Discrete Math for Computer Science - Problems

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Contents

CHAPTER 1

Speaking Mathematically

Section 1.1 - Variables, Statements

(1) Fill in the blanks using a variable or variables to rewrite

Given any real number, there is a real number that is smaller.

(a) Given any real number _______ , there is a real number ________ such that ________.

- **Solution:** Given any real number x, there is a real number y such that $y < x$.

Section 1.2 - The Language of Sets

(1) Consider the following sets

$$\begin{aligned}
A &= \{a, b, \{a\}, \{c, \heartsuit\}, \{\{a\}, 1\}\} \\
B &= \{x \in \mathbb{Z} \mid -1 < x \leq 3\} \\
C &= \{x \in \mathbb{R} \mid -1 < x \leq 3\} \\
D &= \{x \in \mathbb{N} \mid -1 < x \leq 3\} \\
E &= \{a, b, 1, \heartsuit\} \\
F &= \{a, 1\} \\
G &= \{a, \{a\}\} \\
H &= \{z \in \mathbb{Z} \mid 0 \leq x < 4\}
\end{aligned}$$

(a) Which sets are equal to each other?
- **Solution:** Clearly A, E, F, G are not related to the other sets Now

$$\begin{aligned}
B &= \{0, 1, 2, 3\}, \\
C &= (-1, 3] \text{ this is interval notation from Calculus} \\
D &= \{1, 2, 3\} \\
H &= \{0, 1, 2, 3\}
\end{aligned}$$

so only $B = H$.

(b) Is $a \subset A$?
- **Solution:** No, a is an element, not a set.

(c) Is $a \in A$?
- **Solution:** Yes.

(d) Is $\{a, b\} \in A$?
- **Solution:** No because $\{a, b\}$ is a set, not an element.

(e) Is $\{a, b\} \subset A$?
- **Solution:** Yes

(f) Is $\{c, \heartsuit\} \subset A$?
- **Solution:** No

(g) Is $\{c, \heartsuit\} \in A$?
- **Solution:** Yes

(h) Is $\{a, \{\{a\}, 1\}\} \subset A$?
- **Solution:** Yes

(i) How many elements are in A?
- **Solution:** 5

(j) Is $E \subset A$?
- **Solution:** No

(k) Is $F \subset A$?
- **Solution:** No

(l) Is $G \subset A$?
- **Solution:** Yes

(2) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.

(a) $A \times B$
- **Solution:** We have

$$A \times B = \{(1, a), (1.b), (2, a), (2.b), (3, a), (3, b)\}$$

6 elements

(b) $B \times A$
- **Solution:** We have

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

6 elements

(c) Is $A \times B = B \times A$?

- **Solution:** No, they clearly have different elements.

(d) $B \times B$

- **Solution:** We have

$$B \times B = \{(a,a), (a,b), (b,a), (b,b)\}$$

4 elements

(e) $A \times B \times B$

- **Solution:** We have that all possible combinations are

$$\begin{aligned} A \times B \times B = \{&(1,a,a), (1,a,b), (1,b,a), (1,b,b) \\ &(2,a,a), (2,a,b), (2,b,a), (2,b,b) \\ &(3,a,a), (3,a,b), (3,b,a), (3,b,b)\} \end{aligned}$$

12 elements

Section 1.3 - The Language of Relations and Functions

(1) Let $A = \{1, 2, 3\}$ and $B = \{-2, -1, 0\}$ and define a relation R from A to B as follows: For every $(x, y) \in A \times B$,

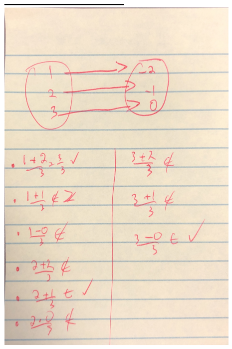
$$(x, y) \in A \times B \text{ means that } \frac{x - y}{3} \text{ is an integer}$$

(a) Is $3R0$? Is $1R(-1)$? Is $(2, -1) \in R$? Is $(3, -2) \in R$?
- **Solution:** By checking
- $\frac{3-0}{3} = 1 \in \mathbb{Z}$ yes
- $\frac{1+1}{3} \notin \mathbb{Z}$ no
- $\frac{2+1}{3} = 1 \in \mathbb{Z}$ yes
- $\frac{3+2}{3} \notin \mathbb{Z}$ no

(b) Write R as a set of ordered paired.
- **Solution:** By checking all possibilities you get

$$R = \{(1, -2), (2, -1), (3, 0)\}$$

(c) Write the domain and co-domain or R.
- **Solution:** The domain is $A = \{1, 2, 3\}$ and co-domain is $B = \{-2, -1, 0\}$.

(d) Draw an arrow diagram for R.
- **Solution:**
- 

(e) Is R a function?
- **Solution:** Yes because **every** element of the domain is mapped to a **unique** element in B.

