

MTH 197 DEFINITIONS

Definition 1. A universal statement says that a certain property is true for all elements of a set.

Definition 2. An existential statement says some property is true for at least one object.

Definition 3. A conditional statement says that if something is true, then something else has to be true.

Definition 4. A universal conditional statement is both universal and conditional.

Definition 5. A universal existential statement is a two-part statement whose first part is universal and second part is existential.

Definition 6. An existential universal statement is a two-part statement whose first part is existential and second part is universal.

Definition 7. The set of integers is the set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

Definition 8. The set of rational numbers is

$$\mathbb{Q} = \left\{ x = \frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

Definition 9. An ordered pair is an object of the form (a, b) , where a and b are any objects.

Definition 10. Two ordered pairs are equal, $(a, b) = (c, d)$, provided that $a = c$ and $b = d$.

Definition 11. If A and B are sets, then the Cartesian product of A and B , denoted by $A \times B$ is the set of ordered pairs (a, b) , where $a \in A$ and $b \in B$.

Definition 12. Let A and B be sets. Then A is called a subset of B , written $A \subseteq B$, provided that every element of A is an element of B .

Definition 13. Let A and B be sets. A relation R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in R , we say “ x is related to y ” and write xRy .

The set A is called the domain of R and set B is called the co-domain.

Definition 14. A function F from A to B is a relation from A to B satisfying:

- (1) For all $x \in A$, there exists $y \in B$ such that $(x, y) \in F$.
- (2) For all $x \in A$ and $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

Definition 15. A statement (or proposition) is a sentence that is true or false but not both.

Definition 16. Let p and q be statements.

- (1) $\sim p$ is read “not p ” and is called the negation of p . It is the statement defined by

$$\sim p : \begin{cases} \text{TRUE} & \text{when } p \text{ is FALSE} \\ \text{FALSE} & \text{when } p \text{ is TRUE} \end{cases}$$

- (2) $p \wedge q$ is read “ p and q ” and is called the conjunction of p and q . It is the statement defined by

$$p \wedge q \text{ is } \begin{cases} \text{TRUE} & \text{when both } p \text{ and } q \text{ are TRUE} \\ \text{FALSE} & \text{otherwise (i.e., at least one is FALSE)} \end{cases}$$

- (3) $p \vee q$ is read “ p or q ” and is called the disjunction of p and q . It is the statement defined by

$$p \vee q \text{ is } \begin{cases} \text{TRUE} & \text{when at least one of } p \text{ and } q \text{ is TRUE} \\ \text{FALSE} & \text{otherwise (i.e., both } p \text{ and } q \text{ are FALSE)} \end{cases}$$

Definition 17. A statement form (or propositional form) is a well-formed expression made up of statement variables (p, q, r, \dots) and logical connectives ($\sim, \wedge, \vee, \dots$).

Definition 18. Two statement forms P and Q are logically equivalent, denoted $P \equiv Q$, if they have identical truth values for every possible assignment of truth values to their statement variables.

Definition 19.

- A tautology is a statement form that is always true, regardless of the truth values assigned to the its variables.
- A contradiction is one that is always false, regardless of the truth values assigned to its variables.

Definition 20. The conditional of p by q is “ p implies q ” and denoted by $p \rightarrow q$. It is the statement defined by

$$p \rightarrow q \text{ is } \begin{cases} \text{FALSE} & \text{when } p \text{ is TRUE and } q \text{ is FALSE} \\ \text{TRUE} & \text{otherwise} \end{cases}$$

Definition 21. The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Definition 22. The converse of $p \rightarrow q$ is $q \rightarrow p$.

Definition 23. The biconditional of p and q is written $p \leftrightarrow q$ (or p iff q , or $p \Leftrightarrow q$), read “ p if and only if q ,” and it means $p \rightarrow q$ and $q \rightarrow p$.

Definition 24. An argument form is valid if when the premises are all true, then the conclusion is true, no matter what statements are substituted for the statement variables in the premises. A valid argument form is called a rule of inference.

Definition 25. The following rule of inference is called modus ponens

$$\begin{array}{l} \text{If } p, \text{ then } q. \\ p \\ \therefore q \end{array}$$

Definition 26. The following rule of inference is called modus tollens

$$\begin{array}{l} \text{If } p, \text{ then } q. \\ \sim q \\ \therefore \sim p \end{array}$$

Definition 27. A predicate is a sentence that contains a finite number of variables and becomes a statement (T or F) when specific values are substituted for the variables. The domain of a predicate (variable) is the set of all values that may be substituted in place of the variable.

