Locally Compact Path Spaces

S. B. Niefield
Union College, Schenectady, NY 12308
niefiels@union.edu

August 13, 2004

Abstract

It is shown that the space $X^{[0,1]}$, of continuous maps $[0,1] \to X$ with the compact-open topology, is not locally compact for any space X having a nonconstant path of closed points. For a T_1-space X, it follows that $X^{[0,1]}$ is locally compact if and only if X is locally compact and totally path-disconnected.

AMS Classification: 54C35, 54E45, 55P35, 18B30, 18D15

Key Words: locally compact, compact-open topology, path space

1 Introduction

There were two definitions of *local compactness* under consideration in the 1970's. One definition hypothesized the existence of a compact neighborhood of each point, and hence included all compact spaces. The other required that each point have arbitrarily small compact neighborhoods. Although the two are equivalent for Hausdorff spaces, the latter definition is easily seen to be stronger, as there are compact spaces that are not locally compact in the latter sense. For example, take any space X which is not locally compact, add a point $*$, and define $U \subseteq X \cup \{*\}$ to be open if (1) $U = X \cup \{*\}$ or (2) $* \notin U$ and U is open in X.

Local compactness arises in the consideration of suitable topologies on function spaces as discussed by R.H. Fox in his 1945 paper "On topologies for function spaces" [2]. A space Y is called *exponentiable* if, for every space Z, there is a topology on the function space Z^Y so that continuous maps $X \times Y \to Z$ correspond, in the obvious way, to continuous maps $X \to Z^Y$, for every space X. Local compactness (in the stronger sense) is precisely what is needed for exponentiability of Hausdorff spaces [8] (or more generally, sober spaces [4]), and the topology on Z^Y is the compact-open topology. The question of local compactness of these spaces arose in recent work by the author on homotopy pullbacks and exponentiability [11]. It turns out that $[0,1]^{[0,1]}$ is not locally compact by either definition, and the proof works for a much wider class of spaces.

We begin, in Section 2, by proving that $X^{[0,1]}$ is not locally compact, if X has a nonconstant path of closed points (Theorem 2.2). We then apply this result to characterize the local compactness of $X^{[0,1]}$, for T_1-spaces X (Theorem 2.6). We conclude, in Section 3, by showing that the proof of Theorem 2.2 can be used to obtain a new proof of the well-known result that the category of compact Hausdorff spaces is not closed under exponentiation (i.e., is not *cartesian closed*).

Finally, I would like to thank the referees for their useful suggestions.

2 Path Spaces

Recall that the compact-open topology on $X^{[0,1]}$ is generated by the sets of the form $\langle K, W\rangle = \{\sigma \in X^{[0,1]} | \sigma(K) \subseteq W\}$, where K is compact in $[0,1]$ and W is open in X. We will consider the open sets $\langle t, W\rangle = \langle \{t\}, W\rangle$, where $t \in [0,1]$. Note that a path has $\langle t, W\rangle$ as a neighborhood if its graph passes through the opening in the fence corresponding to W on the vertical axes.

Lemma 2.1 *If the path component of a closed point x in X is nontrivial, then the same holds for any neighborhood W of x. Moreover, every nonconstant path from x in X contains a nonconstant path from x in W.*

Proof. Suppose the path component of the closed point x is nontrivial, and let σ be a path in X such that $\sigma(0) = x$ and $\sigma(1) \neq x$. Since $\sigma^{-1}(W)$ is locally path-connected as an open subspace of $[0,1]$, the component of 0 in $\sigma^{-1}(W)$ is of the form $[0,m)$, for some m. Then $\sigma(m) \neq x$ since $\sigma(m) \notin W$, and so σ is not constant on $[0,m)$, for otherwise $[0,m) = \sigma^{-1}(\{x\}) \cap [0,m]$, which is closed in $[0,m]$ since $\{x\}$ is closed. Then $t \mapsto \sigma(mt)$ is a nonconstant path from x in W whose image is contained in that of σ. □

Theorem 2.2 *If X has a nonconstant path consisting of closed points, then $X^{[0,1]}$ is not locally compact by either definition.*

Proof. Suppose σ is a nonconstant path of closed points in X, and let $x = \sigma(0)$. It suffices to show that the constant x-valued path $\sigma_x \colon [0,1] \to X$ has no compact neighborhood in $X^{[0,1]}$.