CHAPTER 2

The Language of Compound Statements

Section 2.1 - Logical Form and Logical Equivalence

(1) Make a truth table for $(p \wedge q) \vee (\sim p \wedge r)$.

- **Solution:** We have

p	q	r	$p \wedge q$	$\sim p$	$\sim p \wedge r$	$(p \wedge q) \vee (\sim p \wedge r)$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	T	T
F	T	F	F	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	F	F

(2) Suppose

$$p = \text{“}\frac{17}{95} \in \mathbb{Z}\text{”},$$
$$q = \text{“}\{95\} \in \{17, \{95\}\}\text{”}$$
$$r = \text{“}(u, l) \in \{P, A\} \times \{u, l\}\text{”}.$$

Use the table you made for Problem 1 to find out if $(p \wedge q) \vee (\sim p \wedge r)$ is True or False.

- **Solution:**
- Clearly p is false, q is true, and r is false.
- Hence using the table:

p	q	r	$p \wedge q$	$\sim p$	$\sim p \wedge r$	$(p \wedge q) \vee (\sim p \wedge r)$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	T	T
F	**T**	**F**	**F**	**T**	**F**	**F**
F	F	T	F	T	T	T
F	F	F	F	T	F	F

- Then

$$(p \wedge q) \vee (\sim p \wedge r) \text{ is false .}$$

Section 2.2- Conditional Statements

(1) Find out if $p \wedge q \to r$ is logically equivalent to $\sim p \vee \sim q \vee r$.

- **Solution:**

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge q \to r$	$\sim p \vee \sim q \vee r$
T	T	T	F	T	T	T	T
T	T	F	F	T	T	F	F
T	F	T	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	F	F	T	T
F	F	F	T	F	F	T	T

same truth values

-
- So yes $p \wedge q \to r \equiv \sim p \vee \sim q \vee r$.

Section 2.3 - Valid and Invalid Arguments

- None. See textbook HW.

CHAPTER 3

The logic of Quantified Statements

Section 3.1 Introduction to Predicates and quantified Statements I.

- See textbook problems

Section 3.2 Predicates and Quantified Statements II

(1) Write the negation of each statement

(a) $\forall x, y \in \mathbb{R}$, $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$.

- **Solution:**
- $\exists x, y \in \mathbb{R}$ such that $xy > \frac{x^2}{2} + \frac{y^2}{2}$

(b) $\forall x \in \mathbb{R}$, if $x > 2$ then $x^4 > 16$.

- **Solution:**
- Recall that an if/then statement is equivalent to $(p \to q) \equiv (\sim p \vee q)$. Thus to negate an if/then statement, we use de Morgan's law:

$$\sim (p \to q) \equiv p \wedge \sim q$$

- Thus the negation: "$\exists x \in \mathbb{R}$ such that $x > 2$ and $x^4 \leq 16$".

(c) $\forall x \in \mathbb{R}$, if $0 < x < y$ then $x^2 < y^2$.

- **Solution:**
- Recall that an if/then statement is equivalent to $(p \to q) \equiv (\sim p \vee q)$. Thus to negate an if/then statement, we use de Morgan's law:

$$\sim (p \to q) \equiv p \wedge \sim q$$

- Thus the negation: "$\exists x, y \in \mathbb{R}$ such that $0 < x < y$ and $x^2 \geq y^2$".

(d) $\exists n \in \mathbb{Z}^+$ such that $3n > 5$.

- **Solution:**
- $\forall n \in \mathbb{Z}^+$, $3n \leq 5$.

(2) Write the converse and the contrapositive of

(a) $\forall x \in \mathbb{R}$, if $x > 2$ then $x^4 > 16$.

- **Solution:**

(i) Converse: $\forall x \in \mathbb{R}$, if $x^4 > 16$ then $x > 2$.

(ii) Contrapositive: $\forall x \in \mathbb{R}$, if $x^4 \leq 16$ then $x \leq 2$.

(b) $\forall x > 0$, if $x < y$ then $x^2 < y^2$.

- **Solution:**

(i) Converse: $\forall x > 0$, if $x^2 < y^2$ then $x < y$.

(ii) Contrapositive: $\forall x > 0$, if $x^2 \geq y^2$ then $x \geq y$.

Section 3.3 - Statements with Multiple Quantifiers.

(1) Figure if the statement is true or false.

$$\text{“}\exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R},\ x = y + 1\text{”}$$

- **Solution:**
- This is false.
- Because it is saying that there exists a special real number x that is equal to

$$x = y + 1$$

for **every** real number y. But there is no number x that can be equal to multiple numbers at the same time.

(2) Write the negation of the following statement:

$$\text{“}\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R},\ \text{if } |x - x_0| < \delta \text{ then } |f(x) - L| < \epsilon\text{”}.$$

This statement is actually the formal mathematical definition of $\lim_{x \to x_0} f(x) = L$.

