Orbi Mapping Spaces

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Outline

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   Groupoid Maps
   Morita Equivalence
   Orbimaps

Orbi Mapping Spaces
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Orbispaces

Definition

- An orbispace is a Morita equivalence class of orbigroupoids.
- An orbigroupoid $\mathcal{G}$ is a groupoid in the category of paracompact Hausdorff spaces

\[
\begin{aligned}
G_1 \times_{s,G_0,t} G_1 & \xrightarrow{m} G_1 \\
\pi_1 & \quad \pi_2 \\
G_1 & \xrightarrow{i} G_1 \\
\pi_2 & \quad \pi_1 \\
G_0 & \xleftarrow{u} G_0 \\
& \xrightarrow{s} t
\end{aligned}
\]

such that the source and target maps are étale, and the diagonal $(s, t): G_1 \to G_0 \times G_0$ is proper (closed with compact fibers).
Example 1: The Silvered Interval

The circle $S^1$ with the $\mathbb{Z}/2$-action by reflection.

The source map is defined by projection, the target by the action.
Example 2: The Order 3 Cone

\[
\begin{array}{c}
\text{id} \\
\text{2/3} \\
\text{1/3} \\
\end{array}
\]

\[
\begin{array}{c}
\text{morphisms} \\
\text{id} \\
\text{objects} \\
\end{array}
\]
Example 3: The Teardrop
Example 4: The Order 2 Corner $V_4 \rtimes \mathbb{D}$
Example 5: The Order 3 Corner $D_6 \rtimes \mathbb{D}$
Example 6: $G$-Points $*_G$
Example 7: $G$-Lines
Definition

A **morphism** $\varphi : G \to H$ of topological groupoids is a pair of maps

$$\varphi_0 : G_0 \to H_0 \text{ and } \varphi_1 : G_1 \to H_1,$$

which commute with all the structure maps.
$\mathbb{Z}/2$-Points of the Order 2 Corner

What are the groupoid maps $*_{\mathbb{Z}/2} \to V_4 \ltimes \mathbb{D}$?

- For any $X \in \mathbb{D}$, $\varphi^X_0(P) = X$ and $\varphi^X_1(\bar{0}) = \varphi^X_1(\bar{1}) = (X, \text{id})$.
- For any $X$ on the horizontal (vertical) axis of $\mathbb{D}$, $\psi^X_0(P) = X$ and $\psi^X_1(\bar{1}) = (X, \tau)$ ($\psi^X_1(\bar{1}) = (X, \sigma)$).
- $\chi(P) = O$ and $\chi(\bar{1}) = (O, \rho)$.

\[ \begin{array}{c}
\text{id} \\
\rho = \sigma\tau \\
\sigma \\
\end{array} \]

\[ \begin{array}{c}
\tau \\
\end{array} \]
What are the groupoid maps $\ast \mathbb{Z}/2 \rightarrow V_4 \ltimes \mathbb{D}$?

For any $X \in \mathbb{D}$, $\varphi_0^X(P) = X$ and $\varphi_1^X(0) = \varphi_1^X(1) = (X, \text{id})$.

For any $X$ on the horizontal (vertical) axis of $\mathbb{D}$, $\psi_0^X(P) = X$ and $\psi_1^X(1) = (X, \tau)$ ($\psi_1^X(1) = (X, \sigma)$).

$\chi(P) = O$ and $\chi(1) = (O, \rho)$. 
**Z/2-Points of the Order 2 Corner**

- What are the groupoid maps $*_{\mathbb{Z}/2} \to V_4 \rtimes \mathbb{D}$?

  ![Diagram](image)

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  - $\chi(P) = O$ and $\chi(1) = (O, \rho)$. 

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**Z/2-Points of the Order 2 Corner**

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Paths
2-Cells

**Definition**
A 2-cell $\alpha : \varphi \Rightarrow \psi$ is a map $\alpha : G_0 \rightarrow H_1$, such that

$$s \circ \alpha = \varphi_0, \quad t \circ \alpha = \psi_0,$$

which satisfies the naturality condition, i.e., for each $g \in G_1$,

$$\varphi_0(sg) \xrightarrow{\alpha(sg)} \psi_0(sg)$$

$$\varphi_1(g) \downarrow \quad \downarrow \psi_1(g)$$

$$\varphi_0(tg) \xrightarrow{\alpha(tg)} \psi_0(tg)$$

commutes in $\mathcal{H}$, $m(\psi_1(g), \alpha(sg)) = m(\alpha(tg), \varphi_1(g))$. 
2-Cells Between Paths
Here are two paths with a unique 2-cell between them.

