Tangent categories are locally Cartesian differential categories

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WHAT IS THIS TALK ABOUT?

Answer: The algebraic/categorical foundations for abstract differential geometry.

Tangent categories - introduction

A tangent category is a category $\mathbb{X}$ with an endofunctor T with a natural transformation

$$p : T(A) \longrightarrow A$$

which satisfies certain properties (more below) making $T(A)$ behave like a *tangent bundle* over A.

Tangent categories includes all standard examples from differential geometry but, in addition, models of synthetic differential geometry (SDG), models from combinatorics, and models from Computer Science.

Tangent categories - introduction

- Originally introduced by Rosicky:
  Abstract tangent functors.
  Diagrammes 12, Exp. No. 3, (1984)
  (One citation in 30 years!!)
- With Geoff Crutwell generalized to include the combinatoric and Computer Science examples:
  Differential structure, tangent structure, and SDG.
  To appear in Applied Categorical Structures, 2013.
  - Generalize to additive
    (i.e. commutative monoid – no negation)
  - Clean up the formulation (added proofs)
  - Expanded on the links to SDG and differential manifolds
  - Describe the link to Cartesian differential categories

Tangent categories - introduction

THIS TALK:

- More evidence the axiomatization is right!
- Revisiting the link to Cartesian differential categories ...
- Differential bundles and the structure of tangent spaces ....
- Main result:
  Local logic is given by Cartesian differential categories!

Tangent categories are not easy to manipulate

... a key tool to facilitate their development?

Tangent categories: introduction

The definition of tangent categories:

- Additive bundles $q : E \longrightarrow M$ ...
- The transformations:
  - Tangent spaces: $p : T(A) \longrightarrow A$ (being stable)
  - The vertical lift $\ell : T(A) \longrightarrow T^2(A)$
  - The canonical flip $c : T^2(A) \longrightarrow T^2(A)$
- The coherences ...
- An exactness condition: the universality of the vertical lift.

Tangent categories: additive bundles

An **additive bundle** over $M \in \mathbb{X}$ consists of:

- A map $E \xrightarrow{q} M$ such that pullbacks along q exist;
- Maps $+ : E_2 \longrightarrow E$ and $0 : M \longrightarrow E$, with $+q = \pi_0 q = \pi_1 q$ and $0q = 1$ such that this operation is associative, commutative, and unital; that is, each of the following diagrams commute:

$$
\begin{tikzcd}
E_3 \arrow[r, "\langle\langle\pi_0,\pi_1\rangle+,\pi_2\rangle"] \arrow[d, "\langle\pi_0,\langle\pi_1,\pi_2\rangle+\rangle"'] & E_2 \arrow[d, "+"] \\
E_2 \arrow[r, "+"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E_2 \arrow[d, "\langle\pi_1,\pi_0\rangle"'] \arrow[dr, "+"] & \\
E_2 \arrow[r, "+"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[d, "\langle q0,1\rangle"'] \arrow[dr, equal] & \\
E_2 \arrow[r, "+"'] & E
\end{tikzcd}
$$

A bundle over M is a commutative monoid object in the slice category $\mathbb{X}/M$, $q : E \longrightarrow M$, such that q is **stable**, in the sense that the functor $q \times_M \_$ exists.

Tangent categories: additive bundles

A **bundle morphism** $(f,g) : q \longrightarrow q'$ is a commutative square:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "g"'] & M' \end{tikzcd}$$

An **additive** bundle morphism preserves addition:

$$\begin{tikzcd} E_2 \arrow[r, "\langle \pi_0 f, \pi_1 f \rangle"] \arrow[d, "+"'] & F_2 \arrow[d, "+"] \\ E \arrow[r, "f"'] & F \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "g"] \arrow[d, "0"'] & N \arrow[d, "0"] \\ E \arrow[r, "f"'] & F \end{tikzcd}$$

NOTE: Bundle morphisms are not assumed additive ...

Tangent categories: additive bundles

The category of additive bundles, bun($\mathbb{X}$), is a fibration over $\mathbb{X}$, in which the additive bundle morphisms sit as a subfibration:

$$P : \mathrm{bun}(\mathbb{X}) \longrightarrow \mathbb{X}; \quad \begin{array}{ccc} E & \xrightarrow{q} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ E' & \xrightarrow[q']{} & M' \end{array} \quad \mapsto \quad \begin{array}{c} M \\ \downarrow{\scriptstyle g} \\ M' \end{array}$$

... the stability of the projection map $q : M \longrightarrow E$ is essential to give Cartesian maps!

