Discrete Math for Computer Science - Problems

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Contents

CHAPTER 1

Speaking Mathematically

Section 1.1 - Variables, Statements

(1) Fill in the blanks using a variable or variables to rewrite

Given any real number, there is a real number that is smaller.

(a) Given any real number ______ , there is a real number _______ such that _______.

Section 1.2 - The Language of Sets

(1) Consider the following sets

$$A = \{a, b, \{a\}, \{c, \heartsuit\}, \{\{a\}, 1\}\}$$
$$B = \{x \in \mathbb{Z} \mid -1 < x \leq 3\}$$
$$C = \{x \in \mathbb{R} \mid -1 < x \leq 3\}$$
$$D = \{x \in \mathbb{N} \mid -1 < x \leq 3\}$$
$$E = \{a, b, 1, \heartsuit\}$$
$$F = \{a, 1\}$$
$$G = \{a, \{a\}\}$$
$$H = \{z \in \mathbb{Z} \mid 0 \leq x < 4\}$$

(a) Which sets are equal to each other?
(b) Is $a \subset A$?
(c) Is $a \in A$?
(d) Is $\{a, b\} \in A$?
(e) Is $\{a, b\} \subset A$?
(f) Is $\{c, \heartsuit\} \subset A$?
(g) Is $\{c, \heartsuit\} \in A$?
(h) Is $\{a, \{\{a\}, 1\}\} \subset A$?
(i) How many elements are in A?
(j) Is $E \subset A$?
(k) Is $F \subset A$?
(l) Is $G \subset A$?

(2) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.
(a) $A \times B$
(b) $B \times A$
(c) Is $A \times B = B \times A$?
(d) $B \times B$
(e) $A \times B \times B$

Section 1.3 - The Language of Relations and Functions

(1) Let $A = \{1, 2, 3\}$ and $B = \{-2, -1, 0\}$ and define a relation R from A to B as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in A \times B \text{ means that } \frac{x-y}{3} \text{ is an integer}$$

(a) Is $3R0$? Is $1R(-1)$? Is $(2, -1) \in R$? Is $(3, -2) \in R$?
(b) Write R as a set of ordered paired.
(c) Write the domain and co-domain or R.
(d) Draw an arrow diagram for R.
(e) Is R a function?

CHAPTER 2

The Language of Compound Statements

Section 2.1 - Logical Form and Logical Equivalence

(1) Make a truth table for $(p \wedge q) \vee (\sim p \wedge r)$.
(2) Suppose

$$p = \text{“}\frac{17}{95} \in \mathbb{Z}\text{”}, \\ q = \text{“}\{95\} \in \{17, \{95\}\}\text{”} \\ r = \text{“}(u, l) \in \{P, A\} \times \{u, l\}\text{”}.$$

Use the table you made for Problem 1 to find out if $(p \wedge q) \vee (\sim p \wedge r)$ is True or False.

Section 2.2- Conditional Statements

(1) Find out if $p \wedge q \rightarrow r$ is logically equivalent to $\sim p \vee \sim q \vee r$.

Section 2.3 - Valid and Invalid Arguments

- None. See textbook HW.

CHAPTER 3

The logic of Quantified Statements

Section 3.1 Introduction to Predicates and quantified Statements I.

- See textbook problems

Section 3.2 Predicates and Quantified Statements II

(1) Write the negation of each statement
   (a) $\forall x, y \in \mathbb{R}$, $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$.
   (b) $\forall x \in \mathbb{R}$, if $x > 2$ then $x^4 > 16$.
   (c) $\forall x \in \mathbb{R}$, if $0 < x < y$ then $x^2 < y^2$.
   (d) $\exists n \in \mathbb{Z}^+$ such that $3n > 5$.
(2) Write the converse and the contrapositive of
   (a) $\forall x \in \mathbb{R}$, if $x > 2$ then $x^4 > 16$.
   (b) $\forall x > 0$, if $x < y$ then $x^2 < y^2$.

Section 3.3 - Statements with Multiple Quantifiers.

(1) Figure if the statement is true or false.

$$\text{``}\exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R},\ x = y + 1\text{''}$$

(2) Write the negation of the following statement:

$$\text{``}\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R},\ \text{if } |x - x_0| < \delta \text{ then } |f(x) - L| < \epsilon\text{''}.$$

This statement is actually the formal mathematical definition of $\lim_{x \to x_0} f(x) = L$.

CHAPTER 4

Elementary Number Theory and Methods of Proof

Section 4.1/4.2 Direct Proof and Counterexample I. and II.