Note that these paths have the same image in the quotient space.
2-Cells Between Paths

These two paths do not have a 2-cell between them, although they have the same image in the quotient space.
2-Cells Between Paths

And these two paths have two 2-cells between them.
2-Cells Between $G$-Points

If $\varphi(P) = \psi(P)$, then 2-cells

$$\alpha : \varphi \Rightarrow \psi : *_G \Rightarrow \mathcal{H}$$

correspond to elements $h \in \mathcal{H}_{\psi(P)}$ such that $h\psi(g)h^{-1} = \varphi(g)$ for all $g \in G$,
There are two $\mathbb{Z}/3$ points of the order-3-corner with a non-trivial map on groups:

$\varphi_0(P) = O$, $\varphi_1(\bar{1}) = \rho$ and $\psi_0(P) = O$, $\psi_1(\bar{1}) = \rho^2$.

There are three transformations from one to the other (corresponding to the three reflections) and three transformations from each point to itself (corresponding to the rotations).
Essential Equivalences, I

An essential equivalence \( \phi: G \to H \) satisfies the following two properties:

1. **(Essentially surjective)**

   \[
   G_0 \times_{H_0} H_1 \longrightarrow H_0
   \]

   is an open surjection,

   \[\phi\] may not be surjective on objects, but every object in \( H \) is isomorphic to an object in the image of \( G \).
Essential Equivalences, II

2 (Fully faithful)

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\phi} & H_1 \\
\downarrow & & \downarrow \\
(s,t) & \downarrow & (s,t) \\
G_0 \times G_0 & \xrightarrow{\phi \times \phi} & H_0 \times H_0
\end{array}
\]

is a pullback,

The local isotropy structure is preserved.
Morita Equivalent Groupoids

- Two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$ are called **Morita equivalent** if there exists a third orbigroupoid $\mathcal{K}$ with essential equivalences

$$\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}.$$ 

- This is an equivalence relation on groupoids, because essential equivalences of topological groupoids are stable under weak pullbacks (iso-comma-squares).
Examples, I

A line segment can be presented as

```
---   \rightarrow
\ \ \ |     \ \ \ |
\ \ morphisms
\ \ \ |
\ \ |   \ \ |
\ |     |
\ |     |
\ |     |
\ |     |
```

```
---   \rightarrow
\ \ \ |     \ \ \ |
\ \ objects
\ \ \ |
```

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Examples, II

It can also be presented as

\[
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
\begin{array}{c}
\text{morphisms} \\
\hline
\end{array}
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
\begin{array}{c}
\text{morphisms} \\
\hline
\end{array}
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
\begin{array}{c}
\text{morphisms} \\
\hline
\end{array}
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
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\text{morphisms} \\
\hline
\end{array}
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
\begin{array}{c}
\text{morphisms} \\
\hline
\end{array}
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
\begin{array}{c}
\text{morphisms} \\
\hline
\end{array}
\begin{array}{c}
\text{objects} \\
\hline
\end{array}
\begin{array}{c}
\text{morphisms}
\end{array}
\]
Examples, III

Or as:

```
objects

morphisms

objects
```
Examples, IV

Here is our order 3 cone again.

\[
\begin{align*}
\bullet & \quad 2/3 \\
\bullet & \quad 1/3 \\
\bullet & \quad \text{id} \\
\bullet & \quad \text{objects}
\end{align*}
\]
Examples, IV

And here is a Morita equivalent presentation

![Diagram](image-url)
The Bicategory of Orbispaces

Theorem

There is a bicategory of fractions $\text{OrbiGrpd}(W^{-1})$ of orbispaces where:

- **objects** are orbigroupoids;
- **morphisms** \((\text{generalized maps or orbimaps})\) are spans $\mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ where \(w\) is an essential equivalence;
- **2-cells** are equivalence classes of diagrams of the form

\[
\begin{array}{c}
\mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{\phi} \mathcal{H} \\
\downarrow \alpha_1 \downarrow \nu_1 \\
\mathcal{L} \xrightarrow{\nu_2} \mathcal{K}' \xrightarrow{\phi'} \mathcal{H}
\end{array}
\]
The Bicategory of Orbispaces

Theorem

There is a bicategory of fractions $\text{OrbiGrpd}(W^{-1})$ of orbispaces where:

- **objects** are orbigroupoids;
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- **2-cells** are equivalence classes of diagrams of the form

\[
\begin{array}{ccc}
\mathcal{G} & \xleftarrow{\nu_1} & \mathcal{K} & \xrightarrow{\phi} & \mathcal{H} \\
\downarrow{\nu} & & & & \downarrow{\phi} \\
\mathcal{L} & \xleftarrow{\nu_2} & \mathcal{K}' & \xrightarrow{\phi'} & \mathcal{H}
\end{array}
\]
An Example

A map from $I$ to $\mathcal{X}$ (i.e., a path in $\mathcal{X}$):

Replacing $I$ by a Morita equivalent orbigroupoid allows us to jump from one chart to another.
Is the category of orbigroupoids with orbimaps Cartesian closed? I.e., can we define an orbi mapping groupoid $\text{OMap}(G, H)$?


