Fifty Years of Functorial Semantics

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Dear Susan, dear students and colleagues, I thank you warmly for making possible this honor and pleasure.

There are several authors who, when citing me mention only my 1963 thesis. Did I really disappear after that? Perhaps through comments and questions we can arrive at a more explicit historical knowledge of the subject, which in turn may be of help in future developments.

Is the basic theme of Functorial Semantics, namely the application of categorical methods to the study of general algebraic systems and their relationships, indeed still active after 50 years? We can certainly answer 'yes'. For example, Adamek, Rosicky, and Vitale produced a book on the subject only a couple of years ago and some young people are reading it. There was one open question of a basic nature, which I thought should be clarified for any comprehensive treatment. In effect I trusted that both the solution to that general question, as well as the production of a book further disseminating these ideas, would be carried out by my able colleagues who were already contributing substantially to the subject. This trust turned out to be well justified.

Let me recall what that general question was: It could be succinctly described as finding a presentation of the dual doctrine to the doctrine of finite products. Here I use the term 'doctrine', due to Jon Beck, signifying 'something like a theory, but higher'. One interpretation of 'higher' was to treat the logic of (a) higher types, but (b) not simply in terms of predicates on them, using instead a fibrational model of actual proofs, as opposed to the mere existence of proofs; these conceptions were embodied in my 1968 notion of Hyperdoctrine (AMS), which has figured in some later work, for example in Bart Jacobs' thesis (1990), on comprehension categories.

However, in the present connection, I am referring to doctrines in the sense of 'theories whose models are theories', and much more specifically to 2-monads on the category of categories; these are referred to as 'equational doctrines' in Springer Lecture Notes 80, (1969). Let us concentrate on those equational doctrines for which the category S of small sets is an algebra. In particular, the doctrine D whose algebras are categories with finite products is appropriate to the (many-sorted) theories of general algebraic systems in the traditional sense. Because there are 2-dimensional versions of enrichment, and commutativity, et cetera, we can rather freely make use in the category of categories of the hom-tensor operations on bimodules coming from Cartan-Eilenberg 1956. For example, the dual doctrine of any equational doctrine D is the one whose value at any category C is

$$D^* (C) = D\text{-Hom}(S^{\wedge}C,S).$$

This is clearly the 'algebraic structure' of all D-algebraic categories D-Hom(A,S) as A ranges over the category of algebraic theories (= the 2-category of D-algebras).

In other words, the standard lifting describes the maximum structure that all algebraic categories have, insofar as structure is to be described by the meta-doctrine of equational doctrines; an algebraic category is always more than a mere category, because of the way it arises; indeed it is a D* algebra, so that if we wish to represent or approximate some given category by algebraic categories, a first step would be to verify that it be a D* algebra. That may not be sufficient because properties in a still stronger logic may need to be invoked; on the other hand, categorical equivalence ('recovery' or 'descent') is not the only goal of such investigations, because incomplete invariants are often the tool of choice, as for example in algebraic topology. The dual doctrine D* is a very definite thing, but to work with it, a presentation is helpful: obviously small limits, small filtered colimits, and (thanks to Fred Linton) reflexive coequalizers are ingredients of the D*; moreover, these commute and distribute with each other in fairly obvious ways. But are these ingredients sufficient and do these relations generate all the relations; in other words, does this constitute a presentation of D*? The answer took a while, Linton being followed several years later by Pedicchio and Wood, whose results are described, together with much more, in the book by Adamek, Rosicky, & Vitale.

The use of 2-monads above is of course an extension of the version of algebraic semantics and algebraic structure discovered by Beck, by combining the notion of algebraic theory, as extended by Linton, with the Eilenberg-Moore theory of triples (later called monads). In the case where adjoints exist, the structure in monad form is obtained by simply composing a given functor with its adjoint. But note that the case above is also very special in that the functor whose structure is being extracted is itself a dualization functor, so that the monad/doctrine/theory is actually a double-dualization, where the second dualization is strongly constrained by naturality with respect to the domain category (here is a striking formal resemblance to functional analysis, except that the basic 'quantities' are small abstract sets rather than real numbers).

