A PERSONAL TRIBUTE TO BILL LAWVERE

MARTA BUNGE

This year marks the 50th anniversary of Lawvere's thesis on algebraic theories and of the far-reaching idea of algebraic structure as a functor adjoint to semantics. Around the time of this important event, Lawvere was also thinking about several other questions in category theory, motivated by his manifold interests from logic to physics, passing through algebra. In fact, so much so, that it was only in 1969 that he made public the open problems arising from his 1963 thesis.

I met Bill Lawvere in Jerusalem at the 1964 International Congress for Logic and Philosophy of Science, where he gave an invited talk about ETCS (*Elementary Theory of the Category of Sets*). Having learnt that I was a new student of Peter Freyd's still in search of a thesis topic, Lawvere immediately produced a list of several possible such. One of them appealed to me particularly since it meant emulating Freyd's abelian categories, but modelling them on sets rather than on abelian groups. This, in turn, led, first, to my participating in the Eckmann Seminar at the E.T.H. Zurich during the academic year 1965-66, then to my thesis (*Categories of Set-Valued Functors*, University of Pennsylvania, 1966), presented at the first Oberwolfach Meeting in Categories, and finally, thanks to Jim Lambek, to a job at McGill University in Montréal where I have remained until now.

The topic of my 1966 thesis was the study of "diagrammatic categories", or categories equivalent to one of the form $Set^{\mathbb{C}}$ for $\mathbb{C}$ a small category. After ETCS, it became desirable to axiomatize categories with variation based on a model *Set* of ETCS. One of the motivations for so doing was to investigate independence of Lawvere's axioms for *Set*. A more important motivation was based on the observation that "all" concrete categories can be expressed as full subcategories of diagrammatic ones, thus benefitting in principle from a more powerful machinery. To this end, I developed a theory of what I called "regular categories", but which included dual ("coregularity") axioms. I proved that a locally small category $\mathscr{X}$ is equivalent to a diagrammatic category $Set^{\mathbb{C}}$ if and only if $\mathscr{X}$ is a cocomplete regular atomic category. By "atomic", I meant generated by the "atoms", or objects A of $\mathscr{X}$ for which the functor $\mathrm{Hom}_{\mathscr{X}}(A, -) : \mathscr{X} \to Set$ preserves colimits. The fruitful analogy, which Lawvere had suggested, was to think of fields of sets 2^{Λ} as the complete atomic Heyting algebras, replacing 2 by *Set*, and the preordered set Λ by a small category $\mathbb{C}$. At the Zurich Seminar during 1965-66, Lawvere lectured on the then new subject of triples (monads, standard constructions). This led me to an alternative characterization of diagrammatic categories in terms of adjoint triples. This material I later extended to the relative case, whereby *Set* was replaced by an arbitrary symmetric monoidal closed category $\mathscr{V}$.

The influence that Bill Lawvere has had on my work did not stop with my thesis. In

fact, several of my areas of research since then, and until very recently, have started with an idea, a paper, or a specific question from him. Prominent among these is my (ongoing) work on Topos Theory and applications of it to various areas of mathematics. I proceed to describe just three of these developments, namely, Stacks, SDG, and Distributions, each having taken about a decade of my life to develop, yet all three being far from complete[1].

The notion of a *stack* is central to non-abelian cohomology and was first developed by Giraud in terms of sites for Grothendieck toposes. During his lectures in Perugia in 1973 and in Montreal in 1974, Lawvere proposed adding an axiom to topos theory whereby every category object in an elementary topos admits a (representable) stack completion for its regular epimorphisms topology, hence, unlike the known theory, independent of a site. In collaboration with Bob Paré in 1979, I developed a theory of intrinsic stacks in the context of indexed categories over an arbitrary base topos, while considerably simplifying the subject. This led in turn to my construction, also in 1979, of the stack completion of any category object in a topos. Stacks have appearead often in my work, for instance, in my 2004 paper on the fundamental groupoid of a locally connected topos, where the stack property distinguishes it from its Galois groupoid, and in my 2013 paper on a notion of tight completion that I came up with in order to answer a question of Lawvere's concerning the similarities between the stack and the Cauchy completions. This required a previous development – namely that of a theory of indexed enriched categories that Lawvere had advocated already in his Perugia lectures. One of the desirable applications of it would be to develop a theory of internal metric spaces in a petit topos as time-parameterized sets, suggested by Einstein but bypassed because of lack of sufficient mathematical machinery. Stacks are (or should be) a key ingredient of any attempt to give a foundation of mathematics based on category theory.