- **Solution:**
- Recall that an if/then statement is equivalent to $(p \to q) \equiv (\sim p \vee q)$. Thus to negate an if/then statement, we use de Morgan's law:

$$\sim (p \to q) \equiv p \wedge \sim q$$

- Thus the negation:

$$\text{“}\exists \epsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in \mathbb{R} \text{ such that } |x - x_0| < \delta \text{ and } |f(x) - L| \geq \epsilon\text{”}.$$

CHAPTER 4

Elementary Number Theory and Methods of Proof

Section 4.1/4.2 Direct Proof and Counterexample I. and II.

(1) Prove the statement: "For any integer n, m, if n is even and m is odd then $2n + 5m$ is odd"

- **Solution:**
- **Sketch of Proof:**
- Given: $n, m \in \mathbb{Z}$, n even, m odd which means

$$n = 2k, \text{ for some } k \in \mathbb{Z}$$
$$m = 2l + 1, \text{ for some } l \in \mathbb{Z}$$

- WTS: Want to show $2n + 5m$ is odd, which basically means

$$2n + 5m = 2(\text{some integer}) + 1$$

- Do it:
  - Try adding

$$\begin{aligned} 2n + 5m &= 2(2k) + 5(2l + 1) \\ &= 4k + 10l + 5 \\ &= (4k + 10l + 4) + 1 \\ &= 2(2k + 5l + 2) + 1 \end{aligned}$$

  - Woohoo! Found the "some integer", which is $x = 2k + 5l + 2$.
- **Formal Proof:**

Proof. Suppose $n, m \in \mathbb{Z}$, n even, m odd. By the definition of even and odd, this means

$$n = 2k, \text{ for some } k \in \mathbb{Z}$$
$$m = 2l + 1, \text{ for some } l \in \mathbb{Z}.$$

By substitution,

$$\begin{aligned} 2n + 5m &= 2(2k) + 5(2l + 1) \\ &= 4k + 10l + 5 \\ &= (4k + 10l + 4) + 1 \\ &= 2(2k + 5l + 2) + 1 \end{aligned}$$

Let $x = 2k + 5l + 2$. Since $\mathbb{Z}$ is closed under multiplication and addition, then $x \in \mathbb{Z}$. Thus

$$2n + 5m = 2x + 1$$

and since $x \in \mathbb{Z}$ then by the definition of off, $2n + 5m$ is odd. □

(2) Prove the statement: "For all integers n, if $n > 6$ then $n^2 - 25$ is composite".

- **Solution:**
- **Sketch of Proof:**
- Given: $n \in \mathbb{Z}$ and $n > 6$
- WTS: Want to show $n^2 - 25$ is composite.
  - The definition of composite is: m is composite provided $m > 1$ and

$$m = rs$$

for some integers r, s with $1 < r < n$ and $1 < s < n$.

  - So we need to show $n^2 - 25$ is bigger than 1 and factors like

$$n^2 = (\text{some integer})\,(\text{some other integer})$$

    where the integers are strictly betwee 1 and $n^2 - 25$.

- Do it:
  - Try factoring

$$n^2 - 25 = (n-5)\,(n+5)$$

- **Formal Proof:**

PROOF. Suppose $n \in \mathbb{Z}$ and $n > 6$. Then clearly $n^2 - 25 > 1$. Also

$$n^2 - 25 = (n-5)\,(n+5)$$

and since $n > 6$ then

$$1 < (n-5) < \left(n^2 - 25\right) \text{ and } 1 < (n+5) < \left(n^2 - 25\right).$$

Thus by letting $r = n - 5$ and $s = n + 5$ we've shown that

$$n^2 - 25 = rs$$

where $1 < r < n$ and $1 < s < n$, and r, s are integers since $\mathbb{Z}$ is closed under subtration and addition. Thus by the definition of composite, $n^2 - 25$ is also composite. □

Section 4.3 Direct Proof and Counterexample III. Rational Numbers

(1) Prove the statement: "If r, s are rational numbers then $r - s$ is rational "
- **Solution:**
- **Sketch of Proof:**
- Given: $r, s \in \mathbb{Q}$, which means
$$r = \frac{a}{b}, \text{ where } a, b \in \mathbb{Z}, b \neq 0$$
$$s = \frac{c}{d}, \text{ where } c, d \in \mathbb{Z}, d \neq 0.$$
- WTS: Want to show $r - s$ is rational. That means,
$$r - s = \frac{\text{integer}}{\text{some integer not equal to zero}}$$
- Do it:
  - Try subtracting
$$r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$
  - Woohoo! Found the numerator and denominator.
- **Formal Proof:**

PROOF. Suppose $r, s \in \mathbb{Q}$, which means
$$r = \frac{a}{b}, \text{ where } a, b \in \mathbb{Z}, b \neq 0$$
$$s = \frac{c}{d}, \text{ where } c, d \in \mathbb{Z}, d \neq 0.$$
By substitution,
$$r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}.$$
Since $\mathbb{Z}$ is closed under subtration, multiplication, then $n = ad - cb$ and $m = bd$ are integers. Moreover, by the Zero-Product Property since $b, d \neq 0$ then $m = bd \neq 0$. Thus we've shown
$$r - s = \frac{n}{m} \text{ where } n, m \in \mathbb{Z}, \text{and } m \neq 0,$$
which means $r - s$ is a rational number. □