What is this notion of algebraic structure? Refuting the idea that an algebraic theory can only arise syntactically, it simultaneously refutes the idea that algebraic theories constitute a 'new' or 'alternative approach' or 'categorical counterpart' to universal algebra; in fact they constitute an essential feature that was long implicitly present, for example in the study of cohomology operations. Of course it is often helpful if, moreover, a presentation can be found for it. Algebraic structure results simply from the application of the general notion of Natural Structure within the doctrine of algebraic theories. In that case, one could say that it is just a particular case of the fact that for any given object in a category with products, the clone of its finite powers constitutes a single-sorted algebraic theory. But the other crucial aspect of this construction is that the given object is in a functor category, so that the basic naturality condition may severely reduce the size of its algebraic theory, even reducing it to manageable dimensions because a presentation can often be found for it. As a very simple example, suppose that U is a functor whose codomain is the category of finite sets (here U is the traditional symbol used in this context, though it may not resemble any 'underlying set' for the objects X in the domain of U). How much information about the objects X can be extracted from applying the functor U? The first naïve answer is of course that with each object is associated a natural number, namely n(X) = the cardinality of U(X). However, consider the group G of automorphisms of the functor U. The functor U lifts to a new functor Φ to finite G-sets; knowledge of G-sets tells us immediately that the mere n is canonically expressible as a linear combination of other invariants, one for each (conjugacy class of) subgroup(s); the coefficients provide a much more refined measurement for distinguishing objects of the domain category. Going beyond the doctrine of group actions, we could consider not necessarily invertible natural endomorphisms, including for example idempotents, so that the basic measurement U is still further refined; or we could go to arbitrary finitary operations, dealing thus with algebraic structure.

Natural structure with respect to a given doctrine is thus a precise mathematical model for a very general scientific process of concept formation. Observing a domain of individuals that form a collective due to definite mutual relations, and recording these observations as structure that varies in a natural way with respect to those mutual relations, leads to the emergence of general concepts that are abstracted from all the individuals, but that may then be applicable to a larger population, and in terms of which a more precise analysis of the individuals becomes possible.

The mutual relations need not be composable, since in any case the naturality condition that they impose depends on only finitely many at a time; expansion of the sampling graph, causes the partial structures to converge by approximation to the full structure. Since the process of extracting structure is a left adjoint to a contravariant functor, the final result is an inverse limit of the approximations. On the other hand, if the mutual relations can indeed be construed as a category, then there is the possibility that the basic measurement functor might be representable, so that the whole natural structure would be entirely determined by the mutual relation of a small set of special individuals, such as the Eilenberg-Mac Lane spaces in the case of cohomology operations.

The lifted Φ itself sometimes has a left adjoint, that can be considered as the best attempt at descent. It is of interest even if it is not actually inverse. In the case of dual doctrines it is given by the formula D*Hom (( ), S), applied to any given D*-algebra (ie to any algebraic category in the case of the specific D for finite products).

The deeper categorical study of Syntax (in terms of pairs of signatures, etc) has been somewhat neglected due to the two oversimplified views (a) that only Syntax matters, or (b) (the 'new revolutionary' claim) that it does not matter at all. The syntactical presentation of a theory is not unique; moreover even the style of what is meant by presentation is subject to some choice: the usual choice for single-sorted algebraic theories of course involves the functor that assigns to each theory the 'signature', that is the sequence of sets whose nth term is the set of all n-ary operations provided by the theory. That functor has a left adjoint that assigns to every signature the associated free algebraic theory, whose own underlying signature is much bigger. Like many adjoint pairs this one can be factored into two stages, with the intermediate category incorporating a second signature of names for equational axioms and a way of relating the latter to the signature of generators. Here again there are several styles of how this relation could be expressed, so that the factorization is not unique, but in any case the second stage of the factorization involves taking a coequalizer in the category of algebras (or of theories if appropriate). Such procedures are most successful if the adjoint pair is monadic. A categorical study of the various styles of such factorization, as well as of the underlying syntax functors, might be useful. For example, instead of giving two arbitrary maps ('left and right-hand sides of an equation') from relation names to the composite generator names, one could arrange that every basic relation is of the special form stating that a complex expression is to be identified with a mere variable (i.e. a generator) so that all desired axioms can be derived from the fact that two different complex expressions can be identified with the same simple expression. Moreover, an important improvement would exploit the special role of reflexive relations in general algebra, by including in the presentation structure itself a definite proof of each trivial equation $x = x$. (The inclusion of such axioms is motivated by the preservation of finite products under the process of extraction of connected components of reflexive spatial objects such as graphs or globes.)

An old conjecture of William Boone suggests that among the presentations under the category of algebraic theories there would be the same analogous relation to recursivity as is known to hold under the category of groups (Higman) or under the category of first-order theories (Craig & Vaught): A finitely-generated theory can be described by a recursive set of axioms if and only if it can be embedded monomorphically in a larger theory that is finitely-presented. It should be possible to settle this conjecture.

The noble goal of general theory is to illuminate the path of further struggles, both practical and theoretical. That goal can be obscured by the allure and attraction of a 'new' or 'even more universal' theory, deviating us from the path. We are therefore especially happy to have seen that universal concepts, that had been generalized by Birkhoff and others from the work of Galois, Dedekind, Hilbert, and Noether, have again been particularized by our colleagues in a categorical way that illuminates commutative algebra itself.