quad (10)$$

Let $\mathcal{D}$ consist of all $m$-tuples $d = (d_1, \ldots, d_m)$ of integers with $0 \leq d_j \leq \sum_{i=1}^{r} c_{ij}$. For each $d \in \mathcal{D}$, $X(d)$ is the set of all $t = (t_1, \ldots, t_r) \in I^r$ with $d_j \leq \ell_j(t) \leq 1 + d_j$, for all $j$. Note that since each $t_i$ is in the interval $[0, 1]$, the $X(d)$ cover $I^r$.

If $t = \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) \in X(d)$, then $\ell_j(a) \in [d_j q, q + d_j q]$, and Eq. (4) shows that

$$\varphi_I(t) = q^{-2} \deg \left( \prod_{j=1}^{m} g_j^d \prod_{j=1}^{m} g_j^{\ell_j(a) - d_j q} \right) = q^{-2} \deg \left( I^{[q]} : \prod_{j=1}^{m} g_j^d \right) + q^{-2} \deg \left( \prod_{j=1}^{m} g_j^{\ell_j(a) - d_j q} \right) = \psi_d(\ell(t) - d), \quad (11)$$

where $\psi_d$ is the $m$-dimensional $p$-fractal

$$\left( \frac{b_1}{q}, \ldots, \frac{b_m}{q} \right) \mapsto \deg \left( I : \prod_{j=1}^{m} g_j^{b_j} \right) + q^{-2} \deg \left( \prod_{j=1}^{m} g_j^{b_j} \right). \quad (12)$$
Let $V$ be a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{F}^m$ containing the $\psi_d$. Let $V^*$ consist of all maps $\varphi : \mathcal{F}^r \to \mathbb{Q}$ whose restriction to each $X(d)$ is $t \mapsto \psi(\ell(t) - d)$, for some $\psi \in V$. To complete the proof of Theorem 1 it suffices to show that $V^*$ is finite dimensional and $p$-stable. This is done in the following lemma:

**Lemma 3.8** Let $\ell_1, \ldots, \ell_m$ be linear forms with nonnegative integer coefficients, and $\ell = (\ell_1, \ldots, \ell_m)$. Let $\mathcal{D}$ consist of all integer vectors $(d_1, \ldots, d_m)$ with $0 \leq d_j \leq \ell_j(1, \ldots, 1)$. For each $d = (d_1, \ldots, d_m) \in \mathcal{D}$, $X(d)$ is the set of all $t \in \mathcal{F}^r$ with $d_j \leq \ell_j(t) \leq 1 + d_j$. Suppose $V$ is a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{F}^m$, and let $V^*$ consist of all $\varphi : \mathcal{F}^r \to \mathbb{Q}$ whose restriction to each $X(d)$ is $t \mapsto \psi(\ell(t) - d)$, for some $\psi \in V$. Then $V^*$ is a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{F}^r$.

**Proof.** Suppose that $\psi : d \mapsto \psi_d$ is in $V^\mathbb{Q}$, the finite dimensional space of functions from $\mathcal{D}$ to $V$. Then there is at most one $\varphi : \mathcal{F}^r \to \mathbb{Q}$ such that for each $d$ in $\mathcal{D}$ the restriction of $\varphi$ to $X(d)$ is $t \mapsto \psi_d(\ell(t) - d)$. We say that $\psi$ is compatible if such a $\varphi$ exists. The compatible $\psi$ form a subspace of $V^\mathbb{Q}$. Since each $\varphi \in V^*$ is attached to a compatible $\psi$, $V^*$ is finite dimensional.

Let $\varphi \in V^*$ and $b = (b_1, \ldots, b_r)$ with $0 \leq b_i < p$. Then $T^{p\varphi}(t) = \varphi(t^*)$, where $t^* = p^{-1}(t + b)$. Fix $d$. Suppose $t \in X(d)$, and write $d_j + \ell_j(b)$ as $p\ell_j(c_j)$, for $0 \leq c_j < p$. Then $\ell_j(t^*) = p^{-1}(\ell_j(t) - d_j + c_j) + d_j^*$, which lies in $[d_j^*, 1 + d_j^*]$, so that $t^* \in X(d^*)$. Consequently there is a $\psi \in V$ such that $T^{p\varphi}(t) = \psi(\ell(t^*) - d^*) = \psi(p^{-1}(\ell(t) - d + c)) = (T^{p\varphi})\psi(\ell(t) - d)$. So the restriction of $T^{p\varphi}$ to $X(d)$ is $t \mapsto \psi^*(\ell(t) - d)$ for some $\psi^*$ in $V$. As this holds for all $d$, $T^{p\varphi}$ is in $V^*$, and $V^*$ is $p$-stable. □

### 4 Theorem 2

Throughout this section we assume that $k$ is finite and $A = k[x, y]$. To relate Theorems 1 and 2 we introduce reflections:

**Definition 4.1** Suppose $\varphi : \mathcal{F}^r \to \mathbb{Q}$ is a function, and $S$ is a subset of $\{1, \ldots, r\}$. Let $\psi(t_1, \ldots, t_r) = \varphi(t_1^*, \ldots, t_r^*)$, where $t_i^* = 1 - t_i$ if $i \in S$ and $t_i^* = t_i$ otherwise. We say that $\psi = R_S(\varphi)$, and that $\psi$ is a reflection of $\varphi$.

**Lemma 4.2** A reflection of a $p$-fractal is again a $p$-fractal.

**Proof.** Let $V$ be a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{F}^r$, and let $V^* = R_S(V)$, then $V^*$ is finite dimensional. Suppose $b = (b_1, \ldots, b_r)$, with $0 \leq b_i < p$, and let $c = (c_1, \ldots, c_r)$, where $c_i = p - 1 - b_i$ if $i \in S$, and $c_i = b_i$
otherwise. An easy calculation shows that $T_{p|b}(R_S(\varphi)) = R_S(T_{p|c}\varphi)$. So $V^*$ is $p$-stable, and the elements of $V^*$ are $p$-fractals. □

**Definition 4.3** Let $u = (u_1, u_2, u_3)$ be a triple of nonzero elements of $A$ with $\deg(u_1, u_2, u_3) < \infty$. Then $\varphi_u : I^3 \to \mathbb{Q}$ is the function with $\varphi_u \left( \frac{a_1}{a_1}, \frac{a_2}{a_2}, \frac{a_3}{a_3} \right) = q^{-2} \deg(u_1, u_2^q, u_3^q)$.

**Proposition 4.4** Suppose that $\deg(u_1, u_2) < \infty$. Then $\varphi_u$ is a $p$-fractal.

**Proof.** Let $I = (u_1, u_2)$, and let $\varphi_I : I^3 \to \mathbb{Q}$ be the map $\left( \frac{a_1}{a_1}, \frac{a_2}{a_2}, \frac{a_3}{a_3} \right) \mapsto q^{-2} \deg(I^q, u_1^q u_2^q u_3^q)$, which by Theorem 1 is a $p$-fractal. Lemma 2.7, applied twice, shows that

\[
\deg(u_1^a, u_2^a, u_3^a) = \deg(u_1^q, u_2^q, u_1^{q-a_1} u_2^{q-a_2} u_3^{q-a_3}) - \\
\deg(u_1^{q-a_1}, u_2^q) - \deg(u_1^q, u_2^{q-a_2}) - \\
\deg(u_1^q, u_2^{q-a_1}, u_2^{q-a_2} u_3^{q-a_3}) - \\
(q^2 - a_1 a_2) \deg(u_1, u_2). \quad (13)
\]

Dividing by $q^2$ and taking $S$ to be $\{1, 2\}$ we find that

\[
\varphi_u(t_1, t_2, t_3) = (R_S(\varphi_I))(t_1, t_2, t_3) - (1 - t_1 t_2) \deg(u_1, u_2), \quad (14)
\]

and lemma 4.2 gives the proposition. □

The restriction $\deg(u_1, u_2) < \infty$ in proposition 4.4 is in fact unnecessary, but it takes some work to eliminate it. Let $x_1, \ldots, x_m, y_1, \ldots, y_m$ denote the coordinate functions $\mathcal{I}^{2m} \to \mathbb{Q}$. For each subset $D$ of $\{1, \ldots, m\}$ let $X_D$ consist of all points $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ with $x_i \geq y_i$ for $i \in D$ and $y_i \geq x_i$ for $i \notin D$. Note that the $X_D$ cover $\mathcal{I}^{2m}$. Let $V$ be a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{I}^m$ stable under the reflection maps.

**Definition 4.5** $V^*$ consists of all functions $\varphi : \mathcal{I}^{2m} \to \mathbb{Q}$ with the following property: for each $D$ there is a $\psi \in V$ such that the restriction of $\varphi$ to $X_D$ is $\psi(|x_1 - y_1|, \ldots, |x_m - y_m|) +$ a degree 2 polynomial in $x_1, \ldots, x_m, y_1, \ldots, y_m$ with rational coefficients.

**Lemma 4.6** $V^*$ is finite dimensional and $p$-stable.

**Proof.** The proof that $V^*$ is finite dimensional is an easy variant of the argument used in the proof of Lemma 3.8. Suppose that $\varphi \in V^*$, and let $(b, c) = (b_1, \ldots, b_m, c_1, \ldots, c_m)$, $0 \leq b_i, c_i < p$. Then $T_{p|(b,c)}\varphi(x_1, \ldots, x_m, y_1, \ldots, y_m) =$
\( \varphi(x_1, \ldots, x_m, y_1, \ldots, y_m) \), where \( x_i^* = \frac{x_i + h_i}{p} \) and \( y_i^* = \frac{y_i + c_i}{p} \). Fix \( D \) and suppose that \((x_1, \ldots, x_m, y_1, \ldots, y_m) \in X_D \). Let \( D^* \) consist of all \( i \) with \( b_i > c_i \) together with all \( i \in D \) with \( b_i = c_i \). When \( b_i > c_i \), or when \( b_i = c_i \) and \( x_i \geq y_i, x_i^* \geq y_i^* \). When \( b_i < c_i \), or when \( b_i = c_i \) and \( y_i \geq x_i, y_i^* \geq x_i^* \). Thus \((x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_m^*) \in X_{D^*} \), and there is a \( \psi \in V \) with \( T_p(b,c) \varphi(x_1, \ldots, x_m, y_1, \ldots, y_m) = \varphi(x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_m^*) = \psi(z_1^*, \ldots, z_m^*) \) + a quadratic polynomial in the \( x_i^* \) and \( y_i^* \), where \( z_i^* = |x_i^* - y_i^*| \).

Now let \( z_i = |x_i - y_i| \) and \( S \) consist of all \( i \in D \) with \( b_i < c_i \), together with all \( i \notin D \) with \( b_i > c_i \). Let \( e_i = |b_i - c_i| - 1 \) if \( i \in S \) and \( e_i = |b_i - c_i| \) otherwise. Then

\[
z_i^* = \frac{(1 - z_i) + e_i}{p} \quad \text{if} \quad i \in S, \quad \text{and} \quad z_i^* = \frac{z_i + e_i}{p} \quad \text{otherwise. (15)}
\]

(Suppose for example that \( i \in D \) and \( b_i < c_i \), so that \( i \in S \). Then \( z_i^* = y_i^* - x_i^* = \frac{c - y - x}{p} \); the other cases are equally easy.) Now (15) tells us that \( \psi(z_1^*, \ldots, z_m^*) = (R_S(T_p b) \psi)(z_1, \ldots, z_m) \), and by our assumptions on \( V \), this is \( \eta(z_1, \ldots, z_m) \), with \( \eta \in V \). So the restriction of \( T_p(b,c) \varphi \) to \( X_D \) is \( \eta(|x_1 - y_1|, \ldots, |x_m - y_m|) + \) a degree 2 polynomial in \( x_1, \ldots, x_m, y_1, \ldots, y_m \), and \( T_p(b,c) \varphi \) is in \( V^* \). □

**Definition 4.7** A partition \( P \) is an ordered triple \((L, M, N)\) of disjoint subsets of \( \{1, \ldots, m\} \) covering \( \{1, \ldots, m\} \).

Suppose now that \( h_1, \ldots, h_m \) are pairwise prime irreducible elements of \( A \), and \( P = (L, M, N) \) is a partition.

