# Talk on Sheaf Representation

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Note: I didn't actually use slides, but if I had they would be something like the following.

I want to thank Bill Lawvere as so many of the issues, problems and puzzles that I enjoy working on derive their significance from his rich storehouse of ideas.

# Introduction

We look at the limit closure of a full subcategory, $\mathcal{A}$, of a complete category $\mathcal{C}$. In this talk, $\mathcal{C}$ will be the category of commutative rings (with identity) and $\mathcal{A}$ a class of domains such that every field $F$ is a subring of a field in $\mathcal{A}$.

The limit-closure of $\mathcal{A}$ is then reflective and determines a topology on the spectrum of any ring in $\mathcal{C}$ such that, given some first-order conditions, there is a canonical sheaf over $\mathrm{Spec}(R)$.

# Notation

Let $\mathcal{A}$ be a full subcategory of domains as above.

1. Let $\mathcal{K}$ be the limit-closure of $\mathcal{A}$ (in $\mathcal{C}$ = commutative rings with 1). (then $\mathcal{K}$ is a reflective subcategory on $\mathcal{C}$)
2. Let $\mathcal{B}$ be the subcategory of domains in $\mathcal{K}$.
3. For each domain $D$, let $F$ be a field in $\mathcal{A}$ with $D \subseteq F$ and let $G(D)$ denote the smallest subdomain of $F$ which is in $\mathcal{B}$ and which contains $D$. (Then $G(D)$ is, in effect, independent of the choice of $F$.)
4. If $R$ is in $\mathcal{C}$, and $P, Q \in \mathrm{Spec}(R)$, we say that $P \sqsubseteq Q$ if the canonical map $R/P \longrightarrow R/Q$ extends to an $R$-homomorphism $G(R/P) \longrightarrow G(R/Q)$.

# Topologies on the Spectrum

Let $\mathrm{Spec}(R)$ be the set of all prime ideals of $R$. For each $r \in R$ let $N(r) = \{P \in \mathrm{Spec}(R) \mid r \in P\}$.

1. Zariski Topology on $\mathrm{Spec}(R)$: Smallest topology for which every $N(r)$ is closed.
2. Domain Topology on $\mathrm{Spec}(R)$: Smallest topology for which every $N(r)$ is open.
3. Patch Topology on $\mathrm{Spec}(R)$: Smallest topology for which every $N(r)$ is clopen.
4. The $\sqsubseteq$-topology: Defined on the next slide.

# The $\sqsubseteq$ topology and the sheaf over it

We define the $\sqsubseteq$-topology on $\mathrm{Spec}(R)$ so that $V \subseteq \mathrm{Spec}(R)$ is open if and only if it is open in the patch topology and up-closed in the $\sqsubseteq$ ordering ($P \in V$ and $P \sqsubseteq Q$ imply $Q \in V$).

If $\mathcal{B}$ is determined by first-order conditions, there is a canonical sheaf over $\mathrm{Spec}(R)$ (with the above topology) and with stalk $G(R/P)$ at the prime $P$.

Example: $\mathcal{A}$ = all fields. Then $\mathcal{K}$ = regular semiprime rings. Stalk at $P$ is $\mathbf{Q}(R/P)$, the field of fractions (or quotient field) of $R/P$. (The global sections of the sheaf is known to be the reflection of $R$.)

Example: $\mathcal{A}$ = all domains. Then the topology on $\mathrm{Spec}(R)$ is the domain topology. The canonical sheaf has stalk $R/P$ at $P$. (Its global sections is known to be the reflection of $R$ into the limit closure of the domains.)

# A key Proposition

Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in $n$ variables with coefficients in $R$. We say that $f = 0$ has a solution in $G(R/P)$ if there exist $t_1, \ldots, t_n$ in $G(R/P)$ such that $f(t_1, \ldots t_n) = 0$. Let $V$ be the set of prime ideals for which $f = 0$ has a solution in $G(R/P)$. Then $V$ is open in the $\sqsubseteq$-topology on $\mathrm{Spec}(R)$.

Sketch of Proof:

It is readily shown that $V$ is up-closed in the $\sqsubseteq$ order on $\mathrm{Spec}(R)$, so it remains to show that $V$ is open in the patch topology. If not there exists $P \in V$ and an ultrafilter $\mathbf{u}$ on $W$ (the complement of $V$) such that $\mathbf{u}$ converges (in the patch topology) to $P$. Let $K_{\mathbf{u}}$ be the corresponding ultraproduct, given by the quotient map $\{G(R/P) \mid P \in V\} \longrightarrow K_{\mathbf{u}}$.

# Diagram

$$\begin{tikzcd}
 & \prod_{Q \in W} G(R/Q) \arrow[dr] & \\
R \arrow[ur] \arrow[dr] & & K_{\mathbf{u}} \\
 & R/P \arrow[r] & G(R/P) \arrow[u]
\end{tikzcd}$$

Since $\mathbf{u}$ converges to $P$, the map from $R$ to the ultraproduct $K_{\mathbf{u}}$ has kernel $P$. By definition of $G(R/P)$ the map from $R$ factors through $R \longrightarrow G(R/P)$ as shown.

# Sketch of proof, continued

Since $P \in V$ there exists $t_1, \ldots, t_n$ in $G(R/P)$ which is a solution for $f = 0$. The above map sends this to a solution in the ultraproduct $(\prod_{Q \in W} R/Q)_{\mathbf{u}})$ which means there exists $U \in \mathbf{u}$ such that $Q \in U$ implies $G(R/Q)$ has such a solution. But then $Q \in V$ which contradicts $Q \in W$. QED

Corollary:

If $S$ is a finitely presented $R$-algebra, then the set of all prime ideals $P$ for which there exists an $R$-homomorphism $S \longrightarrow G(R/P)$ is open in the $\sqsubseteq$ topology.

# The local sections of the sheaf

Let $E$ be the disjoint union of $G(R/P)$ for $P \in \mathrm{Spec}(R)$. Let $\pi : E \longrightarrow \mathrm{Spec}(R)$ be the map for which $\pi^{-1}(P) = G(R/P)$.

Let $S$ be an $R$-algebra. Say that $s \in S$ is in the **dominion** of $R$ if any pair of $R$-homomorphisms $S \longrightarrow T$ agree at $s$. Let $\zeta \in G(R/P)$ be given. By using the first-order conditions, we can show that every map $R \longrightarrow G(R/P)$ factors as $R \longrightarrow S \longrightarrow G(R/P)$ where $S$ is a finitely presented $R$-algebra with a distinguished element $s$ in the dominion of $R$ which maps to $\zeta$. By the above proposition, it follows that the map $R \longrightarrow G(R/P)$ factors through $S$ in a neighborhood of $P$. Then the images of the special element $s \in S$ trace out what we define as a local section over the neighborhood of $P$. We give $E$ the smallest topology for which every local section is continuous.

# Conclusions

Then $E$ is a sheaf of $R$-algebras over $\mathrm{Spec}(R)$.

The ring of global sections of this sheaf is in $\mathcal{K}$.

As far as we know, this ring is the reflection of $R$ into $\mathcal{K}$ and we have proved that it is the reflection whenever the $\sqsubseteq$-topology coincides with the domain topology or the patch topology.