(1) Prove the statement: "For any integer n, m, if n is even and m is odd then $2n + 5m$ is odd"
(2) Prove the statement: "For all integers n, if $n > 6$ then $n^2 - 25$ is composite".

Section 4.3 Direct Proof and Counterexample III. Rational Numbers

(1) Prove the statement: "If r, s are rational numbers then $r - s$ is rational "
(2) Prove the statement: " If $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ then nr is a rational number "

Section 4.4 Direct Proof and Counterexample IV: Divisibility Properties

(1) Prove the statement: " For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$"

Section 4.5 Direct Proof and Counterexample V: Division Into Cases; the Quotient-Remainder Theorem.

(1) Use the Quotient-Remainder Theorem with $d = 2$ to show that: "The square of any integer can be written as either $4k$ or $4k + 1$ for some integer k."

Section 4.6 - Direct Proof and Counterexample VI: Floor and Ceiling.

- See textbook problems.

Section 4.7 - Indirect Argument: Contradiction and Contrapositive.

(1) Prove the statement "The square root of any positive irrational number is irrational"

Section 4.8 - Two Classical Theorems

(1) Prove the statement "Suppose x is irrational and $n, m \in \mathbb{Z}$ and $n \neq 0$. The $nx + m$ is irrational. "

CHAPTER 5

Sequences, Mathematical Induction, and Recursion

Section 5.1 - Sequences

- do all the book problems

Section 5.2 - Mathematical Induction I: Proving formulas

(1) Prove that

$$\sum_{i=1}^{n}(5i-4)=\frac{n\,(5n-3)}{2}.$$

(2) Prove

$$\sum_{k=1}^{n}k^2=\frac{n\,(n+1)\,(2n+1)}{6},$$

for all $n\geq 1$.

Section 5.3 - Mathematical Induction: Applications

(1) Prove that

$$9 \mid ((10)^n - 1) \text{ for all integers } n \geq 0.$$

(2) Use induction to prove the following: Suppose $a_1, a_2, \ldots$ is a sequence defined by the recursion

$$\begin{cases} a_1 = 18 \\ a_n = 9a_{n-1} \quad \text{for } n \geq 2. \end{cases}$$

Prove: $a_n = 2 \cdot 9^n$, for all integers $n \geq 1$.

Section 5.4 - Strong Induction

(1) Use strong induction to prove the following: Suppose $a_1, a_2, \ldots$ is sequence defined by the recursion relation:

$$\begin{cases} a_1 = 7 \\ a_2 = 14 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2. \end{cases}$$

Prove: The sequence a_n is divisible by 7, for all integers $n \geq 1$.

(2) Use strong induction to prove the following: Suppose $c_1, c_2, \ldots$ is sequence defined by the recursion relation:

$$\begin{cases} c_1 = 2 \\ c_2 = 5 \\ c_n = c_{n-1} \cdot c_{n-2} \quad \text{for } n \geq 3. \end{cases}$$

Prove: The sequence c_n is even for $n \geq 3$.

CHAPTER 6

Set Theory

Section 6.1 - Set Theory: Definitions and the Element Method of Proof

(1) Find the power set of $\mathcal{P}(\{x, y, z\})$.

Section 6.2 - Set Proofs; properties of sets

(1) **Prove:** For all sets A, B and C if $A \subseteq B$ then $(A \cup C) \subseteq (B \cup C)$.
(2) **Prove:** For all sets A, B and C if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.
(3) **Prove:** For all sets A, B if $A \subseteq B$ then $A \cap B = A$.

CHAPTER 7

Functions

Section 7.1 - Functions defined on general sets

- Do all book problems

Section 7.2 - One-to-one and Onto Functions.

(1) Show that $f : \mathbb{Z} \to \mathbb{Z}$ defined by
$$f(n) = -3n + 1$$
is one-to-one by not onto.

(2) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = 3x + 7$$
is bijective.

(3) Can a function $f : \{0, 1\} \to \{0, 1, 2\}$ ever be onto? Prove or give an example.
- **Solution:**

CHAPTER 8

Properties of Relations

Section 8.1 - Relations on Sets

- Do book problems

Section 8.2 - Reflexivity, Symmetry, and Transitivity.

(1) Define a relation R on $\mathbb{Z}$ by

$$mRn \text{ if and only if } 5 \mid (m-n).$$

Prove R is an equivalence relation by showing the following three parts.