**Definition 4.8** \( \alpha_P \) is the function \( \mathcal{S}^m \to \mathbb{Q} \) with

\[
\alpha_P \left( \frac{a_1}{q}, \ldots, \frac{a_m}{q} \right) = q^{-2} \text{deg} \left( \prod_{i \in L} h_i^{a_i} \prod_{i \in M} h_i^{a_i} \prod_{i \in N} h_i^{a_i} \right).
\]

**Proposition 4.9** The \( \alpha_P \) are \( p \)-fractals.

**Proof.** Let \( I = (\prod_{i \in L} h_i, \prod_{i \in M} h_i) \), \( \varphi_I : \mathcal{S}^m \to \mathbb{Q} \) be the \( p \)-fractal with \( \varphi_I \left( \frac{a_1}{q}, \ldots, \frac{a_m}{q} \right) = q^{-2} \text{deg} \left( I^{[a]} \prod_{i=1}^m h_i^{a_i} \right) \), and \( S = L \cup M \). Arguing as in the proof of proposition 4.4 we find that \( \alpha_P(t_1, \ldots, t_m) = (R_S(\varphi_I))(t_1, \ldots, t_m) - \sum_{i \in L} \sum_{j \in M} \text{deg}(h_i, h_j)(1 - t_i t_j) \), and lemma 4.2 gives the proposition. □

**Definition 4.10** \( \beta_P : \mathcal{S}^{2m} \to \mathbb{Q} \) is the function \( \left( \frac{a_1}{q}, \ldots, \frac{a_m}{q}, \frac{b_1}{q}, \ldots, \frac{b_m}{q} \right) \mapsto q^{-2} \text{deg}(F, G, H) \), where \( F = \prod_{i \in L} h_i^{a_i} \prod_{i \in M} h_i^{b_i} \), \( G = \prod_{i \in M} h_i^{a_i} \prod_{i \in N} h_i^{b_i} \) and \( H = \prod_{i \in N} h_i^{a_i} \prod_{i \in L} h_i^{b_i} \).

**Proposition 4.11** The \( \beta_P \) are \( p \)-fractals.
Now let $V^u$ be the $\mathbb{Q}$-subspace of $\mathbb{F}^m$ spanned by all the $\alpha_P$ and their reflections, together with their transforms under the operators $T_{Plc}$. Then $V$ is finite dimensional, by proposition 4.9 and lemma 4.2, and stable under the $T_{Plc}$ and reflections. Construct $V^*$ as in definition 4.5. Then $\beta_P$ is in $V^*$, and lemma 4.6 gives the proposition. \hfill \Box

We shall use proposition 4.11 and the following lemma to prove Theorem 2 in general.

**Lemma 4.12**

1. Let $\eta : \mathcal{I}^l \to \mathcal{I}^m$ be a map $(t_1, \ldots, t_l) \mapsto (x_1, \ldots, x_m)$, where each $x_j$ is some $t_i$. If $\varphi : \mathcal{I}^m \to \mathbb{Q}$ is a $p$-fractal, then so is $\varphi \circ \eta : \mathcal{I}^l \to \mathbb{Q}$.
2. Let $q^*$ be a power of $p$ and $s_1, \ldots, s_l$, integers with $0 \leq s_i \leq q^*$. If $\varphi : \mathcal{I}^l \to \mathbb{Q}$ is a $p$-fractal, then so is the map $\psi : \mathcal{I}^l \to \mathbb{Q}, (t_1, \ldots, t_l) \mapsto \varphi\left(s_1 t_1 \frac{q}{q^*}, \ldots, s_l t_l \frac{q}{q^*}\right)$.

**Proof.** Let $V$ be a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathbb{F}^m$ containing $\varphi$. Then the $\psi \circ \eta$, $\psi \in V$, form a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{I}^l$ containing $\varphi \circ \eta$, and we get (1).

To prove (2), let $V$ be a finite dimensional $p$-stable $\mathbb{Q}$-subspace of $\mathcal{I}^l$ containing $\varphi$. Let $l_i = s_i T_i$, and define $V^*$ as in lemma 3.8. If $t = (t_1, \ldots, t_l) \in X(d)$, then $(T_{q^*}\varphi)(\ell(t)) = \varphi\left(s_1 t_1 \frac{q}{q^*}, \ldots, s_l t_l \frac{q}{q^*}\right) = \psi(t)$. So $\psi$ is in $V^*$, and lemma 3.8 shows that $\psi$ is a $p$-fractal. \hfill \Box

We now prove Theorem 2 in general. Suppose $\deg(u_1, u_2, u_3) < \infty$, and let $h_1, \ldots, h_m$ be the pairwise prime irreducible factors of $u_1 u_2 u_3$. Define $L, M$
Now fix a power $q^r$ of $p$ with $q^r \geq \max\{\nu_i, s_i\}$. Let $x_i = \frac{x_{i1}}{q^s}, \frac{x_{i2}}{q^s}$ or $\frac{x_{i3}}{q^s}$, according as $i \in L$, $M$ or $N$, and $y_i = \frac{x_{i4}}{q^s}, \frac{x_{i5}}{q^s}$ or $\frac{x_{i6}}{q^s}$, according as $i \in M$, $N$ or $L$. Proposition 4.11 and lemma 4.12 show that the map $\psi : J^3 \rightarrow \mathbb{Q}$, $(t_1, t_2, t_3) \mapsto \beta p(x_1, \ldots, x_m, y_1, \ldots, y_m)$, is a $p$-fractal. Now $\psi \left( \frac{a_1}{q}, \frac{a_2}{q}, \frac{a_3}{q} \right) = (qq^r)^{-2} \deg(u_1^{a_1}, u_2^{a_2}, u_3^{a_3}) = (q^r)^{-2} \varphi_a \left( \frac{a_1}{q}, \frac{a_2}{q}, \frac{a_3}{q} \right)$, and since $\psi$ is a $p$-fractal, so is $\varphi_a$.

5 Operations on Ideal Classes

Let $h_1, \ldots, h_r$ be pairwise prime irreducible elements of $A = \mathbb{k}[x, y]$, $h = \prod_{i=1}^r h_i$ and $B = A/(h)$. We shall denote the images of elements and ideals of $A$ in $B$ by $\bar{g}$, $\bar{I}$, etc. Let $X$ be the set of ideal classes of $B$ containing a non-zero-divisor. We will often refer to “the class of an ideal $I$ of $A$”, by which we mean the equivalence class of $\bar{I}$.

Suppose that $\text{char} \mathbb{k} = p > 0$, $b_1, \ldots, b_r$ are fixed integers with $0 \leq b_i < p$, and that $I$ is an ideal of $A$ with $\deg I < \infty$. Let $J = \left( I^{[b]} : \prod_{i=1}^r h_i^{b_i} \right)$. Then:

1. Replacing $I$ by $(I, h)$ doesn’t change the image $\bar{J}$ of $J$ in $B$. For $J$ is replaced by $\left( J, \prod_{i=1}^r h_i^{p-b_i} \right)$, and since $b_i < p$, $\bar{J}$ is unchanged.
2. Suppose $u \in A$ is not divisible by any $h_i$. Replacing $I$ by $(uI, h)$ replaces $J$ by $\left( u^p J, \prod_{i=1}^r h_i^{p-b_i} \right)$. So $\bar{J}$ is multiplied by $\bar{u}^p$.

**Definition 5.1** $\tau_{plb} : X \rightarrow X$ is the function taking the class of $\bar{I}$ to the class of $\bar{J}$, where $\deg I < \infty$ and $J = \left( I^{[b]} : \prod_{i=1}^r h_i^{b_i} \right)$.

(1) and (2) above show that the class of $\bar{J}$ is independent both of the choice of $\bar{I}$ in its equivalence class and of the pull-back of $\bar{I}$ to an ideal $I$ of $A$. So $\tau_{plb}$ is a well-defined function.

Lemmas 3.4 and 3.6 give us the following:

**Lemma 5.2** If $\tau_{plb}$ maps the class of $\bar{I}$ to the class of $\bar{J}$, then $p^2 T_{plb} \varphi_I = \varphi_J + c + \sum_{i=1}^r c_i t_i$, for some integers $c$ and $c_i$. $\square$

Suppose now that we know the finitely many functions $\tau_{plb} : X \rightarrow X$ (and the
integers $c$ and $c_i$ of lemma 5.2 for a set of ideal class representatives, but this information is easily come by). Then lemma 5.2 gives us the action of all the $T_{q|b}$ on the space spanned by the $\varphi_I$ and 1, and allows us to recover a set of functional equations determining any $\varphi_I$, as in remark 2.3. In some cases one only needs to deal with a subset of $X$.

**Definition 5.3** Let $d$ be a positive integer. $X_d \subset X$ consists of the equivalence classes of ideals which can be generated by $d$ or fewer elements.

**Lemma 5.4** Each $X_d$ is stable under the operators $\tau_{p|b}$.

(In particular, in order to reconstruct the map $\mathcal{I}^r \to \mathbb{Q}$, $\left(\frac{a_1}{q}, \ldots, \frac{a_s}{q}\right) \mapsto q^{-2} \deg (x^{a_1}, y^{a_s}, \prod_{i=1}^r h_i^{a_i})$, one essentially only needs to know the functions $\tau_{p|b}: X_2 \to X_2$.)

**Proof.** Suppose $I$ is generated by the images of $u_1, \ldots, u_d \in A$. Replacing $u_i$ by $u_i/\gcd(u_1, \ldots, u_d)$ we may assume that $I = (u_1, \ldots, u_d)$ has finite degree. The Hilbert Syzygy Theorem shows that the module of relations between $u_1, \ldots, u_d$ and $\prod_{i=1}^r h_i^{a_i}$ can be generated by $d$ elements. It follows that $J = (I^{[|p]}: \prod_{i=1}^r h_i^{b_i})$ can be generated by $d$ elements, and that $\bar{J}$ represents an element of $X_d$. □

Besides the functional equations arising from the $\tau_{p|b}$ via lemma 5.2, there are others that arise from “reflection” operators on ideal classes. (These are defined in all characteristics.) Suppose for example that $I \supset (h)$ with $\deg I < \infty$. Fix $w \in I$ not divisible by any $h_i$, and let $J$ be the colon ideal $(\langle w, h \rangle : I)$. If $w'$ is a second element of $I$ not divisible by any $h_i$, then the images of $(\langle w, h \rangle : I)$ and $(\langle w', h \rangle : I)$ in $B$ evidently lie in the same ideal class.

**Definition 5.5** $R : X \to X$ is the map induced by $I \mapsto J$.

**Lemma 5.6** Suppose $K \supseteq (f, g)$ are ideals of $A$ with $\deg (f, g) < \infty$. Then $\deg ((f, g) : K) = \deg (f, g) - \deg K$.

**Proof.** Let $\bar{A} = A/(f, g)$. Then $((f, g) : K)/(f, g) \cong \text{Hom}_A(A/K, \bar{A})$. Since $\bar{A}$ is an Artinian Gorenstein ring, the length of this module is the same as the length of $A/K$ [1, Prop. 3.2.12], giving the result. □

**Proposition 5.7** Let $I$ and $J$ be as above, with char $k = p > 0$. Then

$$\varphi_J(1 - t_1, \ldots, 1 - t_r) = \varphi_I(t_1, \ldots, t_r) + \deg J - \sum_{i=1}^r \deg (w, h_i)t_i. \quad (17)$$
Proof. Let \( u = \prod_{i=1}^{r} h_i^{a_i} \) and \( v = \prod_{i=1}^{r} h_i^{q-a_i} \), with \( 0 \leq a_i \leq q \). Eq. (3) shows that \( \deg(J^{[q]}, v) = \deg J^{[q]} - \deg (J^{[q]} : v) \). Now the ideal \( (J^{[q]} : v) \) is \( \left( (w^q, h^q) : I^{[q]} \right) : v = \left( ((w^q, h^q) : v) : I^{[q]} \right) \). Since \( v \) is prime to \( w \), this is \( (w^q, u) : I^{[q]} = ((w^q, u) : (I^{[q]}, u)) \), and lemma 5.6 shows that its degree is \( \deg(w^q, u) - \deg(I^{[q]}, u) \). So

\[
\deg(J^{[q]}, v) = \deg J^{[q]} - \deg(w^q, u) + \deg(I^{[q]}, u) = q^{2} \deg J - \sum_{i=1}^{r} qa_i \deg(w, h_i) + \deg(I^{[q]}, u),
\]

and dividing by \( q^{2} \) we get (17). \( \square \)

Corollary 5.8 Let \( I \) and \( J \) be ideals of \( A \) whose images in \( B \) lie in classes corresponding to one another under \( R : X \rightarrow X \). Then \( \varphi_{J}(1-t_{1}, \ldots, 1-t_{r}) = \varphi_{I}(t_{1}, \ldots, t_{r}) + a \) linear combination of 1 and the coordinate functions \( t_{i} \). \( \square \)

Remark 5.9 The map \( R : X \rightarrow X \) is an involution—\( R \circ R = \text{id} \). For since \( A \) is Gorenstein and \( w, h \) is a system of parameters, \( ((w, h) : ((w, h) : I)) = I \).

Remark 5.10 \( R \) maps \( X_{2} \) to \( X_{2} \). For any \( C \subseteq X_{2} \) is represented by some ideal \( (w, g) \) of \( A \) with \( w \) not divisible by any \( h_{i} \). Then \( R(C) \) is represented by \( ((w, h) : (w, g)) = ((w, h) : g) \). Using the Hilbert Syzygy Theorem, as in the proof of lemma 5.4, we see that \( ((w, h) : g) \) is generated by 2 elements.