This is the pattern we will follow for differential bundles ...

Tangent categories: the definition

$\mathbb{X}$ has **tangent structure**, $\mathbb{T} = (T, p, 0, +, \ell, c)$, in case:

- **tangent functor**: a natural transformation $p : T(M) \longrightarrow M$ which is T-stable (i.e. each $T^n(p)$ is stable and T preserves all such pullbacks);
- **tangent bundle**: natural transformations $+ : T_2(M) \longrightarrow T(M)$ and $0 : M \longrightarrow T(M)$ making each $p_M : T(M) \longrightarrow M$ an additive bundle;
- **vertical lift**: natural transformation $\ell : T(M) \longrightarrow T^2(M)$ such that $(\ell_M, 0_M) : (p_M, +, 0) \longrightarrow (T(p_M), T(+), T(0))$ is an additive bundle morphism;
- **canonical flip**: natural transformation $c : T^2 \longrightarrow T^2$ such that $(c_M, 1_{T(M)}) : (T(p_M) \longrightarrow (p_{T(M)}, +_{T(M)}, 0_{T(M)})$ is an additive bundle morphism.

Tangent categories: the coherences

This data must satisfy **coherences** for ℓ and c:

$$c^2 = 1 \qquad \ell c = \ell$$

and the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, "\ell"] \arrow[d, "\ell"'] & T^2 \arrow[d, "T(\ell)"] \\ T^2 \arrow[r, "\ell_T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T(c)"] \arrow[d, "c_T"'] & T^3 \arrow[r, "c_T"] & T^3 \arrow[d, "T(c)"] \\ T^3 \arrow[r, "T(c)"'] & T^3 \arrow[r, "c_T"'] & T^3 \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "\ell_T"] \arrow[d, "c"'] & T^3 \arrow[r, "T(c)"] & T^3 \arrow[d, "c_T"] \\ T^2 \arrow[rr, "T(\ell)"'] & & T^3 \end{tikzcd}$$

Tangent categories: the "universality" of lift

... and one exactness condition:
Universality of vertical lift: the following is a pullback

$$\begin{tikzcd} T_2(M) \arrow[r, "v:=\langle \pi_0\ell, \pi_1 0_T\rangle T(+)"] \arrow[d, "\pi_0 p = \pi_1 p"'] & T^2(M) \arrow[d, "T(p)"] \\ M \arrow[r, "0"'] & T(M) \end{tikzcd}$$

We shall refer to the pair $(\mathbb{X}, \mathbb{T})$ as a **tangent category**.

Having tangent structure is not a property: a given category can be a tangent category in more than one way!

Tangent categories: the “universality” of lift

How is $v := \langle \pi_0 \ell, \pi_1 0_T \rangle T(+)$ defined?

$$
\begin{tikzcd}
 & T_2(M) \arrow[dl] \arrow[ddr] \arrow[rr, dotted, "\ell \times_M 0"'] \arrow[rrr, bend left, "v"] & & T(T_2(M)) \arrow[r, "T(+)"'] \arrow[dl] \arrow[ddr] & T^2(M) \arrow[dddl, "T(p)"] \\
T(M) \arrow[rr, "\ell"] \arrow[ddr, "p"'] & & T^2(M) \arrow[ddr, "T(p)"'] & & \\
 & & T(M) \arrow[rr, "0"'] \arrow[dl, "p"] & & T^2(M) \arrow[dl, "T(p)"] \\
 & M \arrow[rr, "0"'] & & T(M) &
\end{tikzcd}
$$

Tangent categories: examples

Here are some examples of tangent categories:

(i) Finite dimensional smooth manifolds: usual tangent bundle.

(ii) Convenient manifolds with the kinematic tangent bundle.

(iii) Any Cartesian differential category is a tangent category, with $T(A) = A \times A$ and $T(f) = \langle Df, \pi_1 f \rangle$.

(iv) The infinitesimally linear objects in any model of SDG gives a *representable* tangent category.

(v) The opposite of finitely presentable commutative rigs has "representable" tangent structure: given by exponentiating with $\mathbb{N}[\varepsilon] := \mathbb{N}[x]/(x^2 = 0)$, the "rig of infinitessimals".

(vi) The opposite of a category with representable tangent structure also has tangent structure.

(vii) The category of C_∞-rings has tangent structure.

Differential bundles

Vector bundles are an important tool in differential geometry: *differential bundles* are the analogous tool in abstract differential geometry.