(2) Prove the statement: " If $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ then nr is a rational number "
- **Solution:**
- **Sketch of Proof:**
- Given: $n \in \mathbb{Z}$ and that $r \in \mathbb{Q}$, which means
$$r = \frac{a}{b}, \text{ where } a, b \in \mathbb{Z}, b \neq 0.$$
- WTS: Want to show nr is rational. That means,
$$nr = \frac{\text{integer}}{\text{some integer not equal to zero}}$$
- Do it:
  - Try substituting
$$nr = n\frac{a}{b} = \frac{na}{b}$$
  - Woohoo! Found the numerator and denominator.

- **Formal Proof:**

Proof. Suppose $n \in \mathbb{Z}$ and that $r \in \mathbb{Q}$, which means

$$r = \frac{a}{b}, \text{ where } a, b \in \mathbb{Z}, b \neq 0.$$

By substitution,

$$\begin{aligned} nr &= n\frac{a}{b} \\ &= \frac{na}{b} \end{aligned}$$

Since $\mathbb{Z}$ is closed under multiplication, then $m = na$ is an integer. Thus we've shown

$$nr = \frac{m}{b} \text{ where } m, b \in \mathbb{Z}, \text{and } b \neq 0,$$

which means nr is a rational number. ☐

Section 4.4 Direct Proof and Counterexample IV: Divisibility Properties

(1) Prove the statement: " For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$"

- **Solution:**
- **Sketch of Proof:**
- Given: $a, b \in \mathbb{Z}$ and that $a \mid b$ which means "a divides b". By the formal definition of divides,

$$b = ka \text{ for some } k \in \mathbb{Z}.$$

- WTS: Want to show $a^2 \mid b^2$ That means, I want to show (using the definition) that

$$b^2 = (\text{some integer})\, a^2.$$

- Do it:
  - Try substituting

$$\begin{aligned} b^2 &= (ka)^2 \\ &= k^2 a^2 \end{aligned}$$

  - Woohoo! Found the "some integer" which is k^2.
- **Formal Proof:**

PROOF. Suppose $a, b \in \mathbb{Z}$ and that $a \mid b$. By the definition, this means

$$b = ka \text{ for some } k \in \mathbb{Z}.$$

By substitution,

$$\begin{aligned} b^2 &= (ka)^2 \\ &= k^2 a^2 \end{aligned}$$

Let $l = k^2$. Since $\mathbb{Z}$ is closed under multiplication, then $l = k \cdot k \in \mathbb{Z}$. Thus we've shown

$$b^2 = la^2 \text{ where } l \in \mathbb{Z},$$

which means $a^2 \mid b^2$. □

Section 4.5 Direct Proof and Counterexample V: Division Into Cases; the Quotient-Remainder Theorem.

(1) Use the Quotient-Remainder Theorem with $d = 2$ to show that: "The square of any integer can be written as either $4k$ or $4k + 1$ for some integer k."

- **Solution:**
- **Sketch of Proof:**
- What do we know? In this case, the problem tell us you use the Q-R statement with $d = 2$: For any integer $n \in \mathbb{Z}$ there exists a unique $q, r \in \mathbb{Z}$ such that

$$n = 2q + r, \text{ where } 0 \le r < 2.$$

- But the only integers r that satify $0 \le r < 2$ are $r = 0, 1$. Thus this splits n into 2 cases:
- Either
  - Case 1: $n = 2q$, or
  - Case 2: $n = 2q + 1$,
- WTS: What do you want to show? I want to show n^2 can be written as (a) $n^2 = 4k$ or (b) $n^2 = 4k + 1$ for some k. We need to find k!
- Try to prove it:
  - Case 1: If $n = 2q$, then

$$n^2 = (2q)^2 = 4q^2 = 4\underbrace{\left(q^2\right)}_{k}$$

and the k is equal to $k = q^2$. Thus n^2 can be written in the type (a):

$$n^2 = 4k.$$

  - Case 2: If $n = 2q + 1$, then

$$\begin{aligned} n^2 &= (2q+1)^2 = 4q^2 + 4q + 1 = 4\left(q^2 + q\right) + 1 \\ &= 4\underbrace{\left(q^2 + q\right)}_{k} + 1 \end{aligned}$$

and the k is equal to $k = q^2 + q$. Thus n^2 can be written in the type (b):

$$n^2 = 4k + 1$$

- **Formal Proof:**

PROOF. Suppose $n \in \mathbb{Z}$. Using the Q-R Theorem with $d = 2$, there exists a unique $q, r \in \mathbb{Z}$ such that

$$n = 2q + r, \text{ where } 0 \le r < 2.$$

But the only integers r that satify $0 \le r < 2$ are $r = 0, 1$. Thus this splits n into 2 cases:

$$n = 2q \text{ or } = 2q + 1 \text{ for some integer } q.$$

Case 1: If $n = 2q$, then

$$n^2 = (2q)^2 = 4q^2 = 4\underbrace{\left(q^2\right)}_{k}.$$

Let $k = q^2$. Since $\mathbb{Z}$ is closed under multiplication, then $k \in \mathbb{Z}$. Thus n^2 can be written as

$$n^2 = 4k, \text{ for some integer } k = q^2.$$

Case 2: If $n = 2q + 1$, then

$$\begin{aligned} n^2 &= (2q+1)^2 = 4q^2 + 4q + 1 = 4\left(q^2 + q\right) + 1 \\ &= 4\underbrace{\left(q^2 + q\right)}_{k} + 1. \end{aligned}$$

Let $k = q^2 + q$. Since $\mathbb{Z}$ is closed under multiplication and addition, then $k \in \mathbb{Z}$. Thus n^2 can be written as

$$n^2 = 4k + 1, \text{ for some integer } k.$$

□

Section 4.6 - Direct Proof and Counterexample VI: Floor and Ceiling.

- See book problems.

Section 4.7 - Indirect Argument: Contradiction and Contrapositive.

(1) Prove the statement "The square root of any positive irrational number is irrational"
- **Solution:**

PROOF. We want to show that if $x > 0$ is irrational then $\sqrt{x}$ is irrational. We prove this by contradiction.

Suppose $\sqrt{x}$ is not irrational, meaning that it is rational. This means $\sqrt{x} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. Squaring both sides we get

$$x = \frac{a^2}{b^2}.$$

Since $\mathbb{Z}$ is closed under multipliaction then $a^2, b^2 \in \mathbb{Z}$ and $b^2 \neq 0$. This shows x is rational, which is a contradiction since we know x is irrational. Thus $\sqrt{x}$ should have never been assumed to be rational. Hence $\sqrt{x}$ is irrational. □

Section 4.8 - Two Classical Theorems

(1) Prove the statement "Suppose x is irrational and $n, m \in \mathbb{Z}$ and $n \neq 0$. The $nx + m$ is irrational. "

- **Solution:**

PROOF. Let x be irrational and $n, m \in \mathbb{Z}$ and $n \neq 0$.We prove that $nx + m$ is irrational by contradicition. Suppose not. Meaning, suppose that $nx + m$ is rational. Then this means

$$nx + m = \frac{a}{b}, \text{ for some } a, b \in \mathbb{Z}, b \neq 0.$$

Rearraning to solve for x we obtain,

$$x = \frac{a}{bn} - \frac{m}{n}$$

which is legal since $n \neq 0$. Moreover we have that

$$x = \frac{a - mn}{bn}.$$

But since $\mathbb{Z}$ is closed under multiplication, addition and subtraction then this means that $a - mn \in \mathbb{Z}$ and $bn \in \mathbb{Z}$. Hence we've shown x to be rational, but this is a contradicition. Hence we never should have assumed that $nx + m$ was rational. Hence $nx + m$ is irrational. □

CHAPTER 5

Sequences, Mathematical Induction, and Recursion

Section 5.1 - Sequences

- do all the book problems

Section 5.2 - Mathematical Induction I: Proving formulas

(1) Prove that
$$\sum_{i=1}^{n}(5i-4)=\frac{n\,(5n-3)}{2}.$$

- **Solution:**

Proof. We prove $P(n)$ for all $n \geq 1$ where $P(n)$ is the statement
$$\sum_{i=1}^{n}(5i-4)=\frac{n\,(5n-3)}{2}$$

Base Case: We show $P(1)$ is true. The left hand side is
$$5-4=1$$
while the right hand side
$$\frac{1(5-3)}{2}=\frac{2}{2}=1.$$
Since they are equal, the formula is true for $n=1$.

Inductive Hypothesis: Suppose $P(k)$ is true. That is, suppose
$$\sum_{i=1}^{k}(5i-4)=\frac{k\,(5k-3)}{2}. \quad (\star)$$

Inductive Step: Using $(\star)$, we show the formula is true for $n=k+1$. That is, we want to show
$$\sum_{i=1}^{k+1}(5i-4)=\frac{(k+1)\,(5\,(k+1)-3)}{2}$$
simplifying the RHS we have
$$\begin{aligned}\text{RHS}=\frac{(k+1)\,(5\,(k+1)-3)}{2}&=\frac{(k+1)\,(5k+2)}{2}\\&=\frac{5k^2+7k+2}{2}.\end{aligned}$$
Thus we want to show
$$\sum_{i=1}^{k+1}(5i-4)=\frac{5k^2+7k+2}{2} \quad (\star\star).$$
Starting with the LHS we have
$$\begin{aligned}\sum_{i=1}^{k+1}(5i-4)&=\underbrace{\sum_{i=1}^{k}(5i-4)}+(5\,(k+1)-4)\\&=\frac{k\,(5k-3)}{2}+(5\,(k+1)-4)\,,\ \text{by inducive hypothesis }(\star)\\&=\frac{k\,(5k-3)}{2}+(5k+1)\\&=\frac{k\,(5k-3)+10k+2}{2}\\&=\frac{5k^2-3k+10k+2}{2}\\&=\frac{5k^2+7k+2}{2},\end{aligned}$$
which proves $(\star)$ is true.