Definition 5.11 Let \( S \) be a subset of \( \{1, \ldots, r\} \). Then \( O_{S} \) is the class of \( (f, g) \), where \( f = \prod_{i \in S} h_{i} \) and \( g = \prod_{i \in S^{c}} h_{i} \). When \( S \) is the empty set, \( O_{S} \) is called the principal class, and is simply denoted by \( O \). If \( S = \{i\} \), we shall write \( O_{i} \) for \( O_{S} \).

Remark 5.12 If \( S \subseteq \{1, \ldots, r\} \), then \( R(O_{S}) = O_{S} \). For when calculating \( R(O_{S}) \) we may take \( w = f + g, \) with \( f \) and \( g \) as in definition 5.11. Then \( R(O_{S}) \) is represented by \( ((f + g, f g) : (f, g)) = ((f + g, f g) : g) \). Since \( g \) and \( f + g \) have no common factor, \( ((f + g, f g) : g) = (f + g, f) = (f, g) \).

The following proposition describes the relation between magnifications and reflections in characteristic \( p \):

Proposition 5.13 Suppose \( \text{char} \mathbb{k} = p > 0 \), and let \( a = (a_{1}, \ldots, a_{r}) \) and \( b = (b_{1}, \ldots, b_{r}) \), with \( a_{i}, b_{i} \geq 0 \) and \( a_{i} + b_{i} + 1 = p \). Then \( \tau_{p}^{b} R = R \tau_{p}^{a} \).

Proof. Suppose \( C \subseteq X \), and let \( I \subseteq A \) be a representative of \( C \) with \( \deg I < \infty \). We shall prove that \( R \tau_{p}^{b} R(C) = \tau_{p}^{a} (C) \). Choose \( w \in I \) prime
to \( h \). Set \( g = \prod_{i=1}^{r} h_i^{a_i} \). Then it’s easily seen that \( R\tau_{p[h]}R(C) \) is represented by \( ((w^p, h) : ((w^p, gh) : I[\bar{p}])) \), while \( \tau_{p[h]}(C) \) is represented by \( ((I[\bar{p}] : g), h) \). We’ll show that these ideals are equal. In fact, since \( A \) is Gorenstein and \( w^p, gh \) is a system of parameters, \( ((w^p, gh) : ((w^p, gh) : I[\bar{p}])) = (I[\bar{p}], gh) \). Applying \(( - : g )\) to both sides we get the desired identity. \( \square \)

Besides the “total reflection” map \( R \), there are partial reflection maps \( R_i : X_2 \to X_2 \), for which analogs to corollary 5.8 and proposition 5.13 hold when \( \text{char} \, k > 0 \). (This was shown in a more restricted setting by the second author [5].) In this section we’ll define \( R_i \) on a subset \( X_2^{(i)} \) of \( X_2 \). This will suffice to treat the examples of section 6. The appendix to this paper will define the \( R_i \) on all of \( X_2 \), show that they are commuting involutions with composition equal to \( R \), and prove the analogs to corollary 5.8 and proposition 5.13 in full generality.

As in lemma 3.2, \( L \) is the total ring of fractions of \( B \). Consider the elements \((u, v)\) of \( L^2 \) with \( Lu + Lv = L \). Define an equivalence relation on this set, identifying \((u, v)\) with \((\lambda u, \lambda v)\) when \( \lambda \in L \) is invertible. Denote the set of equivalence classes by \( \mathbb{P}^1(L) \), and denote the equivalence class of \((u, v)\) by \((u \cdot v)\), or for typographical convenience \([u : v]\). The group \( \text{GL}_2(B) \) imbeds in \( \text{GL}_2(L) \) and acts on \( \mathbb{P}^1(L) \) in the obvious way. Suppose now that \([u : v]\) is an element of \( \mathbb{P}^1(L) \). Choose \( \lambda \in L \) with \( \lambda u \) and \( \lambda v \) in \( B \). Then the ideal class of \((\lambda u, \lambda v)\) is independent of the choice of \( \lambda \), and we get a map from \( \mathbb{P}^1(L) \) onto \( X_2 \). Points in the same \( \text{GL}_2(B) \)-orbit evidently yield the same element of \( X_2 \).

**Lemma 5.14** The map that takes \([u : v]\) to the class of \((\lambda u, \lambda v)\), with \( \lambda u, \lambda v \in B \), induces a bijection between the orbits of \( \mathbb{P}^1(L) \) under the action of \( \text{GL}_2(B) \), and the set \( X_2 \).

**Proof.** It suffices to show that points \([b_1 : c_1]\) and \([b_2 : c_2]\) mapping to the same element of \( X_2 \) are in the same \( \text{GL}_2(B) \)-orbit. We may assume that \( b_1, c_1, b_2 \) and \( c_2 \) are in \( B \), and that \((b_1, c_1)\) and \((b_2, c_2)\) are the same ideal \( J \) of \( B \). If \( b_1 \) and \( c_1 \) are a minimal system of generators for \( J \), write \( b_2 = \alpha b_1 + \beta c_1 \) and \( c_2 = \gamma b_1 + \delta c_1 \), with \( \alpha, \beta, \gamma, \delta \in B \). Then \((\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix})\) has invertible reduction modulo \((\bar{x}, \bar{y})\), and so is in \( \text{GL}_2(B) \), and clearly takes \([b_1 : c_1]\) to \([b_2 : c_2]\). If on the other hand \( J \) is principal, we may assume that each of \( b_1 \) and \( b_2 \) generates \( J \). Then \((b_2/b_1, 0)\) lies in \( \text{GL}_2(B) \) and takes \([b_1 : c_1]\) to \([b_2 : c_2]\). \( \square \)

Let \( L_i \) be the field of fractions of \( A/(h_i) \). Then \( L \) identifies with the product of the \( L_i \), and \( \mathbb{P}^1(L) \) identifies with the product of the usual \( \mathbb{P}^1(L_i) \), which we think of as \( L_i \cup \{ \infty \} \) in the usual way. So we may speak of the \( i \)-th coordinate of an element \( P \in \mathbb{P}^1(L) \) and write \( P = (u_1, \ldots, u_r) \), where \( u_i \in L_i \cup \{ \infty \} \). If
Lemma 5.16 Suppose \( R \) is a ring. Then Lemma 5.16 shows that this definition is independent of the choice of \( R \).

To prove this when \( R \) is defined on all \( X_2 \), the leading form of \( h_i \) has degree 1, \( R_i \) is defined on all \( X_2 \).

Definition 5.15 \( X_2^{(i)} \) consists of those \( C \in X_2 \) that can be represented by some \( P \in \mathbb{P}^1(L) \) whose \( i \)th coordinate is 0.

If \( \alpha \) is an element of \( A \) or \( B \), \( \alpha^{(i)} \) will denote the image of \( \alpha \) in \( A/(h_i) \subset L_i \).

We are now in a position to define a reflection map \( R_1 : X_2^{(1)} \to X_2^{(1)} \).

Lemma 5.16 Suppose \( P = (0, u_2, \ldots, u_r) \) and \( Q = (0, v_2, \ldots, v_r) \) are in the same \( \text{GL}_2(B) \)-orbit of \( \mathbb{P}^1(L) \). Set \( P^* = (0, u_2^*, \ldots, u_r^*) \) and \( Q^* = (0, v_2^*, \ldots, v_r^*) \), where \( u_i^* = h_i^{(i)} u_i^{-1} \) and \( v_i^* = h_i^{(i)} v_i^{-1} \). Then \( P^* \) and \( Q^* \) lie in the same \( \text{GL}_2(B) \)-orbit.

Proof. Choose \( \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \) in \( \text{GL}_2(B) \) taking \( P \to Q \). Then \( v_i = \gamma h_i u_i + \beta \), while \( \beta^{(i)} = 0 \). So \( \beta = \bar{h}_1 \epsilon \), for some \( \epsilon \) in \( B \). Then the matrix \( \left( \begin{array}{cc} \delta & \gamma \bar{h}_1 \\ \epsilon & \alpha \end{array} \right) \) lies in \( \text{GL}_2(B) \), and an easy calculation shows that it takes \( P^* \) to \( Q^* \). \( \square \)

Definition 5.17 Suppose \( C \in X_2^{(i)} \) is represented by \( P = (0, u_2, \ldots, u_r) \). Then \( R_i(C) \) is the ideal class in \( X_2^{(1)} \) represented by \( (0, h_i^{(2)} u_2^{-1}, \ldots, h_i^{(r)} u_r^{-1}) \).

Lemma 5.16 shows that this definition is independent of the choice of \( P \), so we have a well-defined map \( R_1 : X_2^{(1)} \to X_2^{(1)} \). Similarly we have \( R_i : X_2^{(i)} \to X_2^{(i)} \). Note that \( R_i \circ R_i = \text{id} \).

Remark 5.18 If the leading form of \( h_i \) has degree 1, \( R_i \) is defined on all \( X_2 \). For let \( [\alpha : \beta] \), with \( \alpha, \beta \in B \), represent \( C \in X_2 \). Since \( B/(\bar{h}_i) \cong A/(h_i) \) is a discrete valuation ring, we may assume that \( \beta \) divides \( \alpha \) in \( B/\bar{h}_i \). Then \( \alpha = \gamma \beta + \delta \bar{h}_i \), with \( \gamma, \delta \in B \), and \([\delta \bar{h}_i : \beta]\) represents \( C \)—hence \( C \in X_2^{(i)} \).

Remark 5.19 We earlier defined classes \( O_S, S \subseteq \{1, \ldots, r\} \), and showed that \( R(O_S) = O_S \). By \( \mathcal{O} \), we claim that \( R_i \) interchanges \( O_S \) and \( O_{S \cup \{i\}} \). It suffices to prove this when \( S = \{1, \ldots, i\} \) and \( i = j + 1 \). Then \( O_S \) is represented by \( \left[ \begin{array}{c} f \\ g \end{array} \right] \), with \( f \) and \( g \) as in definition 5.11. This corresponds to the point \( (\infty, \ldots, \infty, 0, \ldots, 0) \). Then \( R_{j+1}(O_S) \) is represented by \( (0, \ldots, 0, \infty, \ldots, \infty) \), and the result follows. Let us now set \( R_{i_1 i_2 \ldots i_s} = R_{i_1} \circ R_{i_2} \circ \cdots \circ R_{i_s} \). We find that \( R_{i_1 i_2 \ldots i_s} \) takes the principal class \( O \) to \( O_{S_i} \), where \( S = \{i_1, i_2, \ldots, i_s\} \).

Lemma 5.20 Suppose that \( I = (h_1 g, f) \) is an ideal of \( A \) with deg \( I < \infty \). Set \( J = (h_1 f, h_1 g + u f) \), where \( u = \prod_{i=2}^r h_i \). Then
\[ \varphi_f(t_1, t_2, \ldots, t_r) = \varphi_f(1 - t_1, t_2, \ldots, t_r) + \deg(h_1, f)(2t_1 - 1) + \sum_{i=2}^{r} \deg(h_1, h_i)(t_1 + t_i - t_1 t_i). \] (19)

**Proof.** Let

\[ D_1 = q^2 \varphi_f \left( \frac{a_1}{q}, \ldots, \frac{a_r}{q} \right) = \deg \left( h_1^q f^q, h_1^q g^q + u^q f^q, \prod_{i=1}^{r} h_i^{a_i} \right) \] (20)

and

\[ D_2 = q^2 \varphi_I \left( \frac{q-a_1}{q}, \frac{a_2}{q}, \ldots, \frac{a_r}{q} \right) = \deg \left( f^q, h_1^q g^q, h_1^{q-a_1} \prod_{i=2}^{r} h_i^{a_i} \right). \] (21)

We need to calculate \( D_1 - D_2 \). To this end we introduce

\[ D_3 = \deg \left( h_1^{q-a_1} f^q, h_1^q g^q + u^q f^q, \prod_{i=2}^{r} h_i^{a_i} \right) = \deg \left( h_1^{q-a_1} f^q, h_1^q g^q, \prod_{i=2}^{r} h_i^{a_i} \right) \] (22)

(this equality holds because \( \prod_{i=2}^{r} h_i^{a_i} \) divides \( u^q \)) and

\[ D_4 = \deg \left( f^q, h_1^{a_1} g^q, \prod_{i=2}^{r} h_i^{a_i} \right). \] (23)

Now lemma 2.7 gives:

1. \( D_1 - D_3 = \deg(h_1^{a_1}, h_1^q g^q + u^q f^q) = \deg(h_1^{a_1}, u^q f^q) = a_1 q \deg(h_1, f) + \sum_{i=2}^{r} a_1 q \deg(h_1, h_i) \).
2. \( D_3 - D_4 = \deg(h_1^{q-a_1}, \prod_{i=2}^{r} h_i^{a_i}) = \sum_{i=2}^{r} (q a_i - a_1 a_i) \deg(h_1, h_i). \)
3. \( D_2 - D_4 = \deg(h_1^{q-a_1}, f^q) = (q^2 - a_1 q) \deg(h_1, f). \)

So \( D_1 - D_2 = (D_1 - D_3) + (D_3 - D_4) - (D_2 - D_4) = (2a_1 q - q^2) \deg(h_1, f) + \sum_{i=2}^{r} (a_1 q + qa_i - a_1 a_i) \deg(h_1, h_i) \), and dividing by \( q^2 \) we get (19). \( \square \)

Observe now that the points \( [\bar{h}_1 \bar{g} : \bar{f}] \) and \( [\bar{h}_1 \bar{g} + \bar{u} \bar{f}] \) represent the ideal classes of \( I \) and \( J \). The first of these points is \( \left( 0, \frac{h_1^{(2)} g^{(2)}}{f^{(2)}}, \ldots, \frac{h_1^{(r)} g^{(r)}}{f^{(r)}} \right) \), while the second is \( \left( 0, \frac{f^{(2)}}{g^{(2)}}, \ldots, \frac{f^{(r)}}{g^{(r)}} \right) \). So \( R_1 \) takes the class of \( I \) to the class of \( J \).