A differential bundle is an additive bundles with, in addition, a *lift map* satisfying properties similar to those of the vertical lift of the tangent bundle.

The morphisms between differential bundles are just commuting squares, *linear* bundle morphisms must also preserve the lift. An important observation is:

Lemma

Linear bundle morphisms are always additive bundle morphisms.

Differential bundles: the definition

A **differential bundle** in a tangent category consists of

$$\mathsf{q} = (q : E \longrightarrow M, \sigma : E_2 \longrightarrow E, \zeta : M \longrightarrow E, \lambda : E \longrightarrow TE)$$

where λ is called the **lift**, such that

- (E, q, σ, ζ) is an additive bundle;
- $(\lambda, 0) : (E, q, \sigma, \zeta) \longrightarrow (T(E), T(q), T(\sigma), T(\zeta))$ is additive;
- $(\lambda, \zeta) : (E, q, \sigma, \zeta) \longrightarrow (TE, p, +, 0)$ is addtiive;
- **universality of the lift**, that is the following is a pullback:

$$\begin{tikzcd} E_2 \arrow[r, "\mu := \langle \pi_0 \lambda, \pi_1 0 \rangle T(\sigma)"] \arrow[d, "\pi_0 q = \pi_1 q"'] & T(E) \arrow[d, "T(q)"] \\ M \arrow[r, "0"'] & T(M) \end{tikzcd}$$

  where E_2 the pullback of q along itself;
- the equation $\lambda \ell_E = \lambda T(\lambda)$ holds.

Differential bundles

A **morphism of differential bundles** is simply a bundle morphism; that is, a pair of maps $(f, g) : (q, \sigma, \zeta, \lambda) \longrightarrow (q', \sigma', \zeta', \lambda')$ such that $fq' = qg$:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "q"'] & E' \arrow[d, "q'"] \\ M \arrow[r, "g"'] & M' \end{tikzcd}$$

A morphism of differential bundles is **linear** in case, in addition, it preserves the lift, that is $f\lambda' = \lambda T(f)$:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\lambda"'] & E' \arrow[d, "\lambda'"] \\ T(E) \arrow[r, "T(f)"'] & T(E') \end{tikzcd}$$

Differential bundles: examples

(1) Any object has an associated "trivial" differential bundle $1_M = (1_M, 1_M, 1_M, 0_M)$. Any differential bundle over M has a unique linear bundle map to this bundle, $(q, 1_M) : \mathrm{q} \longrightarrow 1_M$, which is the identity on the base:

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "q"'] & M \arrow[d, "1_M"] \\ M \arrow[r, "1_M"'] & M \end{tikzcd}$$

(2) The tangent bundle of each object M, $\mathrm{p}_M = (p : T(M) \longrightarrow M, +, 0, \ell)$, is clearly a differential bundle and any map $f : N \longrightarrow M$ induces a linear map $(T(f), f) : \mathrm{p}_N \longrightarrow \mathrm{p}_M$ between these tangent bundles.

Differential bundles: $\mathrm{Bun}(\mathbb{X})$

Differential bundles of a tangent category $\mathbb{X}$, with their morphisms, form a category: we write this as $\mathrm{Bun}(\mathbb{X})$.

There is an obvious functor:

$$P : \mathrm{Bun}(\mathbb{X}) \to \mathbb{X} : \quad \begin{array}{c} \mathsf{q} = (q, \sigma, \zeta, \lambda) \\ \Big\downarrow{\scriptstyle (f,g)} \\ \mathsf{q}' = (q', \sigma', \zeta', \lambda') \end{array} \quad \mapsto \quad \begin{array}{c} M \\ \Big\downarrow{\scriptstyle g} \\ M' \end{array}$$

The linear morphism carve out a subcategory $\mathrm{LBun}(\mathbb{X}) \subseteq \mathrm{Bun}(\mathbb{X})$.

Differential bundles: Cartesian maps

If $\mathsf{q} := (q, \sigma, \zeta, \lambda)$ is a differential bundle and $f : N \longrightarrow M$ any map then the pullback:

$$\begin{tikzcd} N \times_M E \arrow[r, "f'"] \arrow[d, "f^*(q)"'] & E \arrow[d, "q"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

makes $f^*(q)$ into a differential bundle and

$$(f', f) : f^*(\mathsf{q}) \longrightarrow \mathsf{q}$$

into a linear morphism which is Cartesian over f.