Definition 28. If $P(x)$ is a predicate with domain D , then the truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true. It is denoted by

$$\{x \in D \mid P(x)\}.$$

Definition 29. (takes over Definition 1) A universal statement is one of the form

$$\forall x \in D, Q(x)$$

where $Q(x)$ is a predicate with domain D . It is defined to be true if $Q(x)$ is true for every x in D , and false if $Q(x)$ is false for at least one x in D .

Definition 30. (takes over Definition 2) An existential statement is one of the form

$$\exists x \in D \text{ such that } Q(x)$$

where $Q(x)$ is a predicate with domain D . It is defined to be true if $Q(x)$ is true for at least one x in D , and false if $Q(x)$ is false for every x in D .

Definition 31. The contrapositive of the statement $\forall x \in D, P(x) \rightarrow Q(x)$ is $\forall x \in D, \sim Q(x) \rightarrow \sim P(x)$ and the converse is $\forall x \in D, Q(x) \rightarrow P(x)$

Definition 32. An integer n is even provided that $n = 2k$, for some $k \in \mathbb{Z}$.

Definition 33. An integer n is odd provided that $n = 2k + 1$, for some $k \in \mathbb{Z}$.

Definition 34. An integer n is prime provided that $n > 1$ and for all positive integers r and s , if $n = rs$, then $r = n$ or $s = n$.

An integer n is composite provided that $n > 1$ and $n = rs$, for some integers r, s with $1 < r < n$ and $1 < s < n$.

Definition 35. A real number r is rational provided that $r = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$ s.t. $b \neq 0$. The set of rational numbers is defined by

$$\mathbb{Q} = \left\{ x \in \mathbb{R} \mid x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Definition 36. Let $n, d \in \mathbb{Z}$. Then n is divisible by d provided that $n = dk$ for some $k \in \mathbb{Z}$ and $d \neq 0$.

Definition 37. Given a non-negative integer n and a positive integer d ,

- $n \text{ div } d =$ the integer quotient of n divided by d ,
- $n \text{ mod } d =$ the integer remainder of n divided by d .

Definition 38. Given a real number x , the floor of x , denoted $\lfloor x \rfloor$, is the unique integer n such that $n \leq x < n + 1$.

Definition 39. Given a real number x , the ceiling of x , denoted $\lceil x \rceil$, is the unique integer n such that $n - 1 < x \leq n$.

Definition 40. Given $n \in \mathbb{Z}^+$, let $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$. This is read “ n factorial.” Also, $0! = 1$.

This is a Theorem, not a definition. You are asked to learn the statement of this Theorem.

The Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n and let a be a fixed integer. Suppose the following two properties hold:

- (1) $P(a)$ is true
- (2) $\forall k \geq a$, if $P(k)$ is true, then $P(k + 1)$ is true.

Then $P(n)$ is true for all $n \geq a$.

Definition 41. (takes over Definition 12) Let A and B be sets. Then A is a subset of B , denoted $A \subseteq B$, provided that

$$\forall x(x \in A \rightarrow x \in B)$$

Definition 42. If A and B are sets, then $A = B$ provided that $A \subseteq B$ and $B \subseteq A$.

Definition 43. A is a proper subset of B provided that $A \subseteq B$ and $A \neq B$.

Definition 44. Let A and B be sets. Then:

(1) A intersect B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

(2) A union B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

(3) A minus B is the set $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Definition 45. The empty set, denoted \emptyset , is the set with no elements.

Definition 46. Let A be a set. The power set of A , denoted by $\mathcal{P}(A)$, is the set of all subsets of A , i.e.,

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}.$$

Note: this means, $X \in \mathcal{P}(A) \leftrightarrow X \subseteq A$

Definition 47. Given sets A_1, A_2, \dots and a nonnegative integer n , define:

(1) $\bigcap_{i=1}^n A_i = A_1 \cap \dots \cap A_n = \{x \mid x \in A_i, \text{ for all } i = 1, 2, \dots, n\}$

(2) $\bigcap_{i=1}^{\infty} A_i = \{x \mid x \in A_i, \text{ for all } i \geq 1\}$

(3) $\bigcup_{i=1}^n A_i = A_1 \cup \dots \cup A_n = \{x \mid x \in A_i, \text{ for at least one } i = 1, 2, \dots, n\}$

(4) $\bigcup_{i=1}^{\infty} A_i = \{x \mid x \in A_i, \text{ for at least one } i \geq 1\}$

Definition 48. Suppose f and g are functions from X to Y . Then f equals g , written $f = g$, provided that $f(x) = g(x)$, for all $x \in X$.