**Proposition 5.21** Let \( I \) and \( J \) be ideals of \( A \) whose classes correspond under the map \( R_1 : X_2^{(1)} \to X_2^{(1)} \). Then \( \varphi_f(t_1, t_2, \ldots, t_r) = \varphi_I(1 - t_1, t_2, \ldots, t_r) - \sum_{i=2}^{r} \deg(h_1, h_i)t_1 t_i + \) a linear combination of 1 and the \( t_j \). There are corresponding results for all the reflections \( R_i \).

**Proof.** We may replace \( I \) and \( J \) by ideals in their classes. Choose \( P \) with first coordinate 0 representing the class of \( I \). \( P \) can be taken to be \( [\bar{w} : \bar{f}] \),
with \(w \) and \(f \) in \(A\), having no common factor. Then \(w = h_1 g \) for some \(g \in A\), and \((h_1 g, f)\) is in the class of \(I\). Set \(u = \prod_{i=2}^{r} h_i\). The observation following lemma 5.20 shows that the image of \((h_1 f, h_1 g + uf)\) in \(B\) is in the class of \(J\), and we apply lemma 5.20. \(\square\)

**Proposition 5.22** If the leading forms of \(h_1\) and \(h_2\) have degree 1, and \(C \in X_2\), then \(R_1 \circ R_2(C) = R_2 \circ R_1(C)\).

**Proof.** \(A/(h_2)\) is a discrete valuation ring; let \(\text{ord}\) be the order function attached to its valuation. Suppose first that \(\text{deg}(h_1, h_2) = 1\). We may assume that \(h_1 = x\) and \(h_2 = y\). Take \(P = (0, u_2, \ldots, u_r)\) representing \(C\). If \(\text{ord} u_2 > 0\), some \(f \in A\) maps to \(u_2/x^{(2)}\) in \(A/(h_2)\), and we argue as above. If \(\text{ord} u_2 = 0\), we make use of \(R_1\), as above. Suppose finally that \(0 < \text{ord} u_2 < d\). Since the leading form of \(h_2\) is \(cx\) (\(c \in k\)), \(y^{(2)}\) is a uniformizer in \(A/(h_2)\), and there is an \(H = H(y) \in k[y]\) such that \(x^{(2)} = H^{(2)}\).

Since \(h_2\) is irreducible and divides \(x - H(y)\), they generate the same ideal in \(A\), and we may assume that \(h_2 = x - H(y)\). Since \(y^{(2)}\) is a uniformizer in \(A/(h_2)\), some \(f(y) \in k[y]\) maps to \(u_2\). Since \(\text{ord} u_2 < d = x^{(2)}\), \(u_2\) divides \(x^{(2)}\), and \(H(y) = f(y)g(y)\) for some \(g(y) \in k[y]\). To simplify notation, we now assume that \(r = 3\); the general proof is similar. We also omit the superscripts \((i)\), for ease of notation. Then

\[
P = (0, f(y), u_3) \xrightarrow{R_1} (0, g(y), x_{u_3}) \sim (-g(y), 0, \frac{x-u_3g(y)}{u_3})
\]

\[
\xrightarrow{R_2} (f(y), 0, \frac{x_{u_3} - u_3 H(y)}{x - u_3 g(y)}) \sim (0, -f(y), \frac{u_3 x f(y)}{x - u_3 g(y)}).
\]

(24)

On the other hand,

\[
P \sim (-f(y), 0, u_3 - f(y)) \xrightarrow{R_2} (g(y), 0, \frac{x - H(y)}{u_3 - f(y)}) \sim (0, -g(y), \frac{x - u_3g(y)}{u_3 - f(y)})
\]

\[
\xrightarrow{R_1} (0, -f(y), \frac{xu_3 - xf(y)}{x - u_3 g(y)}).
\]

(25)

giving the proposition. \(\square\)

**Remark 5.23** The appendix constructs reflections \(R_i : X_2(h) \to X_2(h)\) in general. Suppose \(k\) is algebraically closed. In cases we’ve studied, the \(R_i\) and
\( \tau_{p|b} \) seem to be in some sense “algebraic”. Here’s a possible formalization of that sense. One can partition \( X_2(h) \) into finitely many disjoint strata, and put a structure of \( \mathbb{k} \)-variety on each stratum so that:

1. Each \( R_i \) permutes the strata, and induces isomorphisms of corresponding strata.
2. The graph of \( \tau_{p|b} : X_2(h) \to X_2(h) \) is a disjoint union of finitely many sets, each of which is the graph of a morphism from a locally closed subset of a stratum to a stratum.

See in particular example 3 of section 6, where there are 24 one point and 2 one-dimensional strata. It would be interesting to know if the above holds generally.

6 Examples

6.1 Example 1

Let \( \mathbb{k} \) be a field of characteristic 3, and \( h = y^3 - x^4 + x^2y^2 \). Let \( z \) be the element \( \bar{y}/\bar{x} \) in \( L \), the field of fractions of \( B = A/(h) \). Then \( \bar{x} = z^3(1 - z^2)^{-1} \) and \( \bar{y} = z^4(1 - z^2)^{-1} \). Furthermore, \( z^4, z^3 + z^5 \) and \( z^n (n \geq 6) \) all lie in \( B \). Now each orbit of \( GL_2(B) \) acting on \( \mathbb{P}^1(L) = \mathbb{k}[[z]] \cup \{\infty\} \) contains an element \( u \) with ord \( u > 0 \). Modifying \( u \) by adding an element of \( B \) doesn’t change the orbit. By the above we may assume that \( u = 0 \) or that ord \( u = 1, 2 \) or 5. We claim that if ord \( u = i \) (\( i = 1, 2 \) or 5), then \( u \) is in the same orbit as \( z^i \). We may assume that \( u = z^i + (\text{higher powers}) \). If \( i = 5 \), then \( u = z^5 + (\text{an element of } B) \). If \( i = 2 \), we may assume that \( u = z^2 + cz^5 + cz^7 \). Multiplying \( u \) by the invertible element \((1 + c(z^3 + z^5))^{-1}\) of \( B \) gives the result.

If \( i = 1 \), suppose \( u = z + az^2 + (\text{higher powers}) \). Replacing \( u \) by \( u(au + 1)^{-1} \) we may assume that \( a = 0 \). Then, modifying \( u \) by an element of \( B \), we arrange \( u = z + cz^5 \), and we multiply by \((1 + cz)^{-1}\).

Now the points \([\bar{y} : \bar{x}], [\bar{y}^2 : \bar{x}]\) and \([\bar{y}^2 : \bar{x}^2]\) have ords 1, 5 and 2, respectively. It follows from the paragraph above that \( X_2 \) has 4 elements: the classes of \((\bar{y}, \bar{x})\), \((\bar{y}^2, \bar{x})\), \((\bar{y}^2, \bar{x}^2)\) and the principal class \( O \). Denote the first 3 classes by \( m, m^* \) and \( PL \), respectively. It’s easy to see that \( R \) interchanges \( m \) and \( m^* \) and fixes \( PL \). The action of the operators \( \tau_{3|0}, \tau_{3|1} \) and \( \tau_{3|2} \) on \( X_2 \) is quickly determined. For example, \( \tau_{3|0}(m) \) is the class of \(((y^3, x^3) : h^0) = (y^3, x^3)\). Now \([\bar{y}^3 : \bar{x}^3]\) has \( u = z^3 = -z^5 + (\text{an element of } B) \); it follows that \( \tau_{3|0}(m) = m^* \). Also \(((x^3, y^3) : h) = ((x^3, y^3) : x^2y^2) = (x, y) \), so that \( \tau_{3|1}(m) = m \). Finally, \(((x^3, y^3) : h^2) = (1) \), so that \( \tau_{3|2}(m) = O \). Symbolically we write \( m \to [m^* | m | O] \). Furthermore, \( PL \to [O | O | O] \); it follows that \( \varphi(x^2, y^2) \) is
piecewise linear. And of course, \( m^* = R(m) \to [O \mid m^* \mid m] \).

As a result, we get the following functional equations completely determining \( \varphi = \varphi(x,y) \):

1. \( 9 \varphi \left( \frac{t}{3} \right) = \varphi(1-t) + 9t - 1 \);
2. \( 9 \varphi \left( \frac{t+1}{3} \right) = \varphi(t) + 8 \);
3. \( \varphi(t) = 1 \) on \( \left[ \frac{2}{3}, 1 \right] \).

The above equations become simpler if we modify \( \varphi \) slightly. Let \( \psi(t) = 36t - 36t^2 - 1 \). Then \( 9 \psi \left( \frac{t}{3} \right) = \psi(1-t) + 72t - 8 \), while \( 9 \psi \left( \frac{t+1}{3} \right) = \psi(t) + 64 \). Now \( 8 \varphi \) satisfies the same relations. So if we set \( \eta = 8 \varphi - \psi \), then \( 9 \eta \left( \frac{t}{3} \right) = \eta(1-t) \) and \( 9 \eta \left( \frac{t+1}{3} \right) = \eta(t) \). Furthermore, the restriction of \( \eta \) to \( \left[ \frac{2}{3}, 1 \right] \) is \( (6t-3)^2 \). It follows that the value of \( \eta \) at any \( \frac{a}{q} \) is the square of a rational number. So we can write \( 8 \varphi = \psi + \Delta^2 \), where \( \Delta \) satisfies the following simple magnification rules:

1. \( 3 \Delta \left( \frac{t}{3} \right) = \Delta(1-t) \);
2. \( 3 \Delta \left( \frac{t+1}{3} \right) = \Delta(t) \);
3. \( \Delta(t) = 6t - 3 \) for \( \frac{2}{3} \leq t \leq 1 \).

The function \( \Delta \) is easily seen to extend to a continuous function \( [0,1] \to [0,3] \) with the following striking property. On any interval \( \left[ \frac{a}{q}, \frac{a+1}{q} \right] \) it is either linear with slope \( \pm 6 \), or it is a miniature version of \( \Delta \) or \( \Delta(1-t) \), scaled by a factor of \( \frac{1}{q} \). The graph of \( \Delta \) is shown in Figure 1—examples like this motivated the term \( p \)-fractals. (The existence of \( \Delta \) in this case is rather fortuitous, but when
one deals with $\mathbb{k}[x,y]$, homogeneous polynomials $h_i$, and homogeneous ideal classes, then one always gets such $\Delta$. They are related to syzygy gaps—see [5].

6.2 Example 2

Let $\mathbb{k}$ be an arbitrary field, $h_1 = x$, $h_2 = y$, and $h_3 = x + y$. Let $m$ be the class of $(x,y)$. Then every element of $X_2$ is either a reflection of $O$ or a reflection of $m$. For suppose $C \in X_2$. Take $P \in \mathbb{P}^1(L)$ representing $C$. The argument in the proof of proposition 5.22 shows that the orbit of $P$ contains a point $(0,0,u)$ or $(0,\infty,u)$. Replacing $C$ by $R_1(C)$, if necessary, we may assume that $P = (0,\infty,u)$. Translating $P$ by an element of $xA$, we can modify $u$ by an arbitrary element of the maximal ideal of $A/(h_3)$. So we may assume that $u = 0$ or $u = 1$. In the first case, $P$ represents $O_2 = R_2(O)$, and in the second case $P$ represents $m$, since $[\overline{x} : -\overline{y}]$ has coordinates $(0,\infty,1)$.