(a) **Part(a):** Show R is reflexive.
(b) **Part(b):** Show R is symmetric.
(c) **Part(c):** Show R is transitive.

(2) Find all equivalence classes of the equivalence relation R on $A = \{a, b, c, d, 1\}$

$$R = \left\{ \begin{array}{lllll} (a,a) & ,(b,b) & ,(c,c) & ,(d,d) & (1,1) \\ (a,b) & ,(b,a) & & ,(d,1) & (1,d) \end{array} \right\}$$

Section 8.3 - Equivalence Relations

(1) Find the induced relation $R_{\mathcal{P}}$ for each partition $\mathcal{P}$ of the set $A = \{a, b, c, d, e\}$
   (a) $\mathcal{P} = \{\{a, b, c\}, \{d, e\}\}$
   (b) $\mathcal{P} = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$

CHAPTER 9

Probability (based on Lectures Notes)

Section 9.1 - Counting

(1) Suppose a License plate must consist of 7 numbers or letters. How many license plates are there if
  (a) there can only be letters?
  (b) the first three places are numbers and the last four are letters?
  (c) the first three places are numbers and the last four are letters, but there can not be any repetitions in the same license plate?

(2) A school of 50 students has awards for the top math, english, history and science student in the school
  (a) How many ways can these awards be given if each student can only win one award?
  (b) How many ways can these awards be given if students can win multiple awards?

(3) An iPhone password can be made up of any 4 digit combination.
  (a) How many different passwords are possible?
  (b) How many are possible if all the digits are odd?
  (c) How many can be made in which all digits are different or all digits are the same?

(4) An $n-$place Boolean function is a function of the form $f : \{0,1\}^n \to \{0,1\}$. How many $n-$place Boolean functions exist?

(5) There is a class of 25 people made up of 11 guys and 14 girls.
  (a) How many ways are there to make a committee of 5 people?
  (b) How many ways are there to pick a committee of 5 of all girls?
  (c) How many ways are there to pick a committee of 3 girls and 2 guys?

(6) If a student council contains 10 people, how many ways are there to elect a president, a vice president, and a 3 person prom committee from the group of 10 students?

(7) Suppose you are organizing your textbooks on a book shelf. You have three chemistry books, 5 math books, 2 history books and 3 english books.
  (a) How many ways can you order the textbooks if you must have math books first, english books second, chemistry third, and history fourth?
  (b) How many ways can you order the books if each subject must be ordered together?

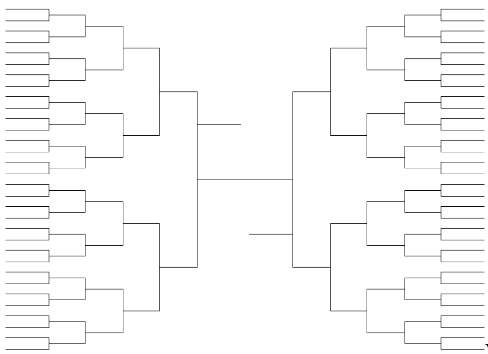
(8) You buy a Powerball lottery ticket. You choose 5 numbers between 1 and 59 (picked on white balls) and one number between 1 and 35 (picked on a red ball).How many ways can you
  (a) win the jackpot (guess all the numbers correctly)?
  (b) match all the white balls but not the red ball?
  (c) match 3 white balls and the red ball?

(9) A couple wants to invite their friends to be in their wedding party. The groom has 8 possible groomsmen and the bride has 11 possible bridesmaids. The wedding party will consist of 5 groomsman and 5 bridesmaids.
  (a) How many wedding party's are possible?
  (b) Suppose that two of the possible groomsmen are feuding and will only accept an invitation if the other one is not going. How many wedding party's are possible?
  (c) Suppose that two of the possible bridesmaids are feuding and will only accept an invitation if the other one is not going. How many wedding party's are possible?
  (d) Suppose that one possible groosman and one possible woman refuse to serve together. How many wedding party's are possible?

(10) There are 52 cards in a standard deck of playing cards. The poker hand is consists of five cards. How many poker hands are there?

(11) There are 30 people in a communications class. Each student must have a one-on-one conversation with each student in the class for a project. How many total one-on-one convesations will there be?

(12) Suppose a college basketball tournament consists of 64 teams playing head to head in a knock-out style tournament. There are 6 rounds, the round of 64, round of 32, round of 16, round of 8, the final four teams, and the finals. Suppose you are filling out a bracket such as this



which specifies which teams will win each game in each round. How many possible brackets can you make?