Hence the formula is true for all $n \geq 1$ by mathematical induction. □

(2) Prove

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

for all $n \geq 1$.

- **Solution:**

Proof. For $n \in \mathbb{N}$ let $P(n)$ be the statement

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Base Case: For $n = 1$, we have that the left hand side is

$$\text{LHS} = 1$$

and the right hand side is

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1.$$

Thus the formula is true for $n = 1$.

Inductive hypothesis: We assume the formula holds for $n = k$. That is, assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}. \quad (\star)$$

is true.

Inductive step: Using the inductive hypothesis, we want to show that the formula holds for $n = k + 1$, that is

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \frac{(k+1)(k+2)(2(k+1)+1)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}. \quad (\star\star) \end{aligned}$$

To prove this, we start with the Left Hand side(LHS),

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \underbrace{1 + 2^2 + 3^2 \cdots + k^2} + (k+1)^2 \\ &= \underbrace{\frac{k(k+1)(2k+1)}{6}} + (k+1)^2, \text{ by inductive hypothesis } (\star) \\ &= \underbrace{\frac{k(k+1)(2k+1)}{6}} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)\left[2k^2 + k + 6k + 6\right]}{6} \\ &= \frac{(k+1)\left[2k^2 + 7k + 6\right]}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Thus we have shown $(\star\star)$.

Thus by induction, the formula holds for all integers $n \geq 1$. □

Section 5.3 - Mathematical Induction: Applications

(1) Prove that

$$9 \mid ((10)^n - 1) \text{ for all integers } n \geq 0.$$

- **Solution:**

PROOF. For $n \geq 0$ let $P(n)$ be the statement

$$9 \mid ((10)^n - 1).$$

Base Case: For $n = 0$, we want to show $9 \mid \left((10)^0 - 1\right)$, which means $9 \mid 0$. But this is true since

$$0 = 9 \cdot 0 \text{ and } 0 \in \mathbb{Z}.$$

Thus the statement is true for $n = 0$.

Inductive hypothesis: We assume the statement holds for $n = k$. That is, assume

$$9 \mid \left((10)^k - 1\right).$$

This means that

$$(10)^k - 1 = 9m \text{ for some } m \in \mathbb{Z}. \quad (\star)$$

Inductive step: Using the inductive hypothesis, we want to show that the formula holds for $n = k + 1$, that is

$$9 \mid \left((10)^{k+1} - 1\right).$$

Meaning, we want to show that

$$(10)^{k+1} - 1 = 6\,(\text{some integer}). \quad (\star\star).$$

To prove this, we start with the Left Hand side(LHS) of $(\star\star)$.

$$\begin{aligned}(10)^{k+1} - 1 &= 10 \cdot \underbrace{(10)^k} - 1\\ &= 10 \cdot \underbrace{(9m+1)} - 1, \text{ by inductive hypothesis } (\star)\\ &= 10 \cdot 9m + 10 - 1\\ &= 10 \cdot 9m + 9\\ &= 9\,(10m + 1).\end{aligned}$$

Thus we have shown $(\star\star)$, with the integer being $10m + 1 \in \mathbb{Z}$ (because $\mathbb{Z}$ is closed under multiplication and addition).

Thus by induction, the statement holds for all integers $n \geq 0$. □

(2) Use induction to prove the following: Suppose $a_1, a_2, \ldots$ is a sequence defined by the recursion

$$\begin{cases} a_1 = 18 \\ a_n = 9a_{n-1} \quad \text{for } n \geq 2. \end{cases}$$

Prove: $a_n = 2 \cdot 9^n$, for all integers $n \geq 1$.

- **Solution:**

PROOF. Let a_n be the sequence defined by the recursion above. For $n \in \mathbb{N}$ let $P(n)$ be the assertion that

$$\text{“}a_n = 2 \cdot 9^n.\text{”}$$

Base Case: For $n = 1$, we have that $a_1 = 18$ by definition, and using the fomula

$$2 \cdot 9^1 = 18$$

hence $a_1 = 2 \cdot 9^1$ and so $P(1)$ holds.

Inductive hypothesis: We assume $P(k)$ holds for $n = k$ where $k \geq 1$. That is, assume

$$a_k = 2 \cdot 9^k. \quad (\star)$$

Inductive step: Using the inductive hypothesis, we want to show that the formula holds for $n = k+1$. Meaning, we want to show that

$$a_{k+1} = 2 \cdot 9^{k+1}. \quad (\star\star).$$

To prove this, we start with the Left Hand side(LHS) of $(\star\star)$: (Recall we can only prove this using **recursion** defined in the problem, and **equation** $(\star)$)

$$\begin{aligned} a_{k+1} &= 9a_k, \text{ by definition of recursion and since } k+1 \geq 2 \\ &= 9 \cdot \underbrace{a_k} \\ &= 9 \cdot \underbrace{2 \cdot 9^k}, \text{ by inductive hypothesis } (\star) \\ &= 9 \cdot 2 \cdot 9^k \\ &= 2 \cdot 9^{k+1}. \end{aligned}$$

Thus we have shown $(\star\star)$..