It’s also easy to see that $m$ is fixed by $R_{12}$, $R_{13}$ and $R_{23}$, and that $R(m) = R_1(m) = R_2(m) = R_3(m)$. When char $\mathbb{k} = p > 0$, we have an action of the operators $\tau_{pib}$ on $X_2$. To describe the action it’s enough to say what these operators do to $m$. This is implicit in the functional equations for $\varphi_{(x,y)}$ developed in [2], or in [3]. Explicitly one finds:

1. If $2b_1 > b_1 + b_2 + b_3$, $\tau_{pib}(m) = O_i$.
2. If $b_1 + b_2 + b_3 > 2p - 2$, $\tau_{pib}(m) = O$.
3. In all other cases, $\tau_{pib}(m) = m$ or $R(m)$, according as $b_1 + b_2 + b_3$ is even or odd.

6.3 Example 3

Suppose that $|\mathbb{k}| > 2$ and that $\lambda \in \mathbb{k} - \{0,1\}$. Let $h_1 = x$, $h_2 = y$, $h_3 = x + y$ and $h_4 = x + \lambda y$. This example, first studied in [5], has a rich theory, still not completely understood. Let $m$ be the class of $(x,y)$. The coordinates of $[\overline{x} : -\overline{y}]$ are $(0,\infty,1,\lambda)$, so this point represents $m$.

Lemma 6.1 If the first 3 coordinates of $P \in \mathbb{P}^1(L)$ are $(0,0,0)$ or $(0,0,\infty)$, then $P$ represents a reflection of $O$ or a reflection of $m$.

Proof. By applying $R_3$, if necessary, we may assume that $P = (0,0,0,u)$. Acting on $P$ by $\left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$, if necessary, we may assume that $u$ has nonnegative ord in $L_4$. We are free to modify $u$ by any $A/(h_4)$-multiple of $[xy(x + y)]^{(4)}$, and so we may assume that $u = 0$ or ord $u \leq 2$. We are also free to multiply $u$ by...
any element of ord 0, and so we may assume that \( u = 0, 1, x(4) \) or \((x^2)^{(4)}\). The four points we get in this way represent \( O, O_4 = R_4(O), R_4(m) \) and \( R_{1234}(m) \), respectively. \( \square \)

**Lemma 6.2** Suppose \( u \) is in the field of fractions of \( A/(h_4) \) with \( \text{ord} \, u \neq 0 \). Then \( P = (0, \infty, 1, u) \) represents a reflection of \( m \).

**Proof.** If \( \text{ord} \, u > 0 \), \( u = [ax]^{(4)} \) with \( a \in A \). Then \( \begin{pmatrix} 1 & 0 \\ 0 & 1 - ax \end{pmatrix} \) takes the point \((1, \infty, 0, 1 - u)\) to \((1, \infty, 0, 1)\), which represents \( R_{23}(m) \). Suppose \( u = v^{-1} \) with \( \text{ord} \, v > 0 \). Then \((\infty, 1, 0, 1)\) is in the orbit of \( P \). Write \( v = [ay]^{(4)} \) with \( a \in A \). Then \( \begin{pmatrix} 1 & 0 \\ 0 & 1 - ay \end{pmatrix} \) takes this last point to \((\infty, 1, 0, 1)\), which represents \( R_{13}(m) \). \( \square \)

**Proposition 6.3** The following are all the \( GL_2(B) \)-orbits in \( \mathbb{P}^1(L) \).

1. The orbits representing the 8 reflections of \( O \) and the 16 reflections of \( m \).
2. The orbits of the points \( P_c = (0, \infty, 1, c), \, c \in k - \{0, 1, \lambda\} \).
3. The reflections of the orbits in (2) under \( R_1 \).

**Proof.** Suppose \( P \in \mathbb{P}^1(L) \) represents \( C \in X_2 \). Replacing \( C \) by \( R_1(C) \), if necessary, we may assume that the first 3 coordinates of \( P \) are \((0, 0, 0), (0, 0, 1)\) or \((0, \infty, 1)\)—see the classification of the orbits in example 2. The first two of these cases are handled by lemma 6.1. In view of lemma 6.2 we may assume that \( P = (0, \infty, 1, u) \), with \( u \) in \( L_4 \) and \( \text{ord} \, u = 0 \). Then \( u = c - cv \), with \( c \neq 0 \) in \( k \) and \( \text{ord} \, v > 0 \). Write \( v = [(x + y)w]^{(4)} \), with \( w \in A \). Acting on \( P \) by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 - xw - ym \end{pmatrix} \) we get \( P_c = (0, \infty, 1, c) \). Finally, we may omit \( P_1 \) and \( P_\lambda \), which represent reflections of \( m \). \( \square \)

The orbits in (2) are parametrized by \( \mathbb{P}^1(k) - \{0, 1, \infty, \lambda\} \). Now \((0, 0, \infty, 1, c) \xrightarrow{R_3} (0, 0, x^{(3)}, \frac{x^{(4)}}{c}) \xrightarrow{R_2} \left(0, \infty, 0, -1, -\frac{1}{\lambda}\right) \sim \left(0, \infty, 1, \frac{1}{c}\right) \). So the reflection map \( R_{12} \) restricted to the orbits in (2) is induced by the colineation \( c \mapsto \frac{1}{c} \) of \( \mathbb{P}^1(k) \) that interchanges 0 and \( \infty \), and also interchanges 1 and \( \lambda \). Similar results hold for the reflection maps \( R_{13} \) and \( R_{23} \). Furthermore, \( R_{1234} \) is the identity map on this set of orbits (as is \( R \)).

When \( \text{char} \, k = p > 0 \), the operators \( \tau_{plb} \) act on \( X_2 \); an action that depends very much on \( \lambda \). Let us restrict our attention to the orbits in (2), and suppose that \( b = (0, 0, 0, 0) \). If the coordinates of \( [\bar{f} : \bar{g}] \) are \((0, \infty, 1, c)\), then those of \( [\bar{f}^p : \bar{g}^p] \) are \((0, \infty, 1, c^p)\). So if \( c^p \neq \lambda \), \( \tau_{plb} \) takes \( P_c \) to \( P_{c^p} \), while if \( c^p = \lambda \), \( P_c \) is mapped to \( m \). So \( \tau_{plb} \) is essentially the morphism \( \mathbb{P}^1(k) \to \mathbb{P}^1(k), \, c \mapsto c^p \). This illustrates remark 5.23.
6.4 Example 4

Let $k$ be an arbitrary field, $h_1 = y^2 - x^3$, $h_2 = y - x$ and $h_3 = x$. As the uniformizers in $L_1$, $L_2$ and $L_3$, we take the images of $z = y/x$, $x$ and $y$, respectively (we shall omit the superscripts $^i$, to simplify notation). Then the images of $x$ and $y$ in $A/(h_1) \times A/(h_2) \times A/(h_3)$ are $(z^2, x, 0)$ and $(z^3, x, y)$. $m$, $m^*$ and $a \in X_2$ will denote the classes of $(x, y)$, $(x^4, y)$ and $(x^3, y)$.

Lemma 6.4

(1) If $\epsilon$ is invertible in $k[\[z^2, z^3\]]$, $(u, \infty, 0)$ and $(\epsilon u, \infty, 0)$ are in the same $GL_2(B)$-orbit.

(2) $(z^3, \infty, 0)$ and $(z, \infty, 0)$ represent $R_2(m^*)$ and $R_2(a)$.

Proof. For (1) we take $w \in A$ mapping to $\epsilon$ in $A/(h_1)$. Then the element \begin{pmatrix} \bar{\omega} & 0 \\ 0 & 1 \end{pmatrix} of $GL_2(B)$ takes $(u, \infty, 0)$ to $(\epsilon u, \infty, 0)$. For (2) note that $[\bar{x}^3 : \bar{y}]$ has coordinates \begin{pmatrix} \bar{z}^5, \bar{x}^3, 0 \end{pmatrix} \sim (z^5 - z^6, 0, 0). So $R_2(m^*)$ is represented by $(z^{-3}, 0, \infty) \sim (z^3, \infty, 0)$. Also, $[\bar{x}^3 : \bar{y}]$ has coordinates \begin{pmatrix} \bar{z}^3, \bar{x}^2, 0 \end{pmatrix} \sim (z^3 - z^4, 0, 0)$, so that $R_2(a)$ is represented by $(z^{-1}, 0, \infty) \sim (z, \infty, 0)$. 

If $c \in k - \{0, 1\}$, let $M(c) \in X_2$ be the class represented by $(1 + cz, \infty, 0)$. Note that $R_{23}$ takes $(u, \infty, 0)$ to \begin{pmatrix} x \\ u(y-x) \end{pmatrix} \sim \begin{pmatrix} 1 \\ u(1-z) \end{pmatrix}$. Since, up to a unit in $k[\[z^2, z^3\]]$, $1 + cz = \frac{1}{(1+(1-c)z)(1-z)}$, $R_{23}$ interchanges $M(c)$ and $M(1-c)$. Now let $H$ consist of the operators $id$, $R_2$, $R_3$ and $R_{23}$ on $X_2$.

Proposition 6.5 Every $C \in X_2$ is either

(1) An image of $O$, $m$, $m^*$ or $a$ under $H$; or

(2) $M(c)$ or $R_3(M(c))$, with $c \in k - \{0, 1\}$.

Proof. Replacing $C$ by $R_3(C)$, if necessary, we may assume that $C$ is represented by some $(u, \infty, 0) \in \mathbb{P}^1(L)$. $R_{23}$ stabilizes the classes in (1) and the classes in (2), so we are free to replace $C$ by $R_{23}(C)$. As we've seen, this replaces $u$ by $\frac{1}{u(1-z)}$; it follows that we may assume that $u$ has nonnegative ord in $L_1$. Acting on $(u, \infty, 0)$ by \begin{pmatrix} 1 & m \bar{w} \\ 0 & 1 \end{pmatrix}, we may change $u$ by an arbitrary $k[\[z^2, z^3\]]$-multiple of $z^2$. In conjunction with lemma 6.4 (1), this allows us to assume that $u = 0$, $z^3$, $z$ or $1 + cz$. When $u = 0$, $z^3$ or $z$, $C$ is $R_2(O)$, $R_2(m^*)$ or $R_2(a)$. When $u = 1 + z$ or $1$, $C$ is $m$ or $R_{23}(m)$. Finally, when $u = 1 + cz$, $c \neq 0$ or $1$, $C = M(c)$. 

□
Remark 6.6 For \( c \neq 0 \) or 1, the ideal \( I(c) = (\overline{x^2}, (1 - c)\overline{y^2} + c\overline{x}\overline{y}) \) represents the class \( M(c) \). The operator \( R \) fixes each \( M(c) \). Computer calculations suggest that for each \( p \) the one parameter family of ideals \( I(c) \) all have the same \( \varphi \). Furthermore, this \( \varphi = \varphi_I \) seems to be a “symmetrization” of \( \varphi(x,y) \), in the sense that \( \varphi_I(t_1, t_2, t_3) = \varphi(x,y)(t_1, t_2, t_3) + 2t_1 + t_2 + t_3 \), whenever \( 2t_1 + t_2 + t_3 \leq 2 \). This curious phenomenon—a family of ideal classes whose associated \( \varphi \)’s are symmetrizations of \( \varphi(x,y) \)—has turned up in other examples.

7 Questions

The results of this paper cast a partial light on a very complex situation. Here are some of the outstanding unresolved questions. Suppose first that \( A = \mathbb{k}[x, y] \), with \( \mathbb{k} \) finite.

(1) Is Theorem 2 valid for \( r > 3 \)?
(2) Do the obvious generalizations of Theorem 1 hold? For example, if \( \deg I < \infty \), is the map \( X^n \to \mathbb{Q}, (\frac{a_1}{q}, \ldots, \frac{a_n}{q}) \mapsto q^{-2} \deg \left( I^{[q]}, f_1^{a_1}, \ldots, f_n^{a_n} \right) \), a \( p \)-fractal? Theorem 1 answers this only when \( n = 1 \).
(3) Suppose \( 0 < i < p \). Then \( 2 \deg \left( I^{[p]}, h^i \right) - \deg \left( I^{[p]}, h^{i+1} \right) - \deg \left( I^{[p]}, h^{i-1} \right) \) is nonnegative and only depends on the class of \( I \) in \( X \). Is there a good upper bound for this number (only depending on the degree of the leading form of \( h \), perhaps)?
(4) More generally, suppose that \( h_1, \ldots, h_r \) are pairwise prime, \( 0 \leq a_i \leq p \) and \( \deg I < \infty \). Set \( d(a_1, \ldots, a_r) = \deg \left( I^{[p]}, \prod_{i=1}^r h_i^{a_i} \right) \). Is there a good upper bound for \( 2d(a_1, \ldots, a_r) - d(a_1 + 1, a_2, \ldots, a_r) - d(a_1 - 1, a_2, \ldots, a_r) \) when \( 0 < a_1 < p \)? In the situation of example 2 the answer is yes. There are six classes in \( X_2 \). When \( I \) is in a reflection of \( O \) one evidently gets 0, and when \( I \) is in \( m \) or \( R(m) \) there is an upper bound of 1. In the situation of example 3 we conjecture that for any \( p \) and \( \lambda \) and any \( C \in X_2 \) one has an upper bound of 2.
(5) Now only assume that \( \mathbb{k} \) has characteristic \( p > 0 \). Are Theorems 1 and 2 still valid? (Attempts to answer this in special cases seem to lead to intractable problems about the iteration of rational functions.)