(13) You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?

Section 9.2 - Introduction to Probability

(1) Suppose a box contains 3 balls : 1 red, 1 green, and 1 blue
   (a) Consider an experiment that consists of randomly selecting 1 ball from the box and then replacing it in the box and drawing a second ball from the box. List all possible outcomes in the sample space.
   (b) Consider an experiment that consists of randomly selecting 1 ball from the box and then drawing a second ball from the box without replacing the first. List all possible outcomes.

(2) Suppose that A and B are mutually exclusive (disjoint) events for which $P(A) = .3$ and $P(B) = .5$.
   (a) What is the probability that A occurs but B does not? (i.e. find $P(A \cap B^c)$)
   (b) What is the probability that neither A nor B occurs? (i.e. find $P(A^c \cap B^c)$)

(3) Forty percent of college students from a certain college are members of neither an academic club nor a greek organization. Fifty percent are members of academic clubs while thirty percent are members of a greek organization. Suppose a student is chosen at random, what is the probability that this students is a member
   (a) of an academic club or a greek organization?
   (b) of an academic club and a greek organization?

(4) In City, 60% of the households subscribe to newspaper A, 50% to newspaper B, 40% to newspaper C, 30% to A and B, 20% to B and C, and 10% to A and C, but none subscribe to all three. (Hint: Draw a Venn diagram)
   (a) What percentage subscribe to exactly one newspaper?
      - This tells us that 30% of households subscribe to exactly one paper.
   (b) What percentage subscribe to at most one newspaper?

Section 9.3 - Computing Probabilities

(1) A pair of fair dice is rolled. What is the probability that the first die lands on a strictly higher value than the second die.

(2) Nine balls are randomly withdrawn from an urn that contains 10 blue, 12 red, and 15 green balls. What is the probability that
   (a) 2 blue, 5 red, and 2 green balls are withdrawn
   (b) at least 2 blue balls are withdrawn.

(3) Suppose 4 valedictorians (from different high schools) were all accepted to the 8 Ivy League universities. What is the probability that they each choose to go to a different Ivy League university?

(4) There are 8 students in a class. What is the probability that at least two students share a common birthday month?

Section 9.4 - Independent Events and Conditional Probability

(1) Let A and B be two *independent* events with $P(A) = .4$ and $P(A \cup B) = .64$. What is $P(B)$?

(2) Two dice are rolled. Let $S_3 = \{\text{sum of two dice equals 3}\}$, $S_7 = \{\text{sum of two dice equals 7 }\}$, and $A_1 = \{\text{at least one of the dice shows a 1}\}$.

   (a) What is $\mathbb{P}(S_3 \mid A_1)$?

   (b) What is $\mathbb{P}(S_7 \mid A_1)$?

   (c) Are S_3 and A_1 indepedent? What about S_7 and A_1?

(3) Suppose you roll two standard, fair, 6-sided dice. What is the probability that the sum is at least 9 given that you rolled at least one 6?

Section 9.5- Bayes's Formula

(1) Suppose Phanuel is a young bachelor. Phanuel goes to a bar 7 nights a week: 3 of the nights at bar A, 2 of the nights at bar B, and 2 of the nights at bar C. After asking, he'll get a girl's number 99 percent of the time at bar A, 96 percent of the time at bar B , and only 85 percent of the time at bar C.
   (a) On a random night of the week, what is the probability that he gets a number?
   (b) Given that he does get a number, what is the probability that it was at bar A?

(2) Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available.
   (a) What is the chance that a purchased bulb will work for longer than 5000 hours? (Hint: Use Law of Total Probability, the numerator of Bayes's Formula)
   (b) Given that a lightbulb works for more than 5000 hours, what is the probability that it came from factory Y? (Hint: Bayes's formula, or just recall the definition of conditional probability)
   (c) Given that a lightbulb work does not work for more than 5000 hours, what is the probability that it came from factory X?

(3) A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from Machine A?

(4) A multiple choice exam has 4 choices for each question. A student has studied enough so that the probability they will know the answer to a question is 0.5, the probability that they will be able to eliminate one choice is 0.25, otherwise all 4 choices seem equally plausible. If they know the answer they will get the question right. If not they have to guess from the 3 or 4 choices. As the teacher you want the test to measure what the student knows. If the student answers a question correctly what's the probability they knew the answer?

(5) A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not actually present. One percent of the population actually has the disease. Calculate the probability that a person actually has the disease given that the test indicates the presence of the disease.