Suppose N is compact in $X^{[0,1]}$ and $\sigma_x \in \langle K, W\rangle \subseteq N$. Since $x \in W$, applying Lemma 2.1, we can assume that the image of σ is a subset of W and $y = \sigma(1)$ is a closed point of X such that $y \neq x$. We claim that

$$\mathcal{C} = \Big\{\langle 0, X \setminus \{x\}\rangle\Big\} \cup \Big\{\langle \tfrac{1}{n}, X \setminus \{y\}\rangle\Big\}_{n>1}$$

is an open cover of N which has no finite subcover.

Let τ be *any* element of $X^{[0,1]}$. If $\tau(0) \neq x$, then $\tau \in \langle 0, X \setminus \{x\}\rangle$. Otherwise, $\tau(0) = x \neq y$, and so $\tau\Big([0, \frac{1}{k})\Big) \subseteq X \setminus \{y\}$, for some k, and it follows that $\tau \in \langle \frac{1}{n}, X \setminus \{y\}\rangle$, for all $n > k$. Thus, $\mathcal{C}$ covers $X^{[0,1]}$, and hence N. Now, suppose some finite subfamily $\mathcal{F}$ covers N, let n be the largest integer such that $\langle \frac{1}{n}, X \setminus \{y\}\rangle \in \mathcal{F}$, and define $\tau \colon [0,1] \to X$ by

$$\tau(t) = \begin{cases} \sigma(nt) & t \leq \frac{1}{n} \\ y & t \geq \frac{1}{n} \end{cases}$$

Then $\tau \in \langle K, W\rangle \subseteq N$ but τ is not in any member of $\mathcal{F}$. Therefore, N is not compact, and it follows that $X^{[0,1]}$ is not locally compact. □

Corollary 2.3 $[0,1]^{[0,1]}$ *is not locally compact (hence, not compact).* □

Corollary 2.4 *If X has a locally closed subspace S with a nonconstant path of points closed in S, then $X^{[0,1]}$ is not locally compact (in the stronger sense).*

Proof. First, note that $S^{[0,1]}$ is locally closed in $X^{[0,1]}$, whenever S is locally closed in X, for suppose $S = U \cap F$, where U is open and F is closed. Then $S^{[0,1]} = U^{[0,1]} \cap F^{[0,1]}$, and $U^{[0,1]}$ is clearly open in $X^{[0,1]}$ since $U^{[0,1]} = \langle [0,1], U \rangle$. To see that $F^{[0,1]}$ is closed, suppose $\sigma \notin F^{[0,1]}$. Then $\sigma(t) \notin F$, for some $t \in [0,1]$, and it easily follows that $\sigma \in \langle t, X \setminus F \rangle \subseteq X^{[0,1]} \setminus F^{[0,1]}$.

Now, suppose $X^{[0,1]}$ is locally compact in the stronger sense. Then so is the locally closed subspace $S^{[0,1]}$, contradicting Theorem 2.2. $\square$

Recall that X is called *totally path-disconnected* if the path components in X are the points. Thus, adding a T_1 assumption, Corollary 2.4 becomes:

Corollary 2.5 *If X has a T_1 locally closed subspace that is not totally path-disconnected, then $X^{[0,1]}$ is not locally compact (in the stronger sense).* $\square$

Finally, if X itself is T_1, then a characterization of local compactness (under either definition) of the path space $X^{[0,1]}$ is given by:

Theorem 2.6 *The following are equivalent for a T_1-space X.*

(1) $X^{[0,1]}$ is locally compact.

(2) X is locally compact and totally path-disconnected.

Proof. Suppose $X^{[0,1]}$ is locally compact. Then every path in X is constant by Theorem 2.2, and so X is totally path-disconnected. Thus, $X^{[0,1]} \cong X$, and it follows that X is locally compact, as well. Conversely, suppose X is locally compact and totally path-disconnected. Then $X^{[0,1]} \cong X$, and so $X^{[0,1]}$ is locally compact. $\square$

Note that Theorems 2.2 and 2.6 can be adapted to the path spaces $P(X, x_0)$ and the loop space $\Omega(X, x_0)$ (in the sense of [13]).

We conclude this section with a simple example showing that $X^{[0,1]}$ can be locally compact without the disconnectivity assumption on X, if we relax the T_1 requirement. First, suppose Y is a locally compact space, and consider $\mathbf{2}^Y$, where $\mathbf{2} = \{0, 1\}$ is the Sierpinski space with $\{0\}$ open but not $\{1\}$. Identifying $\mathbf{2}^Y$ with the set $\mathcal{O}(Y)$ of open subsets of Y, the open set $\langle K, \{0\} \rangle$

is identified with $\uparrow K = \{U \in \mathcal{O}(Y) | K \subseteq U\}$. To show that $\mathbf{2}^Y$ is locally compact in the stronger sense, suppose $U \in \uparrow K$. Since Y is locally compact, there is an open set V and a compact set L such that $K \subseteq V \subseteq L \subseteq U$. Then $U \in \uparrow L \subseteq \uparrow V \subseteq \uparrow K$. Since $\uparrow V$ is compact in $\mathbf{2}^Y$, it follows that $\mathbf{2}^Y$ is locally compact, and it follows that $\mathbf{2}^{[0,1]}$ is locally compact. More generally, if X is *any* injective space, then the compact-open topology agrees with the Scott topology on $X^{[0,1]}$, and the latter is locally compact since $X^{[0,1]}$ is a continuous lattice (see [3] for details).