Turning finally to \( A = \mathbb{k}[x_1, \ldots, x_s] \) with \( s \geq 3 \), we have the result mentioned in the introduction with \( u_i = x_i \) and \( u_{s+1} = \sum_{i=1}^s x_i \). There are further very special results that can be obtained by combining the results of this paper with techniques from [3]; we’ll take these up in a sequel. But otherwise the situation is a mystery. There is no obvious substitute for the object so essentially employed in this paper—the set of ideal classes in \( A/(h) \).

One can still make computer calculations. When \( p = 2 \) these very strongly suggest that the function \( \left( \frac{1}{q}, \frac{1}{q} \right) \mapsto q^{-3} \deg(x^i, y^i, z^i, (xy + xz + yz)^j) \) is a \( p \)-
fractal. It seems that there is a set of 20 or so functional equations of the expected type characterizing this function, but we have no idea how to prove they hold. Experiment in this and related cases certainly suggests that functional equations abound.

A Appendix — by Paul Monsky

Let \( A = \mathbb{k}[x, y] \), where \( \mathbb{k} \) is an arbitrary field. Let \( h_1, \ldots, h_r \) be pairwise prime elements of \( A \), \( h = \prod_{i=1}^r h_i \), and \( B = A/(h) \). Let \( L_i \), \( X \), \( X_2 \) and \( X_2^{(i)} \) will be as in section 5.

In section 5 we introduced involutions \( R_i : X_2^{(i)} \to X_2^{(i)} \). We shall modify our notation slightly, writing \( \rho_i \) in place of \( R_i \). Our goal in this appendix is to define involutions \( R_i : X_2 \to X_2 \) on all of \( X_2 \), to show that the restriction of \( R_i \) to \( X_2^{(i)} \) is \( \rho_i \), and to develop the properties of the \( R_i \).

Lemma A.1 Suppose \( f, g, h \in A \) with \( \deg(f, g, h) < \infty \). Then the module of relations between \( f, g \) and \( h \) is free on 2 generators, \( (\lambda_1, \lambda_2, \lambda_3) \) and \( (\mu_1, \mu_2, \mu_3) \). Furthermore \( (\lambda_1 \mu_2 - \lambda_2 \mu_1) = (h) \).

Proof. The first statement follows from the Hilbert Syzygy Theorem. Also the vector \( (\lambda_2 \mu_3 - \lambda_3 \mu_2, \lambda_3 \mu_1 - \lambda_1 \mu_3, \lambda_1 \mu_2 - \lambda_2 \mu_1) \) is orthogonal to \( (\lambda_1, \lambda_2, \lambda_3) \) and \( (\mu_1, \mu_2, \mu_3) \), and so is a multiple of \( (f, g, h) \) by an element of the field of fractions of \( A \). Since \( A \) is factorial and \( f, g \) and \( h \) have no common factor, it is an \( A \)-multiple of \( (f, g, h) \). Thus \( h \) divides \( \lambda_1 \mu_2 - \lambda_2 \mu_1 \). Finally, since \( (0, h, -g), (h, 0, -f) \) and \( (g, -f, 0) \) are all in the module of relations, \( \lambda_1 \mu_2 - \lambda_2 \mu_1 \) divides \( fh, gh \) and \( h^2 \), and so divides \( h \). ∎

Definition A.2 Suppose \( f, g \in A \) with \( \deg(f, g, h) < \infty \), and \( S \subseteq \{1, \ldots, r\} \). Then:

1. \( h_S = \prod_{i \in S} h_i \).
2. \( M = M_S = M_S(f, g) = \{(u, v) \in A^2 : uf + vg \in (h_S)\} \).

\( M \) is the projection on \( A^2 \) of the module of relations between \( f, g \) and \( h_S \). By lemma A.1 it has a basis of 2 elements, \( (e_1, e_2) \) and \( (e_3, e_4) \). Furthermore \( (e_1 e_4 - e_2 e_3) = (h_S) \).

Suppose \( u \) and \( v \) are in \( A \). If \( h_i \) does not divide both \( u \) and \( v \), then \( \left( \begin{array}{c} u \\ v \end{array} \right) \) may be viewed as a well-defined element of \( \mathbb{P}^1(L_i) = L_i \cup \{\infty\} \).
Lemma A.3 If \( i \in S \), either \( \binom{e_1}{e_3} \) or \( \binom{e_2}{e_4} \) is defined in \( \mathbb{P}^1(L_i) \). If both are defined they are equal. If \( i \notin S \), then \( \binom{e_1}{e_3} \binom{e_2}{e_4} \binom{f}{g} \) is defined in \( \mathbb{P}^1(L_i) \).

**Proof.** If \( i \in S \), then since \( h_i^2 \) does not divide \( e_1 e_4 - e_2 e_3 \), either \( \binom{e_1}{e_3} \) or \( \binom{e_2}{e_4} \) is defined in \( \mathbb{P}^1(L_i) \). Since \( e_1 e_4 \equiv e_2 e_3 \mod h_i \), if both are defined they are equal. If \( i \notin S \), then \( \binom{e_1}{e_3} \binom{e_2}{e_4} \in \text{GL}_2(L_i) \). □

**Definition A.4** \( \langle f, g, e_1, e_2, e_3, e_4 \rangle \) is the point \( P \) of \( \mathbb{P}^1(L) = \prod_{i=1}^r \mathbb{P}^1(L_i) \) with the following coordinates: if \( i \in S \), \( P_i = \binom{e_1}{e_3} \) or \( \binom{e_2}{e_4} \), while if \( i \notin S \), \( P_i = \binom{e_1}{e_3} \binom{f}{g} \).

**Lemma A.5** Fix \( S \subseteq \{1, \ldots, r\} \). Then the element of \( X_2 \) represented by the point \( \langle f, g, e_1, e_2, e_3, e_4 \rangle \) only depends on the point \( \left[ \bar{f} : \bar{g} \right] \) of \( \mathbb{P}^1(L) \).

**Proof.** Suppose \( C_S \) is the ideal class represented by \( \langle f, g, e_1, e_2, e_3, e_4 \rangle \). We need to show:

1. Changing the basis of \( M_S \) does not change \( C_S \).
2. Modifying \( f \) and \( g \) by elements of \( (h) \) does not change \( C_S \).
3. Replacing \( f \) and \( g \) by \( u f \) and \( u g \), where \( u \in A \) is not divisible by any \( h_i \), does not change \( C_S \).

First consider (1). Changing the basis of \( M_S \) amounts to replacing the matrix \( \binom{e_1}{e_3} \binom{e_2}{e_4} \) by \( \binom{a}{b} \binom{e_1}{e_3} \binom{e_2}{e_4} \), with \( \binom{a}{b} \in \text{GL}_2(A) \). The effect of this is to replace \( \binom{e_1}{e_3} \binom{e_2}{e_4} \binom{f}{g} \) by their products on the left by \( \binom{a}{b} \). So \( \langle f, g, e_1, e_2, e_3, e_4 \rangle \) is replaced by \( \binom{a}{b} \langle f, g, e_1, e_2, e_3, e_4 \rangle \), and \( C_S \) is unchanged.

Next consider (2). The modification leaves \( M_S \) unchanged, and we see immediately that \( \langle \text{old } f, \text{ old } g, e_1, e_2, e_3, e_4 \rangle = \langle \text{new } f, \text{ new } g, e_1, e_2, e_3, e_4 \rangle \). Turning to (3), we note that \( au f + bg \in (h_S) \) if and only if \( af + bg \in (h_S) \). So once again \( M_S \) is unchanged, and we see that \( \langle u f, u g, e_1, e_2, e_3, e_4 \rangle = \langle f, g, e_1, e_2, e_3, e_4 \rangle \). □

**Proposition A.6** Fix \( S \subseteq \{1, \ldots, r\} \). Then there is a well-defined map \( R_S : X_2 \to X_2 \) taking the class represented by \( \left[ \bar{f} : \bar{g} \right] \) to the class represented by \( \langle f, g, e_1, e_2, e_3, e_4 \rangle \).

**Proof.** In view of lemma A.5, we only need to show that the class \( C_S \) represented by \( \langle f, g, e_1, e_2, e_3, e_4 \rangle \) is unchanged when \( \left[ \bar{f} : \bar{g} \right] \) is acted on by an element of \( \text{GL}_2(A) \). Suppose that \( \binom{a}{b} \in \text{GL}_2(A) \) takes \( \left[ f^* : g^* \right] \) to \( \left[ \bar{f} : \bar{g} \right] \).
Let \((d_1, d_2) = (e_1 e_2)\). Since \(d_1 f^* + d_2 g^*\) is equal to \(e_1 f + e_2 g\) in \(B\), and \(e_1 f + e_2 g \in (h_S)\), it follows that \(d_1 f^* + d_2 g^*\) is in \((h_S)\). So \((d_1, d_2) \in M_S(f^*, g^*)\).
Similarly, \((d_3, d_4) \in M_S(f^*, g^*)\), and we see easily that \((d_1, d_2)\) and \((d_3, d_4)\) are a basis of \(M_S(f^*, g^*)\). Let \(P\) and \(P^*\) be the points \(\langle f, g, e_1, e_2, e_3, e_4 \rangle\) and \(\langle f^*, g^*, d_1, d_2, d_3, d_4 \rangle\). If \(i \notin S\), \(P_i^* = \left(\frac{d_1}{d_3} \frac{d_2}{d_4}\right) \left(f^* \frac{f}{g}\right) = 0\). If \(i \in S\), \(P_i^* = P_i\). Suppose for example that \(\left(\frac{e_1}{e_3}\right)\) and \(\left(\frac{d_1}{d_3}\right)\) are defined in \(\mathbb{P}^1(L_i)\). Then \(e_1 d_3 - e_3 d_1 = e_1(\alpha e_3 + \gamma e_4) - e_3(\alpha e_1 + \gamma e_2) = \gamma(e_1 e_4 - e_2 e_3)\), which is divisible by \(h_i\). The other cases are treated similarly, and we conclude that \(P = P^*\). 

When \(S\) contains a single element \(i\) we shall write \(R_i\) rather than \(R_{\{i\}}\). We next study the connection between the maps \(R_S\) and \(R_i : X_2 \to X_2\) and the earlier introduced \(\rho_i : X_2^{(i)} \to X_2^{(i)}\). To illustrate what’s coming up, suppose \(r = 4\) and \(C\) is a class represented by some \((0,0,0,u)\). Then \(\rho_i(C)\) is represented by \(\left(0,0,\infty,h_3^{-1}u\right)\), \(\rho_2\rho_3(C)\) is represented by \(\left(0,\infty,\infty,h_2^{-1}h_3^{-1}u\right)\), and \(\rho_1\rho_2\rho_3(C)\) by \(\left(0,0,0,h_1h_2h_3^{-1}u\right)\). It follows similarly that \(\rho_{\sigma(1)}\rho_{\sigma(2)}\rho_{\sigma(3)}(C)\) is defined for every permutation \(\sigma\) of \(\{1,2,3\}\), and is independent of the choice of \(\sigma\).

**Proposition A.7** (1) The restriction of \(R_i\) to \(X_2^{(i)}\) is \(\rho_i\). So if \(C \in X_2^{(i)}\), \(R_i R_i(C) = C\).

(2) Suppose \(C \in X_2\) can be represented by some \(P\) with \(P_i = 0\) for all \(i \in S = \{i_1,i_2,\ldots,i_k\}\). Then \(R_{i_1}R_{i_2}\cdots R_{i_k}(C) = R_S(C)\). In particular, if \(C\) can be represented by a point \(P\) with \(P_i = P_j = 0\), then \(R_i R_j(C) = R_j R_i(C)\).

**Proof.** Suppose \(C\) satisfies the hypothesis of the first sentence of (2). Then \(C\) is represented by some \(P = \left[\begin{array}{c} h_S f \\ g \end{array}\right]\). A basis of \(M_S(h_S f, g)\) is given by \((0,h_S)\) and \((1,0)\). Let \(P^* = \langle f, g, 0, h_S, 1, 0 \rangle\). Then for \(i \in S\), \(P_i = 0\) and \(P_i^* = \left(\frac{0}{1}\right) = 0\). Also if \(i \notin S\), \(P_i = \left(\begin{array}{c} h_S f \\ g \end{array}\right)\) while \(P_i^* = \left(\begin{array}{c} h_S g \\ h_S f \end{array}\right) = \left(\begin{array}{c} g \\ f \end{array}\right)\).