The basic idea of *Synthetic Differential Geometry*, in the form of the Kock-Lawvere axiom, requires, for a topos $\mathscr{E}$ with a ring object R in it, that the subobject D of R, consisting of those elements of square zero, be tiny and representing of tangent vectors at 0 of arrows from R to R. During the period 1981-88, I devoted myself almost totally to SDG, involving students and collaborators (Murray Heggie, Patrice Sawyer, Eduardo Dubuc, Felipe Gago) and participating in the workshops organized by Anders Kock at Aarhus, as well as in related special meetings. Lawvere's intuition of the role of atoms (or "tiny objects") in developing a simple form of Analysis going back to the ideas of Newton and Leibniz, and in the same spirit as in the work of André Weil, was both simple and attractive. In my work with my student Felipe Gago on a synthetic theory of smooth mappings, we used two additional axioms (Bunge-Dubuc 1987) to SDG, to wit, the representability of germs of smooth mappings by the subobject $\Delta = \neg\neg\{0\}$ of R, required to be tiny, and the existence and uniqueness of solutions of ordinary differential equations. However, no well adapted model of SDG is known at present to satisfy both of these axioms. This open problem is, in my view, pivotal for further progress in this

[1]The papers that I refer to in what follows can easily be located by entering into the website http://www.math.mcgill.ca/biblio/author/bunge, itself included in my academic webpage http://www.math.mcgill.ca/people/bunge

fascinating area, which includes a synthetic proof of Mather's theorem on the equivalence of locally stable and infinitesimally stable germs of smooth mappings (Bunge-Gago 1988), as well as Morse theory, developed synthetically in the thesis work of Felipe Gago at McGill.

In his 1966 lecture at Oberwolfach, Lawvere had proposed a theory of *distributions* on presheaf toposes, followed by a 1983 lecture at Aarhus where he posed several questions about distributions on toposes and locales. The notion of a (Lawvere) distribution on an $\mathbf{S}$-bounded topos $\mathscr{X}$, for $\mathbf{S}$ a base topos, is that of an $\mathbf{S}$-cocontinuous functor from $\mathscr{X}$ to $\mathbf{S}$. Implicit in it is the idea of letting the object classifier $\mathscr{R}$ in Top/$\mathbf{S}$ play the role of "the line", so that a distribution is in fact an instance of the double dualization that Lawvere called the Riesz paradigm. Answering an open question of Lawvere's at his Aarhus lecture, I proved in 1990, using forcing topologies, the existence of the "symmetric topos", the classifier of Lawvere distributions on Grothendieck toposes. This was followed by my joint work with Aurelio Carboni in 1995, in which we proved that the "symmetric monad" on the 2-category of locally presentable categories is a Kock-Zoberlein monad, and that it has the *opposite* of the 2-category of Grothendieck toposes as its category of algebras. The subject of distributions and the symmetric topos was then enhanced by the discovery, made for locales by my student Jonathon Funk and then extended to toposes in collaboration with me, that distributions on a Grothendieck topos $\mathscr{X}$ correspond to (our topos version of) Fox complete spreads over $\mathscr{X}$ with a locally connected domain. Our 1996 and 1998 results on "spreads and the symmetric topos" generated a considerable amount of work with several collaborators (Mamuka Jibladze, Thomas Streicher, Susan Niefield, Marcelo Fiore, Steve Lack), culminating in a book (*Singular Coverings of Toposes*, LNM 1890, Springer 2006). Several problems arising from this book have been solved, yet most of them, particularly in the area of branched coverings, remain open.

I conclude my tribute with the following remarks. Over the years, my admiration for Bill Lawvere has only grown, as has my friendship with him and Fatima. From a mathematical point of view, what I most admire is his vast program in which the basic concepts, used to describe and develop a certain field of mathematics or physics, should be simple to grasp, yet intending to capture its essence. I see in Bill, whom I have now known for almost 50 years, an honest, kind and generous person, always ready to share his ideas with anyone willing to listen. An aspect of this is surely his love for teaching, a testimony of which is his delightful book *Conceptual Mathematics* with Steve Schanuel, a book which I studied in detail so as to give it justice in a review that I wrote of it in Spanish.

I wish Bill Lawvere all the years in peace and good health that he may need to complete his program.