Math 17: Final Review

I. Integrals

A. Double Integrals: $\iint_R f(x,y)\; dA$

1. Computations

a. Type 1 $R\colon a \le x \le b, g_1(x) \le y \le g_2(x)$

$$\iint_R f(x,y)\; dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\; dy\, dx$$

b. Type 2 $R\colon c \le y \le d, h_1(y) \le x \le h_2(y)$

$$\iint_R f(x,y)\; dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\; dx\, dy$$

c. Polars $R\colon \theta_1 \le \theta \le \theta_2, r_1(\theta) \le r \le r_2(\theta)$

$$\iint_R f(x,y)\; dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\; r\, dr\, d\theta$$

d. Reverse Order of Integration

Change $dy\, dx$ to $dx\, dy$ (or visa versa) and integrate

2. Applications

a. $V = \iint_R f(x,y)\; dA =$ volume of $G\colon 0 \le z \le f(x,y)$ over R

b. $A = \iint_R 1\; dA =$ area of R

c. $M = \iint_R \delta(x,y)\; dA =$ mass of lamina of density $\delta(x,y)$

B. Triple Integrals: $\iiint_G f(x,y,z)\ dV$

1. Computations

   a. <u>Simple$-xy$</u> $G\colon g_1(x,y) \le z \le g_2(x,y); (x,y)$ in R

   $$\iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\ dz \right)\ dA$$

   b. <u>Simple$-xz$</u> $G\colon g_1(x,z) \le y \le g_2(x,z); (x,z)$ in R

   $$\iint_R \left(\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\ dy \right)\ dA$$

   c. <u>Simple$-yz$</u> $G\colon g_1(y,z) \le x \le g_2(y,z); (y,z)$ in R

   $$\iint_R \left(\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\ dx \right)\ dA$$

   d. <u>Cylindrical Coordinates</u> Do $\iint_R$ in polars

   e. <u>Spherical Coordinates</u> $G\colon \theta_1 \le \theta \le \theta_2, \phi_1 \le \phi \le \phi_2, \rho_1 \le \rho \le \rho_2$

   $$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(x,y,z)\ \rho^2 \sin\phi\ d\rho\, d\phi\, d\theta$$

   with $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho\cos\phi$

2. Applications

   a. $V = \iiint_G 1\ dV =$ volume of G

   b. $M = \iiint_G \delta(x,y,z)\ dV =$ mass of solid of density $\delta(x,y,z)$

C. Line Integrals: $\displaystyle\int_C \mathbf{F}\cdot d\vec{r} \;=\int_C\; f\;dx+g\;dy+h\;dz$

1. Computations

   a. Using $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k};\;\; a\le t\le b$

   (i) $\displaystyle\int_C \vec{F}\cdot d\vec{r} \;=\int_a^b \mathbf{F}\big(x(t),y(t),z(t)\big)\cdot\mathbf{r}'(t)\;dt$

   (ii) $\displaystyle\int_C\; f\;dx+g\;dy+h\;dz \;=$

   $\displaystyle\int_a^b\; f(x,y,z)x'(t)\;dt+g(x,y,z)y'(t)\;dt+h(x,y,z)z'(t)\;dt$

   with $x=x(t),y=y(t),z=z(t)$

   b. For Conservative $\mathbf{F}$ Fundamental Theorem of Work Integrals

   $\displaystyle\int_C \mathbf{F}\cdot d\vec{r} \;=\phi(B)-\phi(A)$; C goes from A to B, $\mathbf{F}=\nabla\phi$

   c. For Closed Curves C (ccw) Boundary Curve of R Green's Th

   $$\int_C\; f\;dx+g\;dy=\iint_R\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dA$$

2. Parametrizations To Know

   a. Graph of a Function $y=f(x);\;\; a\le x\le b$

   $C\colon x=t,y=f(t);\;\; a\le t\le b$

   b. Circles $x^2+y^2=r^2$, counterclockwise

   $x=r\cos t,y=r\sin t;\;\; 0\le t\le 2\pi$

   c. Lines From (x_0,y_0,z_0) to (x_1,y_1,z_1)

   $x=x_0+t(x_1-x_0),\; y=y_0+t(y_1-y_0),\; z=z_0+t(z_1-z_0);\; 0\le t\le 1$

3. Applications $\displaystyle\int_C \mathbf{F}\cdot d\vec{r}=$ Work of $\mathbf{F}$ along C

D. Surface Integrals: (i) $\iint_\sigma f(x, y, z)\ dS$ and (ii) $\Phi = \iint_\sigma \mathbf{F} \cdot \mathbf{n}\ dS$

1. Computations

   a. From $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$; (u, v) in R

   (i) $\displaystyle\iint_\sigma f(x, y, z)\ dS = \iint_R f(x, y, z) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$
   with $x = x(u, v), y = y(u, v), z = z(u, v)$

   (ii) $\displaystyle\Phi = \iint_\sigma \mathbf{F} \cdot \mathbf{n}\ dS = \iint_R \mathbf{F}(x, y, z) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA$
   with $x = x(u, v), y = y(u, v), z = z(u, v)$

   b. Graph of a Function $z = g(x, y); (x, y)$ in R

   (i) $\displaystyle\iint_\sigma f\ dS = \iint_R f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$

   (ii) $\displaystyle\Phi = \iint_R \mathbf{F}(x, y, g(x, y)) \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} + -\frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k} \right) dA$ (upward)

   $\displaystyle= \iint_R \mathbf{F}(x, y, g(x, y)) \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k} \right) dA$ (downward)

   b. Spheres $x^2 + y^2 + z^2 = a^2$, oriented outward

   (i) For $\iint_\sigma f\ dS$, use $x = a \sin\phi \cos\theta, y = a \sin\phi \sin\theta, z = a \cos\phi$
   with $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$, and $\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = a^2 \sin\phi$

   (ii) For Φ, use $\displaystyle\iint_\sigma \mathbf{F} \cdot \mathbf{n}\ dS = \iint_R \mathbf{F}(x, y, z) \cdot (a \sin\phi \langle x, y, z \rangle)\, dA$
   Simplify, then use $x = a \sin\phi \cos\theta, y = a \sin\phi \sin\theta, z = a \cos\phi$

   and $R{:}\ 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$

2. Applications $\iint_\sigma 1\ dS =$ Surface Area of σ

II. Integral Theorems

A. Gauss' Divergence Theorem (for closed surfaces only)

$$\iint_\sigma \mathbf{F} \cdot \mathbf{n} \; dS = \iiint_G \operatorname{div} \mathbf{F} \; dV,$$

where σ is the boundary surface of G (oriented outward)

and $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$ for $\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$

B. Stokes' Theorem (for closed curves only)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \; dS,$$

where C is the boundary curve of σ (oriented relative to σ)

and $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$

III. Vector Fields

A. Sketch a representative sample for $\mathbf{F}(x, y)$ in the plane

B. Conservative Vector Field Theorem (know the equivalent conditions)

C. Compute $\operatorname{div} \mathbf{F}$ and $\operatorname{curl} \mathbf{F}$

IV. Orientation

A. Positive orientation of σ relative to $\mathbf{r}(u, v)$

$$\mathbf{n}(u, v) = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|}$$

B. Positive orientation of the boundary curve C relative to σ

When $\mathbf{n}$ "walks" along C in that direction the surface is on the left