Thus by induction, the statement holds for all integers $n \geq 1$. □

Section 5.4 - Strong Induction

(1) Use strong induction to prove the following: Suppose $a_1, a_2, \ldots$ is sequence defined by the recursion relation:

$$\begin{cases} a_1 = 7 \\ a_2 = 14 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2. \end{cases}$$

Prove: The sequence a_n is divisible by 7, for all integers $n \geq 1$.

- **Solution:**

PROOF. Let a_n be the sequence defined by the recursion relation above. For $n \in \mathbb{N}$ let $P(n)$ be the assertion that

$$\text{“}a_n \text{ is divisible by 7”}.$$

Base Case: (we now have 2 base cases) We need to show $P(1)$ and $P(2)$ are true. Since $a_1 = 7$ and $a_2 = 14 = 2 \cdot 7$ is it clear that both are integers are divisible by 4.

Inductive hypothesis: We assume for $k \geq 2$, $P(1), P(2), \ldots, P(k)$ all hold. That is, assume

$$\begin{aligned} &\text{“}a_1 \text{ is divisible by 7”} \\ &\text{“}a_2 \text{ is divisible by 7”} \\ &\vdots \\ &\text{“}a_{k-1} \text{ is divisible by 7”} \\ &\text{“}a_k \text{ is divisible by 7”} \quad (\star) \end{aligned}$$

Inductive step: Using the inductive hypothesis, we want to show that the statement holds for $n = k + 1$. Meaning, we want to show that

$$\text{“}a_{k+1} \text{ is divisible by 7”}. \quad (\star\star)$$

To prove this, we start with the Left Hand side(LHS) of $(\star\star)$: (Recall we can only prove this using **recursion** defined in the problem, and **statements in** $(\star)$)

Now

$$\begin{aligned} a_{k+1} &= a_k + b_{k-1}, \text{ by definition of recursion} \\ &= \underbrace{a_k} + \underbrace{a_{k-1}} \\ &= \underbrace{7r} + \underbrace{7s}, \text{ for some } r, s \text{ by inductive hypothesis } (\star) \\ &= 7\,(r + s) \end{aligned}$$

This shows that a_{k+1} is divisible by 7. Thus we have shown $(\star\star)$.

Thus by strong induction, the statement holds for all integers $n \geq 1$. □

(2) Use strong induction to prove the following: Suppose $c_1, c_2, \ldots$ is sequence defined by the recursion relation:

$$\begin{cases} c_1 = 2 \\ c_2 = 5 \\ c_n = c_{n-1} \cdot c_{n-2} \quad \text{for } n \geq 3. \end{cases}$$

Prove: The sequence c_n is even for $n \geq 3$.

- **Solution:**

PROOF. Let c_n be the sequence defined by the recursion relation above. For $n \in \mathbb{N}$ let $P(n)$ be the assertion that

$$\text{“}c_n \text{ is even”}.$$

for $n \geq 3$.

Base Case: We need to show $P(3)$ is true. Since $c_1 = 2$ and $c_2 = 5$ then by the recursion we have

$$c_3 = c_2 \cdot c_1 = 5 \cdot 2 = 10$$

which is clearly even. Thus the base case is true.

Inductive hypothesis: We assume for $k \geq 3$, $P(3), P(4), \ldots, P(k)$ all hold. That is, assume

$$\text{“}c_1 \text{ is even”}$$
$$\text{“}c_2 \text{ is even”}$$
$$\vdots$$
$$\text{“}c_{k-1} \text{ is even”}$$
$$\text{“}c_k \text{ is even”} \quad (\star)$$

Inductive step: Using the inductive hypothesis, we want to show that the statement holds for $n = k+1$. Meaning, we want to show that

$$\text{“}c_{k+1} \text{ is even”}. \quad (\star\star)$$

To prove this, we start with the Left Hand side(LHS) of $(\star\star)$: (Recall we can only prove this using **recursion** defined in the problem, and **statements in** $(\star)$)

Now

$$\begin{aligned} c_{k+1} &= c_k \cdot c_{k-1}, \text{ by definition of recursion} \\ &= \underbrace{2m}\cdot\underbrace{2n}, \text{ by inductive hypothesis } (\star) \\ &= 2 \cdot (2mn) \end{aligned}$$

This shows that a_{k+1} is even sinve $2mn \in \mathbb{Z}$. Thus we have shown $(\star\star)$.

