

Math 117: Final Review

I. Integrals

A. Double Integrals: $\iint_R f(x, y) dA$

1. Computations

a. Type 1 $R: a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

b. Type 2 $R: c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

c. Polars $R: \theta_1 \leq \theta \leq \theta_2, r_1(\theta) \leq r \leq r_2(\theta)$

$$\iint_R f(x, y) dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

d. Reverse Order of Integration

Change $dy dx$ to $dx dy$ (or visa versa) and integrate

2. Applications

a. $V = \iint_R f(x, y) dA = \text{volume of } G: 0 \leq z \leq f(x, y) \text{ over } R$

b. $A = \iint_R 1 dA = \text{area of } R$

c. $M = \iint_R \delta(x, y) dA = \text{mass of lamina of density } \delta(x, y)$

B. Triple Integrals: $\iiint_G f(x, y, z) dV$

1. Computations

a. Simple- xy $G: g_1(x, y) \leq z \leq g_2(x, y); (x, y)$ in R

$$\iint_R \left(\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz \right) dA$$

b. Simple- xz $G: g_1(x, z) \leq y \leq g_2(x, z); (x, z)$ in R

$$\iint_R \left(\int_{g_1(x, z)}^{g_2(x, z)} f(x, y, z) dy \right) dA$$

c. Simple- yz $G: g_1(y, z) \leq x \leq g_2(y, z); (y, z)$ in R

$$\iint_R \left(\int_{g_1(y, z)}^{g_2(y, z)} f(x, y, z) dx \right) dA$$

d. Cylindrical Coordinates Do \iint_R in polars

e. Spherical Coordinates $G: \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2, \rho_1 \leq \rho \leq \rho_2$

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(x, y, z) \rho^2 \sin \phi d\rho d\phi d\theta$$

with $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$

2. Applications

a. $V = \iiint_G 1 dV = \text{volume of } G$

b. $M = \iiint_G \delta(x, y, z) dV = \text{mass of solid of density } \delta(x, y, z)$

C. Line Integrals: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f dx + g dy + h dz$

1. Computations

a. Using $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$; $a \leq t \leq b$

(i) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt$

(ii) $\int_C f dx + g dy + h dz =$
 $\int_a^b f(x, y, z)x'(t) dt + g(x, y, z)y'(t) dt + h(x, y, z)z'(t) dt$
with $x = x(t), y = y(t), z = z(t)$

b. For Conservative \mathbf{F} Fundamental Theorem of Work Integrals

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A); C \text{ goes from } A \text{ to } B, \mathbf{F} = \nabla\phi$$

c. For Closed Curves C (ccw) Boundary Curve of R Green's Th

$$\int_C f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

2. Parametrizations To Know

a. Graph of a Function $y = f(x)$; $a \leq x \leq b$

$$C: x = t, y = f(t); a \leq t \leq b$$

b. Circles $x^2 + y^2 = r^2$, counterclockwise

$$x = r \cos t, y = r \sin t; 0 \leq t \leq 2\pi$$

c. Lines From (x_0, y_0, z_0) to (x_1, y_1, z_1)

$$x = x_0 + t(x_1 - x_0), y = y_0 + t(y_1 - y_0), z = z_0 + t(z_1 - z_0); 0 \leq t \leq 1$$

3. Applications $\int_C \mathbf{F} \cdot d\mathbf{r} = \text{Work of } \mathbf{F} \text{ along } C$

D. Surface Integrals: (i) $\iint_{\sigma} f(x, y, z) dS$ and (ii) $\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS$

1. Computations

a. From $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$; (u, v) in R

$$(i) \iint_{\sigma} f(x, y, z) dS = \iint_R f(x, y, z) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

with $x = x(u, v), y = y(u, v), z = z(u, v)$

$$(ii) \Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F}(x, y, z) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$$

with $x = x(u, v), y = y(u, v), z = z(u, v)$

b. Graph of a Function $z = g(x, y)$; (x, y) in R

$$(i) \iint_{\sigma} f dS = \iint_R f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

$$(ii) \Phi = \iint_R \mathbf{F}(x, y, g(x, y)) \cdot \left(-\frac{\partial z}{\partial x} \mathbf{i} + -\frac{\partial z}{\partial y} \mathbf{j} + \mathbf{k} \right) dA \quad (\text{upward})$$

$$= \iint_R \mathbf{F}(x, y, g(x, y)) \cdot \left(\frac{\partial z}{\partial x} \mathbf{i} + \frac{\partial z}{\partial y} \mathbf{j} - \mathbf{k} \right) dA \quad (\text{downward})$$

b. Spheres $x^2 + y^2 + z^2 = a^2$, oriented outward

(i) For $\iint_{\sigma} f dS$, use $x = a \sin \phi \cos \theta, y = a \sin \phi \sin \theta, z = a \cos \phi$

with $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$, and $\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = a^2 \sin \phi$

(ii) For Φ , use $\iint_{\sigma} \mathbf{F} \cdot \mathbf{n} dS = \iint_R \mathbf{F}(x, y, z) \cdot (a \sin \phi \langle x, y, z \rangle) dA$

Simplify, then use $x = a \sin \phi \cos \theta, y = a \sin \phi \sin \theta, z = a \cos \phi$

and $R: 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$

2. Applications $\iint_{\sigma} 1 dS = \text{Surface Area of } \sigma$

II. Integral Theorems

- A. Gauss' Divergence Theorem (for closed surfaces only)

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_G \operatorname{div} \mathbf{F} \, dV,$$

where σ is the boundary surface of G (oriented outward)

and $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$ for $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$

- B. Stokes' Theorem (for closed curves only)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\sigma} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS,$$

where C is the boundary curve of σ (oriented relative to σ)

and $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$

III. Vector Fields

- A. Sketch a representative sample for $\mathbf{F}(x, y)$ in the plane
- B. Conservative Vector Field Theorem (know the equivalent conditions)
- C. Compute $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F}$

IV. Orientation

- A. Positive orientation of σ relative to $\mathbf{r}(u, v)$

$$\mathbf{n}(u, v) = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|}$$

- B. Positive orientation of the boundary curve C relative to σ

When \mathbf{n} “walks” along C in that direction the surface is on the left