- Yes, according to Behrang Noohi, Mapping stacks of topological stacks, arXiv:0809.2373v2 but this is rather abstract.
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  On a notion of maps between orbifolds, I. Function spaces, 
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Orbi Mapping Spaces in Terms of Groupoids

- We want to use the groupoid description of orbispaces to get a description of the orbi mapping spaces as orbigroupoids.

- There are two ways to do this. Both start by first constructing the mapping groupoids for ordinary groupoid homomorphisms.
Let $\mathcal{G}$ and $\mathcal{H}$ be orbi-groupoids. Then the **mapping groupoid**

$$\text{GMap}(\mathcal{G}, \mathcal{H})$$

in the category of groupoids and groupoid homomorphisms is described as follows.

- **Space of Objects** $\text{GMap}(\mathcal{G}, \mathcal{H})_0$ is the subspace of those $f$ in $\text{Top}(G_1, H_1)$ which preserve composition and units: $m(f, f) = fm$ and

  $$u(G_0) \subseteq G_1$$

  $$\xymatrix{ f \ar[d] & \ar[d] \ar[r]^f & \ar[d] \ar[r]^f & \cr u(H_0) & \subseteq & H_1}$$

- **Space of Arrows** $\text{GMap}(\mathcal{G}, \mathcal{H})_1$ is the subspace of those $(f, \alpha)$ in $\text{GMap}(\mathcal{G}, \mathcal{H})_0 \times \text{Top}(G_0, H_1)$ such that

  $$sf = s\alpha s \text{ and } tf = s\alpha t.$$
Theorem

- If $\mathcal{G}$ is a paracompact Hausdorff groupoid such that the space of orbits, $G_0/G_1$, has finitely many connected components and $\mathcal{H}$ is an orbi-groupoid, then $\text{GMap}(\mathcal{G}, \mathcal{H})$ is an orbi-groupoid.

- For topological groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$,

$$\text{TopGpd}(\mathcal{G} \times \mathcal{H}, \mathcal{K}) \cong \text{TopGpd}(\mathcal{G}, \text{GMap}(\mathcal{H}, \mathcal{K})).$$
G-points of various orbi-groupoids

- $\text{GMap}(\ast \mathbb{Z}/2, \mathbb{Z}/2 \rtimes S^1)$ is the disjoint union $\mathbb{Z}/2 \rtimes S^1$ with two copies of $\ast \mathbb{Z}/2$.

- $\text{GMap}(\ast \mathbb{Z}/2, V_4 \rtimes D)$ is the disjoint union of $V_4 \rtimes D$, two $\mathbb{Z}_2$-lines which have both an additional $\mathbb{Z}/2$-action by reflection, and a $V_4$-point.

- $\text{GMap}(\ast \mathbb{Z}/2, D_3 \rtimes D)$ is Morita equivalent to the disjoint union of $D_3 \rtimes D$ and a $\mathbb{Z}/2$-line.

- $\text{GMap}(\ast \mathbb{Z}/3, D_3 \rtimes D)$ is Morita equivalent to the disjoint union of $D_3 \rtimes D$ and a $\mathbb{Z}/3$-point.
The $\mathbb{Z}/2$-points of the usual rectangular billiard orbifold (with four order 2 corners) form a disjoint union of the same orbifold, four silvered $\mathbb{Z}/2$-intervals and a $V_4$-point.

The $\mathbb{Z}/2$-points of an equilateral triangular billiard orbifold (with three order 3 corners) form the disjoint union of the same orbifold together with a $\mathbb{Z}/2$-circle.

What would be the $\mathbb{Z}/6$-points of the equilateral triangular billiard orbifold?
The orbi mapping groupoid - option 1

Let $\mathcal{G}$ and $\mathcal{H}$ be orbi-groupoids. To obtain $\text{OMap}(\mathcal{G}, \mathcal{H})$, the orbi mapping groupoid, we can encode spans $\mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ where $w$ is an essential equivalence, for the space of objects, and equivalence classes of diagrams

$$
\begin{array}{ccc}
\mathcal{G} & \xleftarrow{\nu} & \mathcal{K} \\
\downarrow{\alpha_1} & & \downarrow{\phi} \\
\mathcal{L} & \xrightarrow{\nu_2} & \mathcal{H}
\end{array}
$$

for the space of arrows, and show that this gives us again an orbigroupoid.
The orbi mapping groupoid - option 2

Alternatively, we may obtain $\text{OMap}(\mathcal{G}, \mathcal{H})$ by considering all orbigroupoids $\text{GMap}(\mathcal{K}, \mathcal{H})$ for essential equivalences $\varphi: \mathcal{K} \to \mathcal{G}$, and take a pseudo colimit of these.