Suppose now that \(S = \{i\}\), so that \(h_S = h_i\). Then the calculation of the last paragraph shows that \(\rho_i\) takes the class \(C\) to the class represented by \(P^*\), and we get (1). To prove (2) we may assume that \(S = \{1,\ldots,k\}\). Suppose \(P = (0,\ldots,0,u_{k+1},\ldots,u_r)\). Then the calculation of the last paragraph shows that \(P^* = (0,\ldots,0,u_{k+1}^{-1} h_i,\ldots,u_r^{-1} \prod_{i=1}^k h_i)\). A calculation illustrated in the remarks preceding the proposition then shows that for each permutation \(\sigma\) of \(\{1,\ldots,k\}\), \(\rho_{\sigma(1)}\cdots \rho_{\sigma(k)}(C)\) is the class \(R_S(C)\) represented by \(P^*\). (1) then gives (2). 

Our next goal is to show that the \(R_i\) are commuting involutions of \(X_2\), and that \(R_S\) is the composition of the \(R_i\), \(i \in S\). We begin by treating a special case. Let \(T\) be a subset of \(\{1,\ldots,r\}\), and \(j\) an element of \(T\) such that \(A/(h_j)\)
is regular. Then \( X_2^{(j)} = X_2 \), and proposition A.7 tells us that \( R_j = \rho_j \). Set \( S = T - \{ j \} \).

**Lemma A.8** \( R_T = R_j R_S \).

**Proof.** For ease of notation we assume \( h_j = x \). Each \( C \in X_2 \) is represented by some \( \left[ \bar{x} \bar{f} : \bar{g} \right] \). Since \( A/(x) \) is a discrete valuation ring, we can find a basis \( (e_1, e_2), (e_3, e_4) \) of \( M_S(x, f, g) \) with \( x \) dividing \( e_2 \). Since \( e_1(xf) + e_2g \) is divisible by \( x \) as well as by \( h_S \), \( (e_1, e_2) \) is in \( M_T(x, f, g) \). We see easily that \( (e_1, e_2) \) and \( (e_3x, e_4x) \) are a basis of \( M_T(x, f, g) \). Write \( e_2 = d_2x \) and let \( P \) and \( P^* \) be the points \( (x, f, g, e_1, e_2, e_3, e_4) \) and \( (x, f, g, e_3x, e_4x, e_1, e_2) \) representing \( R_S(C) \) and \( R_T(C) \). Note that \( P_j = \left( \frac{e_1xf + d_2xg}{e_3x + e_4g} \right) = 0 \), while \( P_j^* = \left( \frac{e_3x}{e_1} \right) = 0 \). If \( i \in S \), \( P_i^* = \left( \frac{e_3}{e_1} \right) \) or \( \left( \frac{e_4}{e_2} \right) \), while \( P_i = \left( \frac{e_1}{e_3} \right) \) or \( \left( \frac{e_2}{e_4} \right) \). Also, if \( i \notin T \), \( P_i = \left( \frac{e_1xf + e_2g}{e_3xf + e_4g} \right) \), while \( P_i^* = \left( \frac{x(e_3xf + e_4g)}{e_1xf + e_2g} \right) \). It follows that \( \rho_j \) takes the class of \( P \) to the class of \( P^* \). Since \( R_j = \rho_j \), the lemma follows. \( \square \)

**Lemma A.9** \( R_T = R_S R_j \).

**Proof.** For ease of notation we take \( j = 1 \) and \( h_1 = x \). Choose \( \left[ \bar{x} \bar{f} : \bar{g} \right] \) representing \( C \) and a basis \( (e_1, e_2), (e_3, e_4) \) of \( M_S(x, f, g) \). The observation preceding proposition 5.21 tells us that \( R_1(C) \) is represented by \( \left[ \bar{x} \bar{f} + \bar{h}_2 \cdots \bar{h}_r : \bar{x} \bar{g} \right] \).

Suppose that \((a, b) \in M_S(x, f, g)\). Then \( ax(xf + h_2 \cdots h_r g) + b(xg) = x(axf + bg) + ahg \in (h_T) \). So \((xa, b) \) is in \( M_T(xf + h_2 \cdots h_r g, xg) \). It follows easily that \((x, f, g, e_1, e_2, e_3, e_4) \) and \( (x, f + h_2 \cdots h_r g, xg, e_1, e_2, e_3x, e_4) \) representing \( R_S(C) \) and \( R_T R_1(C) \). We claim that \( P = P^* \). From this it will follow that \( R_T R_1(C) = R_S(C) \), and replacing \( C \) by \( R_1(C) \) and using proposition A.7 (1) we will get the lemma.

To establish the claim note that \( P_i = \left( \frac{e_1xf + e_2g}{e_3xf + e_4g} \right) = \left( \frac{e_2}{e_4} \right) \), while \( P_i^* \) is also \( \left( \frac{e_2}{e_4} \right) \). If \( i \in S \), \( P_i = \left( \frac{e_1}{e_3} \right) \) or \( \left( \frac{e_2}{e_4} \right) \), and the same is true of \( P_i^* \). Finally if \( i \notin T \), \( P_i = \left( \frac{e_1xf + e_2g}{e_3xf + e_4g} \right) \), while \( P_i^* = \left( \frac{e_1xf + e_2g}{e_3xf + e_4g} \right) = P_i \). \( \square \)

Combining lemmas A.8 and A.9 we get:

**Corollary A.10** Suppose \( j \in \{ 1, \ldots, r \} \) with \( A/(h_j) \) regular. Then \( R_j : X_2 \to X_2 \) commutes with all the \( R_S \). \( \square \)

The integral closure of \( A/(h_i) \) in its field of fractions is a discrete valuation ring \( D_i \). Let \( \text{ord}_i \) be the associated order function on \( L_i \).
Lemma A.11 Suppose \( u_i \in D_i, 1 \leq i \leq r, \) and that each \( \text{ord}_i(u_i) \) is large. Then there is some \( u \in A \) with the image of \( u \) in \( L_i \) equal to \( u_i \), for all \( i \).

Proof. Let \( D \) be the integral closure of \( B \) in its total ring of fractions \( L = \prod_{i=1}^{\infty} L_i \). We have seen that there is a \( w \) in \( D \), not a zero-divisor, such that \( wD \subseteq B \). Since \( \text{ord}_i(u_i) \) is large, \( \text{ord}_i(u_i) \geq \text{ord}_i(w) \). So \( (u_1, \ldots, u_r) = w \left( \frac{u_1}{w}, \ldots, \frac{u_r}{w} \right) \in w(\prod_{i=1}^{r} D_i) = wD \subseteq B \), giving the lemma. \( \Box \)

Now fix an element \( C \) of \( X_2 \). Choose a point \( (u_1, \ldots, u_r) \) of \( \prod_{i=1}^{r} \mathbb{P}^{1}(L_i) \) representing \( C \) with no \( u_i = 0 \). Let \( d \) be a large integer and \( g_1 = h_{r+1}, \ldots, g_d = h_{r+d} \) be auxiliary irreducible elements of \( A \) such that each \( A/(g_i) \) is regular and \( h_1, \ldots, h_{r+d} \) are pairwise prime. Set \( h^* = \prod_{i=1}^{r+d} h_i \).

Definition A.12 \( Y_2 \) is the set of ideal classes in \( A/(h^*) \). \( C^* \in Y_2 \) is the ideal class represented by the point \( (u_1, \ldots, u_r, 0, \ldots, 0) \) of \( \prod_{i=1}^{r+d} \mathbb{P}^{1}(L_i) \).

There is an obvious projection map \( \pi : Y_2 \to X_2 \) taking the class of \( (f, g) \) in \( A/(h^*) \) to the class of \( (f, g) \) in \( A/(h) \). This map is compatible with the projection map \( \prod_{i=1}^{r+d} \mathbb{P}^{1}(L_i) \to \prod_{i=1}^{r} \mathbb{P}^{1}(L_i) \), and takes \( C^* \) to \( C \). The definition of the maps \( R_S : X_2 \to X_2 \) and \( R_S : Y_2 \to Y_2 \) immediately gives:

Lemma A.13 Suppose \( S \subseteq \{1, \ldots, r\} \). Then the following diagram commutes:

\[
\begin{array}{ccc}
Y_2 & \xrightarrow{R_S} & Y_2 \\
\downarrow \pi & & \downarrow \pi \\
X_2 & \xrightarrow{R_S} & X_2 \\
\end{array}
\]

\( \Box \)

Definition A.14 For \( 1 \leq i \leq d \), \( T_i : Y_2 \to Y_2 \) is the map \( R_{r+i} \). Since \( R_{r+i} = \rho_{r+i} \), proposition 5.22 shows that the \( T_i \) are commuting involutions of \( Y_2 \). Let \( T \) be their composition.

Lemma A.15 \( T(C^*) \) is represented by a point of \( \prod_{i=1}^{r+d} \mathbb{P}^{1}(L_i) \) whose first \( r \) coordinates are 0.

Proof. \( C^* \) is represented by \( (u_1, \ldots, u_r, 0, \ldots, 0) \). A calculation illustrated in the remarks preceding proposition A.7 shows that \( T(C^*) \) is represented by \( (v_1, \ldots, v_r, 0, \ldots, 0) \), where \( v_i = u_i^{-1} \prod_{j=1}^{d} g_j \). Since \( d \) is large and \( \text{ord}_i(g_j) \geq 1 \), each \( \text{ord}_i(v_i) \) is large. By lemma A.11 there is a \( v \in A \) whose image in \( A/(h_i) \) is \( v_i \), for \( 1 \leq i \leq r \). Acting on \( (v_1, \ldots, v_r, 0, \ldots, 0) \) by \( \begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \) we get the desired point representing \( T(C^*) \). \( \Box \)
Proposition A.16 \( R_i R_i(C) = C \) and \( R_i R_j(C) = R_j R_i(C) \). Consequently the \( R_i \) are commuting involutions of \( X_2 \).

Proof. By corollary A.10, \( T_j : Y_2 \to Y_2 \) commutes with all the \( R_S : Y_2 \to Y_2 \). It follows that the involution \( T \) of \( Y_2 \) commutes with each \( R_S \). By lemma A.15, \( T(C^*) \in Y_2^{(i)} \). So by proposition A.7 (1), \( R_i R_i T(C^*) = T(C^*) \). Applying \( T \) we find that \( R_i R_i(C^*) = C^* \). The commutative diagram of lemma A.13 then shows that \( R_i R_i(C) = C \). Furthermore, by lemma A.15, \( T(C^*) \) is represented by a point whose \( i^{th} \) and \( j^{th} \) coordinates are 0. By proposition A.7 (2), \( R_i R_j T(C^*) = R_j R_i T(C^*) \). Applying \( T \) and using the commutative diagram of lemma A.13 we conclude that \( R_i R_j(C) = R_j R_i(C) \). \( \square \)

Proposition A.17 \( R_S : X_2 \to X_2 \) is the composition of the \( R_i, i \in S \).

Proof. By lemma A.15, \( T(C^*) \) is represented by some \( P \) with \( P_i = 0 \) for all \( i \in S \). Suppose \( S = \{i_1, i_2, \ldots, i_s\} \). Proposition A.7 (2) then shows that \( R_S T(C^*) = R_{i_1} R_{i_2} \cdots R_{i_s} T(C^*) \), and we argue as in the proof of proposition A.16. \( \square \)

Proposition A.18 \( R_1 R_2 \cdots R_r \) is the total reflection map \( R \). (See definition 5.5.)

Proof. Take \( S = \{1, \ldots, r\} \), so that \( h_S = h \). In view of proposition A.17, it suffices to show that \( R_S(C) = R(C) \) for each \( C \in X_2 \). Take \( [f : g] \) representing \( C \) with \( g \) prime to each \( h_i \), and let \( (e_1, e_2), (e_3, e_4) \) be a basis of \( M_S(f, g) \). Then the colon ideal \( (h, g) : f \) is generated by \( e_1 \) and \( e_3 \). So by definition the point \( [\tilde{e}_1 : \tilde{e}_3] \) represents \( R(C) \). Since the point \( P = \langle f, g, e_1, e_2, e_3, e_4 \rangle \) has each \( P_i = \left( \frac{e_i}{e_3} \right) \) and represents \( R_S(C) \) we are done. \( \square \)

Up to now the field \( \mathbb{k} \) has been arbitrary. Suppose now that \( \mathbb{k} \) has positive characteristic \( p \). Then proposition 5.21, proved for \( \rho_i \), generalizes to \( R_i \). Explicitly:

Proposition A.19 Let \( I \) and \( J \) be ideals of \( A \) whose classes correspond under the map \( R_1 : X_2 \to X_2 \). Then \( \varphi_j(t_1, t_2, \ldots, t_r) = \varphi_j(1 - t_1, t_2, \ldots, t_r) - \sum_{i=2}^r c_i t_1 t_i + a \text{ linear combination of } 1 \text{ and the } t_j, \text{ where } c_i = \text{deg}(h_1, h_i). \) There are corresponding results for all the reflections \( R_i \).