3 Exponentiability

As noted in the introduction, local compactness is precisely what is needed for exponentiability of sober spaces. Categories in which every object is exponentiable (i.e., *cartesian closed categories*) provide models for lambda calculus and hence, functional programming languages.

According to [1], a necessary and sufficient condition for exponentiability of any space Y is that the lattice $\mathcal{O}(Y)$ of open subsets is (what later became known in [12] as) a continuous lattice. The author's interest in exponentiability began in 1978 [9] with a generalization of [1] to spaces over a base, and was followed by a series of related papers, the most recent being [10]. For a further discussion of local compactness, function space problems, and the influence of [1], the reader is referred to Isbell's survey paper [6]. For more on function spaces and cartesian closed categories of continuous lattices, see [3].

It is well-known that if one would like to work in a cartesian closed category of spaces which contains all compact Hausdorff spaces (respectively, locally compact spaces), one must move to a larger category, for example *compactly generated spaces* (respectively, *locally compactly generated spaces*), see [7]. Moreover, the topology on the product of spaces in such a category is finer than the usual product topology. It turns out that the open cover constructed in Theorem 2.2 can be used to show that no category of locally compact spaces can be cartesian closed if it contains $[0,1]$. Here is a proof for compact Hausdorff spaces (see also [5], for another). The general case is similar.

Corollary 3.1 *The category of compact Hausdorff spaces is not cartesian* *closed.*

Proof. Since the proof of Theorem 2.2 uses only compact-open sets of the form $\langle t, W\rangle$, to show that $[0,1]^{[0,1]}$ has an open cover with no finite subcover, it suffices to prove that these sets would be open in $[0,1]^{[0,1]}$, if the category of compact Hausdorff spaces were cartesian closed. But, $\langle t, W\rangle = ev_t^{-1}(W)$, where ev_t is the evaluation map at t, which would be continuous since it is given by the composite

$$[0,1]^{[0,1]} \xrightarrow{\langle id,t\rangle} [0,1]^{[0,1]} \times [0,1] \xrightarrow{ev} [0,1]$$

Therefore, the category of compact Hausdorff spaces is not cartesian closed. □

References

[1] B. J. Day and G. M. Kelly, On topological quotients preserved by pullback or products, *Proc. Camb. Phil. Soc.* **67** (1970) 553–558.

[2] R. H. Fox, On topologies for function spaces, *Bull. Amer. Math. Soc.* **51** (1945) 429–432.

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, Continuous Lattices and Domains, Cambridge University Press, Encyclopedia of Mathematics and its Applications **93**, Cambridge University Press, 2003.

[4] K. H. Hofmann and J. D. Lawson, The spectral theory of distributive continuous lattices, *Trans. Amer. Math. Soc.* **246** (1978) 285–310.

[5] K. H. Hofmann and M. W. Mislove, All compact Hausdorff lambda models are degenerate, *Fund. Inform.* **122** (1995), 23–52. 285–310.

[6] J. R. Isbell, General function spaces, products, and continuous lattices, *Proc. Camb. Phil. Soc.* **100** (1986) 193–205.

[7] J. L. Kelley, General Topology, Graduate Texts in Mathematics **27**, Springer-Verlag, 1955.

[8] E. Michael, Bi-quotient maps and cartesian products of quotient maps, *Ann. Inst. Fourier. (Grenoble)* **18** (1968) 287-302.

[9] S. B. Niefield, Cartesianness: topological spaces, uniform spaces, and affine schemes, *J. Pure Appl. Alg.* **23** (1982) 147–167.

[10] S. B. Niefield, Exponentiability of perfect maps: four approaches, *Theory and Applications of Categories* **10** (2002) 127–133.

[11] S. B. Niefield, Homotopy pullbacks, lax limits, and exponentiability, in preparation.

[12] D. S. Scott, Continuous lattices, *Springer Lecture Notes in Math.* **274** (1972) 97–137.

[13] E. H. Spanier, Algebraic Topology, McGraw-Hill, 1966.