The question is: over which diagram?
Essential Equivalences over $\mathcal{G}$

Given an orbigroupoid $\mathcal{G}$, $\text{EssEq}/\mathcal{G}$ is the 2-category with

- **Objects:** Essential equivalences $\varphi: \mathcal{K} \to \mathcal{G}$.
- **Arrows:** $(\psi, \alpha): (\mathcal{K}, \varphi) \to (\mathcal{K}', \varphi')$ as in

$$
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\psi} & \mathcal{K}' \\
\downarrow & & \downarrow \\
\uparrow \varphi & \Rightarrow & \downarrow \varphi'
\end{array}
$$

- **2-Cells:** $\xi: (\psi_1, \alpha_1) \Rightarrow (\psi_2, \alpha_2)$ where $\xi: \psi_1 \Rightarrow \psi_2$ is a 2-cell in groupoids, such that

$$
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\psi_1} & \mathcal{K}' \\
\downarrow & & \downarrow \\
\uparrow \varphi & \Rightarrow & \downarrow \varphi'
\end{array}
\Rightarrow
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\psi_2} & \mathcal{K}' \\
\downarrow & & \downarrow \\
\uparrow \varphi & \Rightarrow & \downarrow \varphi'
\end{array}
$$

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The Grothendieck Construction

For a small 2-category $\mathbb{D}$ and a 2-functor $J: \mathbb{D}^{\text{op}} \to \text{Cat}$, the category $\int_{\mathbb{D}} J$ is defined as follows:

- **Objects:** $(C, x)$ for $C \in \mathbb{D}_0$ and $x \in J(C)_0$.
- **Arrows:** equivalence classes of pairs $(f, \xi): (C, x) \to (C', x')$,

where $f: C \to C'$ in $\mathbb{D}$ and $\xi: x \to Jf(x') = f^*(x')$ in $J(C)$.

- The equivalence relation is generated by: for any 2-cell $a: f \Rightarrow g: C \Rightarrow C'$ in $\mathbb{D}$, and any $x \in J(C)$, $x' \in J(C')$,

$$ (f, \xi: x \to f^*x') \sim (g, (Ja)_x', \circ \xi: x \to g^*x') $$
Properties of $\int_D J$

- There is an oplax cone $z: JD \to \int_D J$ which gives the oplax colimit of the diagram $J: D^{op} \to \text{Cat}$.

- For $J: D^{op} \to \text{Grpd}$, $\int_D J$ will in general be a category rather than a groupoid.

- When we take the groupoid of fractions of $\int_D J$ for $J: D^{op} \to \text{Grpd}$, we obtain the pseudo colimit of the diagram $J: D^{op} \to \text{Grpd}$.
In our case $\mathcal{D} = \text{EssEq}/\mathcal{G}$.

$J: \text{EssEq}/\mathcal{G} \to \text{TopGrpd}$ is defined by

$$J\left(\mathcal{K} \xrightarrow{\varphi} \mathcal{G}\right) = \text{GMap}(\mathcal{K}, \mathcal{H})$$

and $J$ is defined on arrows and 2-cells by composition.

When we apply the Grothendieck construction with the category of fractions on this diagram we obtain a groupoid which has the property that there is an equivalence of categories from the groupoid encoding the bicategory of fractions diagrams to this new groupoid.

However, we still need a definition of the topology in the second description.
The Topological Case

- For $J: \mathcal{D}^{\text{op}} \to \text{TopGrpd}$, we want to do the whole construction inside the world of topological spaces, but we need to use that the diagram $\mathcal{D}$ is a 2-category internal in $\text{Top}$.

- **The bad news** In general, we cannot use a Grothendieck construction to construct a fibration out of a family of fibers, and fibrations are not colimits.

- **The good news** We can construct a fibration by Cartesian products if the fibers are constant over the connected components of $\mathcal{D}_0$, and this is the case in our example.

- **More good news** The Cartesian products defined this way do form the space of objects of a topological category that forms the oplax colimit of the original diagram.
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Final results

- The topological category just defined satisfies the conditions to apply the internal category of fractions construction.

- The resulting groupoid is Morita equivalent to the one obtained from the bicategory of fractions; to be precise, the equivalence of categories between the two groupoids mentioned before becomes an essential equivalence of topological groupoids.

- So the hom-categories in the bicategory of fractions may be viewed as homotopy/pseudo colimits, both in the categorical and in the topological case.