Definition 49. Given a set X , the identity function on X is the function $I_X: X \rightarrow X$ defined by $I_X(x) = x$.

Definition 50. An (n -place) Boolean function is a function $f: \{0, 1\}^n \rightarrow \{0, 1\}$.

Definition 51. Let $f: X \rightarrow Y$. Then f is one-to-one provided that

$$\forall x_1, x_2 \in X, f(x_1) = f(x_2) \rightarrow x_1 = x_2$$

or equivalently,

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$$

Definition 52. Let $f: X \rightarrow Y$. Then f is onto provided that

$$\forall y \in Y, \exists x \in X \text{ s.t. } y = f(x)$$

Definition 53. Let $f: X \rightarrow Y$. Then f is a bijection or one-to-one correspondence provided that f is one-to-one and onto.

Definition 54. Let R be a relation from A to B . The inverse relation R^{-1} from B to A is defined by $R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}$.

Definition 55. A relation on a set A is a relation from A to A .

Definition 56. Suppose R is a relation on a set A . Then

(r) R is reflexive iff $\forall x \in A, x R x$.

(s) R is symmetric iff $\forall x, y \in A, x R y \rightarrow y R x$.

(t) R is transitive iff $\forall x, y, z \in A, x R y$ and $y R z \rightarrow x R z$.

Definition 57. A binary relation R on a set A is an equivalence relation provided that it is reflexive, symmetric, and transitive.

Definition 58. Let R be an equivalence relation on A and $a \in A$. Then the equivalence class of a is denoted by $[a]$ and defined by $[a] = \{x \in A \mid x R a\}$.

Definition 59. Two sets A and B are called disjoint if $A \cap B = \emptyset$.

Definition 60. A finite or infinite collection \mathcal{P} of nonempty subsets A_i of a set A is a partition of A provided that

(1) $\forall x \in A$, there is some A_i in \mathcal{P} such that $x \in A_i$.

(2) For all A_i and A_j in \mathcal{P} , if $A_i \neq A_j$, then A_i and A_j are disjoint.

Definition 61. Let \mathcal{P} be a partition of A . The relation $R_{\mathcal{P}}$ on A induced by \mathcal{P} is defined by

$$(x, y) \in R_{\mathcal{P}} \text{ iff } \exists A_i \in \mathcal{P} \text{ s.t. } x \text{ and } y \text{ are both in } A_i.$$

Definition 62. Let R be a relation on a set X . Then R is antisymmetric provided that $\forall x, y \in X$, if $x R y$ and $y R x$, then $x = y$.

Definition 63. Let R be a relation on X . Then R is a partial order relation provided that R is reflexive, antisymmetric, and transitive.

Definition 64. A set X together with the partial order R on X is called a partially ordered set or poset.

Definition 65. A total order relation on X is a partial order R such that $\forall x, y \in X$, either $x R y$ or $y R x$.

Definition 66. Let (X, \leq) be a poset. A subset $C \subseteq X$ is called a chain provided that $\forall x, y \in C$, $x \leq y$ or $y \leq x$. The length of C is one less than the number of elements in C .

Definition 67. Let (X, \leq) be a poset. Then $a \in X$ is called a:

- (1) maximal element iff $x \leq a$ or x and a are not comparable, for all $x \in X$
- (2) greatest element iff $x \leq a$, for all $x \in X$
- (3) minimal element iff $a \leq x$ or x and a are not comparable, for all $x \in X$
- (4) least element iff $a \leq x$, for all $x \in X$

Definition 68. Suppose \leq and \leq' are partial orders on a set X . Then \leq' is a refinement of \leq provided that $x \leq y \rightarrow x \leq' y$, for all $x, y \in X$.

Definition 69. (Basic Counting Principle) Suppose 2 experiments are to be performed.

If one experiment can result in m possibilities

Second experiment can result in n possibilities

Then together there are mn possibilities

Definition 70. With n objects, a **permutation** is an arrangement/ordering of n objects. There are

$$n(n-1) \cdots 3 \cdot 2 \cdot 1 = n!$$

different **permutations** of the n objects.

Definition 71. If $r \leq n$, then

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

and we say “ n choose r ”, represents the number of possible **combinations** of objects taken r at a time.