Lemma

$P : \mathrm{Bun}(\mathbb{X}) \longrightarrow \mathbb{X}$ is a fibration.

Differential bundles as a tangent category

More is true:

Theorem

$\mathrm{Bun}(\mathbb{X})$ *is a tangent category and the fibration* $P : \mathrm{Bun}(\mathbb{X}) \longrightarrow \mathbb{X}$ *preserves the tangent structure.*

If $\mathsf{q} = (q, \sigma, \zeta, \lambda)$ is a differential bundle then its tangent space is

$$T(\mathsf{q}) = (T(q), T(\sigma), T(\zeta), T(\lambda))$$

QUESTION: what do the fibres look like?

Bundles over M form a tangent category

Theorem

Each fibre of $P : \mathrm{Bun}(\mathbb{X}) \longrightarrow \mathbb{X}$ is a Cartesian tangent category and the substitution functors preserve this tangent structure.

A tangent category is *Cartesian* when it has products and the tangent functor preserves products.
Even if $\mathbb{X}$ is not Cartesian the fibres of this fibration are ...

Tangent obtained by pulling back the "total" tangent structure:

$$\begin{tikzcd} 0_*(T(E)) \arrow[r] \arrow[d] & T(E) \arrow[d, "T(q)"] \\ M \arrow[r, "0"'] & T(M) \end{tikzcd}$$

Pulling back along a zero map preserves functorial and exact structure.

Cartesian Differential Categories

WANT: each fibre to be a Cartesian differential category!

To formulate a cartesian differential category need:

(a) Left additive categories

(b) Cartesian products in left additive categories

(c) Differential structure

A category $\mathbb{X}$ is a **left-additive category** in case:

- Each hom-set is a commutative monoid $(0,+)$
- $f(g+h) = (fg) + (fh)$ and $f0 = 0$.

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

A map h is said to be **additive** if it also preserves the additive structure on the right $(f+g)h = (fh) + (gh)$ and $0h = 0$.

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$$

Additive maps form a subcategory ...

Example

(i) The category whose objects are commutative monoids CMon but whose maps need not preserve the additive structure.

(ii) Real vector spaces with smooth maps.

(iii) The coKleisli category for a comonad on an additive category. (Note: the functor need not be (left-)additive)

Products in left additive categories

A **Cartesian left-additive category** is a left-additive category with products such that:

- the maps π_0, π_1, and Δ are additive;
- whenever f and g are additive then $f \times g$ is additive.

Lemma

The following are equivalent:

(i) *A Cartesian left-additive category;*

(ii) *A left-additive category for which $\mathbb{X}_+$ has biproducts and the the inclusion $\mathcal{I} : \mathbb{X}_+ \longrightarrow \mathbb{X}$ creates products;*

(iii) *A Cartesian category $\mathbb{X}$ in which each object is equipped with a chosen commutative monoid structure $(+_A : A \times A \longrightarrow A, 0_A : 1 \longrightarrow A)$ which is* **canonical** *in the sense that $+_{A\times B} = \langle (\pi_0 \times \pi_0)+_A, (\pi_1 \times \pi_1)+_B \rangle$ and $0_{A\times B} = \langle 0_A, 0_B \rangle$.*

The differential operator

An operator $D_\times$ on the maps of a Cartesian left-additive category

$$\frac{X \xrightarrow{f} Y}{X \times X \xrightarrow[D_\times[f]]{} Y}$$

is a **Cartesian differential operator** in case it satisfies:

[CD.1] $D_\times[f+g] = D_\times[f] + D_\times[g]$ and $D_\times[0] = 0$;

[CD.2] $\langle (h+k), v\rangle D_\times[f] = \langle h, v\rangle D_\times[f] + \langle k, v\rangle D_\times[f]$;

[CD.3] $D_\times[1] = \pi_0$, $D_\times[\pi_0] = \pi_0\pi_0$, and $D_\times[\pi_1] = \pi_0\pi_1$;

[CD.4] $D_\times[\langle f, g\rangle] = \langle D_\times[f], D_\times[g]\rangle$ (and $D_\times[\langle\rangle] = \langle\rangle$);

[CD.5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$.

[CD.6] $\langle\langle f, 0\rangle, \langle h, g\rangle\rangle D_\times[D_\times[f]] = \langle f, h\rangle D_\times[f]$;

[CD.7] $\langle\langle 0, f\rangle, \langle g, h\rangle\rangle D_\times[D_\times[f]] = \langle\langle 0, g\rangle, \langle f, h\rangle\rangle D_\times[D_\times[f]]$

A Cartesian left-additive category with such a differential operator is a **Cartesian differential category**.