Proof. Choose \( (u_1, \ldots, u_r) \) representing the class \( C \) of \( I \) with no \( u_i = 0 \). We make the construction of definition A.12 and use the language of definitions
A.12 and A.14. Then \( R_1(C^*) = TR_1T(C^*) \), and since \( T(C^*) \in Y_2^{(1)} \), this is just \( \rho_{r+1} \cdots \rho_{r+d} \rho_{r+1} \cdots \rho_{r+d}(C^*) \). Let \( I^*, M^*, N^* \) and \( J^* \) be ideals of \( A \) in the classes \( C^*, T(C^*), R_1T(C^*) \) and \( R_1(C^*) \). Each of these gives a function \((t_1, \ldots, t_r, x_1, \ldots, x_d) \rightarrow \varphi(t_1, \ldots, t_r, x_1, \ldots, x_d)\), from \( \mathcal{S}^{r+d} \) to \([0, \infty)\). Now repeated applications of proposition 5.21 show:

1. \( \varphi_{I^*}(1 - t_1, t_2, \ldots, t_r, 0, \ldots, 0) \) and \( \varphi_{M^*}(1 - t_1, t_2, \ldots, t_r, 1, \ldots, 1) \) differ by a linear combination of 1 and the \( t_j \).
2. \( \varphi_{N^*}(t_1, \ldots, t_r, 1, \ldots, 1) \) and \( \varphi_{I^*}(t_1, \ldots, t_r, 0, \ldots, 0) \) differ by a linear combination of 1 and the \( t_j \).
3. \( \varphi_{N^*}(t_1, \ldots, t_r, 1, \ldots, 1) = \varphi_{M^*}(1 - t_1, t_2, \ldots, t_r, 1, \ldots, 1) - \sum_{i=2}^{r} c_it_i + \) linear combination of 1 and the \( t_j \).

So \( \varphi_{J^*}(t_1, \ldots, t_r, 0, \ldots, 0) = \varphi_{I^*}(1 - t_1, t_2, \ldots, t_r, 0, \ldots, 0) - \sum_{i=2}^{r} c_it_i + \) linear combination of 1 and the \( t_j \). Now \( I^*B \) and \( J^*B \) are ideals of \( B \) in the classes of \( \pi(C^*) \) and \( \pi(R_1(C^*)) \); that is to say in the classes of \( C \) and \( R_1(C) \). To prove the proposition we may replace \( I \) and \( J \) by \( I^* \) and \( J^* \); the first sentence of this paragraph completes the proof. \( \square \)

We complete this appendix by giving a relation between the reflection maps \( R_i \) and the operators \( \tau_{pl\alpha} \) which “explains” proposition A.19.

Let \( a = (\lambda, a_2, \ldots, a_r) \) with \( 0 \leq \lambda, a_i < p \). Set \( b = (p - 1 - \lambda, a_2, \ldots, a_r) \), so that we have operators \( \tau_{pl\alpha} \) and \( \tau_{plb} : X_2 \rightarrow X_2 \). Let \( H_0 = \prod_{i=2}^{r} h_i^{a_i} \), \( H_a = h_1^\lambda H_0 \) and \( H_b = h_1^{p-1-\lambda} H_0 \).

Our goal is to show that \( R_1\tau_{plb} = \tau_{pl\alpha} R_1 \). We begin by showing that these two maps have the same effect on any \( C \in X_2^{(1)} \). We may assume that \( C \) is represented by \( [\bar{h}_1 \bar{f} : \bar{g}] \).

**Definition A.20**

\[
W = \{(u, v) \in A^2 : uh_1^\lambda f^p + vg^p \in (H_0)\};
\]

\[
W' = \{(u, v) \in A^2 : uh_1^{\lambda+1} f^p + vg^p \in (H_0)\}.
\]

Lemma A.1 shows that we can find a basis \((d_1, d_2), (d_3, d_4)\) of \( W \) with \( d_1d_4 - d_2d_3 = H_0 \), and a basis \((e_1, e_2), (e_3, e_4)\) of \( W' \) with \( e_1e_4 - e_2e_3 = H_0 \).

**Lemma A.21** There is a matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), with entries in \( A \), such that:

1. \( \alpha \delta - \beta \gamma = h_1 \).
2. \( \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \).
3. \( \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & h_1 \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \).
Proof. (\(h_1e_1, e_2\)) and (\(h_1e_3, e_4\)) clearly lie in \(W\). So we have \((e_1, e_2)(h_1 0 0 1) = (\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (d_1, d_2) d_4)\) for some \((\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \delta)\). Comparing determinants we get (1). Multiplying (2) on the right by \((1 0 0 h_1)\) we find
\[
(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} (e_1, e_2)) = (\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (d_1, d_2) (1 0 h_1)\).
\] (A.2)
Cancelling \((\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\) we get (3). \(\Box\)

**Definition A.22** Define \(w_1, w_2, w_1^*\) and \(w_2^*\) as follows:
\[
\begin{align*}
(d_1, d_2) (h_1^p, f^p) g^p &= (w_1, w_2) H_0; \\
(e_1, e_2) (h_1^p+1 g^p) &= (w_1^*, w_2^*) H_0.
\end{align*}
\]
(Definition A.20 shows that \(w_1, w_2, w_1^*\) and \(w_2^*\) are in \(A\).)

**Lemma A.23** \((\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} (w_1^*/w_2^*) = (w_1/w_2) h_1.\)

Proof. We evaluate the product \((d_1, d_2) (1 0 h_1) (h_1^p+1 g^p)\) in two different ways.
First, it is \(h_1(d_1, d_2) (h_1^p, f^p) g^p = (w_1, w_2) h_1 H_0.\) By lemma A.21 this product is also
\[
(\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} (e_1, e_2) (h_1^p+1 f^p) g^p) = (w_1^*, w_2^*) H_0.\] Dividing by \(H_0\) we get the result. \(\Box\)

**Lemma A.24** \(\tau_{p|b}(C)\) is represented by \([\bar{w}_1^* : \bar{w}_2^*]\).

Proof. \(((h_1^p, f^p, g^p) : H_0) = ((h_1^p+1 f^p, g^p) : H_0).\) Since \((e_1, e_2)\) and \((e_3, e_4)\) are a basis of \(W'\), and \((e_1, e_2) (h_1^p+1 f^p) = (w_1^*, w_2^*) H_0,\) this colon ideal is \((w_1^*, w_2^*)\), giving the lemma. \(\Box\)

**Lemma A.25** \(R_1 \tau_{p|b}(C)\) is represented by a point \(P\) with the following coordinates: \(P_1 = (d_1/d_3),\) while \(P_k = (w_1/w_2)\) for \(k > 1.\)

Proof. Let \(S = \{1\}.\) Lemma A.23 shows that \((\delta, -\beta)\) and \((-\gamma, \alpha)\) both lie in \(M_S(w_1^*, w_2^*).\) Since \(\alpha\delta - \beta\gamma = h_1\) they form a basis of this space. So lemma A.24 shows that \(R_1 \tau_{p|b}(C)\) is represented by \(P = (w_1^*, w_2^*, \delta, -\beta, -\gamma, \alpha).\) If \(k > 1,\)
\[
P_k = (\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} (w_1^*/w_2^*) = (h_1 w_1/w_1 w_2) = (w_1/w_2) \) by lemma A.23. Also \(P_1 = (\begin{pmatrix} \delta \\ -\gamma \end{pmatrix})\) or \((\begin{pmatrix} -\beta \\ \alpha \end{pmatrix}\).
So to conclude the proof it suffices to show that \(\gamma d_1 + \delta d_3\) and \(\alpha d_1 + \beta d_3\) are both 0 in \(A/(h_1)\). But by lemma A.21 (2), \((\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (d_1) = h_1(e_1).\) \(\Box\)
Observe that $R_1(C)$ is represented by $[\tilde{h}_1 \bar{f} + \tilde{h}_2 \cdots \tilde{h}_r : \tilde{h}_1 \bar{g}]$. Thus $\tau_{p|\alpha} R_1(C)$ is the class of the ideal
\[
(((h_1 f + h_2 \cdots h_r)^p, h_1^p g^p) : H_0) = (((h_1 f + h_2 \cdots h_r)^p, h_1^{p-\lambda} g^p) : H_0). \quad (A.3)
\]
We shall calculate this colon ideal. Note that $u(h_1 f + h_2 \cdots h_r)^p + v h_1^{p-\lambda} g^p \in (H_0) \iff u h_1^p f^p + v h_1^{p-\lambda} g^p \in (H_0) \iff (u, v) \in W \iff (u, v)$ is an $A$-linear combination of $(d_1, d_2)$ and $(d_3, d_4)$. Furthermore, 
\[
\left(\begin{array}{c}
d_1 \\
d_2 \\
d_3 \\
d_4 \\
\end{array}\right)
\Rightarrow
\left(\begin{array}{c}
\frac{h_1^{p-\lambda} f^p}{h_1^{p-\lambda} g^p} + \left(h_1^p - h_1^{p-\lambda}\right)
\end{array}\right) = h_1^{p-\lambda} \left(\begin{array}{c}
w_1 \\
w_2 \\
\end{array}\right) H_0 + h_1^{p-\lambda} \left(\begin{array}{c}
d_1 \\
d_2 \\
\end{array}\right) H_0 + h_1^{p-\lambda} \left(\begin{array}{c}
d_3 \\
d_4 \\
\end{array}\right) H_0. \quad (A.4)
\]
\[\square\]

**Proposition A.27** If $C \in X_2^{(1)}$, $\tau_{p|\alpha} R_1(C) = R_1 \tau_{p|\beta}(C)$.

**Proof.** By lemma A.26, $\tau_{p|\alpha} R_1(C)$ is represented by the point $[\tilde{z}_1 : \tilde{z}_2]$, with $z_1 = h_1^{p-\lambda} w_1 + h_2^{p-\alpha} \cdots h_r^{p-\alpha} d_1$ and $z_2 = h_1^{p-\lambda} w_2 + h_2^{p-\alpha} \cdots h_r^{p-\alpha} d_3$. The first coordinate of this point is $\left(\begin{array}{c}d_1 \\
d_3 \\
\end{array}\right)$, while the $k$th coordinate, $k > 1$, is $\left(\begin{array}{c}w_1 \\
w_2 \\
\end{array}\right)$. Now apply lemma A.25. \[\square\]

**Proposition A.28** Proposition A.27 holds for all $C$ in $X_2$.

**Proof.** We make the construction of definition A.12 and adopt the language introduced in definitions A.12 and A.14. Let $a^* = (\lambda, a_2, \ldots, a_r, 0, \ldots, 0)$ and $b^* = (p - 1 - \lambda, a_2, \ldots, a_r, 0, \ldots, 0)$, so we have maps $\tau_{p|\alpha^*}$ and $\tau_{p|\beta^*} : Y_2 \rightarrow Y_2$.

The following diagrams evidently commute:
\[
\begin{array}{ccc}
Y_2 & \xrightarrow{\tau_{p|\alpha^*}} & Y_2 \\
\pi \downarrow & & \downarrow \pi \\
X_2 & \xrightarrow{\tau_{p|\alpha}} & X_2
\end{array}
\quad \begin{array}{ccc}
Y_2 & \xrightarrow{\tau_{p|\beta^*}} & Y_2 \\
\pi \downarrow & & \downarrow \pi \\
X_2 & \xrightarrow{\tau_{p|\beta}} & X_2
\end{array} \quad (A.5)
\]

So it suffices to show that $\tau_{p|\alpha^*} R_1(C^*) = R_1 \tau_{p|\beta^*}(C^*)$. Let $a^{**}$ be $a^*$ with the last $d$ zeros replaced by $p - 1$, and define $b^{**}$ likewise. Since $A/(g_i)$ is regular, $Y_2^{(i)} = Y_2$ for $i = r + 1, \ldots, r + d$. Using the obvious variants of proposition A.27 we find that $\tau_{p|\alpha^*} T = T \tau_{p|\alpha^{**}}$. So $\tau_{p|\alpha^*} R_1(C^*) = \tau_{p|\alpha^*} T R_1 T(C^*) = T \tau_{p|\alpha^{**}} R_1 T(C^*)$. Since $T(C^*) \in Y_2^{(1)}$, proposition A.27 tells us that this is $R_1 T \tau_{p|\beta^{**}} T(C^*)$. Since $\tau_{p|\beta^{**}} T = T \tau_{p|\beta^{**}}$, we get $R_1 \tau_{p|\beta^*}(C^*)$. \[\square\]
References