The differential operator ... again

[CD.1] $D_\times[f+g] = D_\times[f] + D_\times[g]$ and $D_\times[0] = 0$;
(operator preserves additive structure)

[CD.2] $\langle (h+k), v\rangle D_\times[f] = \langle h, v\rangle D_\times[f] + \langle k, v\rangle D_\times[f]$
(always additive in first argument);

[CD.3] $D_\times[1] = \pi_0$, $D_\times[\pi_0] = \pi_0\pi_0$, and $D_\times[\pi_1] = \pi_0\pi_1$
(coherence maps are linear -differential constant);

[CD.4] $D_\times[\langle f, g\rangle] = \langle D_\times[f], D_\times[g]\rangle$ (and $D_\times[\langle\rangle] = \langle\rangle$)
(operator preserves pairing);

[CD.5] $D_\times[fg] = \langle D_\times[f], \pi_1 f\rangle D_\times[g]$ (chain rule);

[CD.6] $\langle\langle f, 0\rangle, \langle h, g\rangle\rangle D_\times[D_\times[f]] = \langle f, h\rangle D_\times[f]$
(differentials are linear[1] in first argument);

[CD.7] $\langle\langle 0, f\rangle, \langle g, h\rangle\rangle D_\times[D_\times[f]] = \langle\langle 0, g\rangle, \langle f, h\rangle\rangle D_\times[D_\times[f]]$
(partial differentials commute);

[1] In the sense of the differential being constant.

Basic example of a differential operator

Real vector spaces with smooth maps are the "standard" example of a Cartesian differential category.

$$\frac{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} f_1(x_1, .., x_n) \\ \vdots \\ f_m(x_1, .., x_n) \end{pmatrix}}{\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}\right) \mapsto \begin{pmatrix} \frac{\mathrm{d}f_1(\tilde{x})}{\mathrm{d}x_1}(x_1) \cdot u_1 + \ldots + \frac{\mathrm{d}f_1(\tilde{x})}{\mathrm{d}x_n}(x_n) \cdot u_n \\ \vdots \\ \frac{\mathrm{d}f_m(\tilde{x})}{\mathrm{d}x_1}(x_1) \cdot u_1 + \ldots + \frac{\mathrm{d}f_m(\tilde{x})}{\mathrm{d}x_n}(x_n) \cdot u_n \end{pmatrix}} \; \mathrm{D}$$

Tangents and differentials ...

Theorem

Every Cartesian differential category is a tangent category with $T(X) = X \times X$ and $T(f) = \langle D[f], \pi_1 f \rangle$.

BUT not every tangent category is a differential category ...

Tangents and differentials ...

When is a Tangent category a differential category?

Theorem

For a Cartesian tangent category the following are equivalent:

(i) *Every object is canonically a differential object*

(ii) *Every object is canonically a differential bundle over the final object*

(iii) *It is canonically a Cartesian differential category.*

A differential bundle over the final object is, in particular, a commutative monoid object with $T(A) \cong A \times A$ which is the presentation as a differential object.

The word *canonically* is the requirement that the structures behave coherently with respect to both the product *and* the tangent functor.

Tangents in the fibres ...

Recall that the tangent structure in the fibres is Cartesian.

Also (it turns out) that every object is *canonically* a differential bundle over the final object. This gives:

Theorem

In the fibration $P : \mathrm{Bun}(\mathbb{X}) \longrightarrow \mathbb{X}$ *each fibre is a Cartesian differential category and the substitution functors preserve this structure.*

Tangents in the fibres ...

One aspect of the proof: recall the tangent obtained by pulling back the "total" tangent structure:

$$\begin{tikzcd}
T_M(E) = 0_*(T(E)) \arrow[r] \arrow[d] & T(E) \arrow[r, "p"] \arrow[d, "T(q)"] & E \arrow[d, "q"] \\
M \arrow[r, "0"] \arrow[rr, bend right, equal] & T(M) \arrow[r, "p"] & M
\end{tikzcd}$$

This pullback is given by the universality of lift!!!

$$\begin{tikzcd}
T_M(E) = E_2 \arrow[r, "v"] \arrow[d, "\pi_1 q"'] & T(E) \arrow[d, "T(q)"] \\
M \arrow[r, "0"'] & T(M)
\end{tikzcd}$$

But this shows $T_M(E) = E_2 = E \times_M E$ which is a key property of a differential object ...