Thus by strong induction, the statement holds for all integers $n \geq 1$. □

CHAPTER 6

Set Theory

Section 6.1 - Set Theory: Definitions and the Element Method of Proof

(1) Find the power set of $\mathcal{P}(\{x, y, z\})$.
- **Solution:**
- We have

$$\mathcal{P}(\{x, y, z\}) = \{\emptyset, \{x, y, z\}, \{x, y\}, \{y, z\}, \{x, z\}$$
$$\{x\}, \{y\}, \{z\}\}.$$

Section 6.2 - Set Proofs; properties of sets

(1) **Prove:** For all sets A, B and C if $A \subseteq B$ then $(A \cup C) \subseteq (B \cup C)$.

- **Solution:**

PROOF. Suppose $A \subseteq B$. We want to show $(A \cup C) \subseteq B \cup C$. Let $x \in (A \cup C)$, this means

$$x \in A \text{ or } x \in C.$$

This splits into two cases.

Case 1: If $x \in A$. Then since $A \subseteq B$ then

$$x \in B,$$

hence

$$x \in B \cup C,$$

as desired.

Case 2: If $x \in C$. Then clearly,

$$x \in B \cup C,$$

as desired.

In either, case we have shown that

$$x \in B \cup C,$$

which shows

$$(A \cup C) \subseteq (B \cup C),$$

as needed. □

(2) **Prove:** For all sets A, B and C if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

- **Solution:**

PROOF. Suppose $A \subseteq B$ and $B \cap C = \emptyset$. We prove $A \cap C = \emptyset$ by contradiction. Meaning, let's suppose that $A \cap C \neq \emptyset$. This means we can find an element $x \in A \cap C$. This means

$$x \in A \text{ and } x \in C.$$

Since $A \subseteq B$, this means that

$$x \in B.$$

Since $x \in B$ **and** $x \in C$, then this shows that

$$x \in B \cap C.$$

But this is a contradiction, because we know that $B \cap C = \emptyset$.

Hence our original assumption that $A \cap C \neq \emptyset$ is false. Thus

$$A \cap C = \emptyset,$$

as needed. □

(3) **Prove:** For all sets A, B if $A \subseteq B$ then $A \cap B = A$.

- **Solution:**

PROOF. Suppose $A \subseteq B$. We want to show $A \cap B = A$ by showing both sets are subsets of each other.

Part (a): We want to show $A \cap B \subseteq A$. Let $x \in A \cap B$. This means

$$x \in A \text{ and } x \in B.$$

Then clearly $x \in A$ by the previous line. Hence we've shown $A \cap B \subseteq A$.

Part (b): We want to show $A \subseteq A \cap B$. Let $x \in A$. But since we already know that $A \subseteq B$ then this means that

$$x \in B.$$

Thus we know that

$$x \in A \text{ and } x \in B,$$

which shows that

$$x \in A \cap B.$$

We just shows that $A \subseteq A \cap B$.

Since we showed that both sets are subsets of each other then they must be equal, as desired. □

CHAPTER 7

Functions

Section 7.1 - Functions defined on general sets

- Do all book problems

Section 7.2 - One-to-one and Onto Functions.

(1) Show that $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(n) = -3n + 1$$

is one-to-one by not onto.

- **Solution:**
- Graph the function first. Then prove:

PROOF. First we show f is one to one: This means we need to show that if $f(n_1) = f(n_2)$ then $n_1 = n_2$. Suppose $f(n_1) = f(n_2)$, then

$$\begin{aligned} f(n_1) = f(n_2) &\iff -3n_1 + 1 = -3n_2 + 1 \\ &\iff -3n_1 = -3n_2 \\ &\iff n_1 = n_2, \end{aligned}$$

as needed. Hence f is one-to-one.

Now we show that f is not onto. We must produce $y \in \mathbb{R}$ such that $f(n) \neq y$ for any $n \in \mathbb{Z}$.

Take $y = 2$, then $y \in \mathbb{Z}$, and assume for contradiction that $f(n) = 2$, for some $n \in \mathbb{Z}$. Then if this is true, then

$$\begin{aligned} 2 = -3n + 1 &\iff 1 = -3n \\ &\iff n = -\frac{1}{3}. \end{aligned}$$

which is a contradiction, since $n = -\frac{1}{3} \notin \mathbb{Z}$. Hence f is not onto. □

(2) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 3x + 7$$

is bijective.

- **Solution:**
- Graph first.

PROOF. First we show f is one to one: This means we need to show that if $f(x) = fy)$ then $x = y$. Suppose $f(x) = f(y)$, then

$$\begin{aligned} f(x) = fy) &\iff 3x + 7 = 3y + 7 \\ &\iff 3x = 3y \\ &\iff x = y, \end{aligned}$$

as needed. Hence f is one-to-one.

Now we show that f is onto. We must show that for every $y \in \mathbb{R}$ we can find an $x \in \mathbb{R}$ such that $f(x) = y$.

Let $y \in \mathbb{Z}$, and take $x = \frac{y-7}{3} \in \mathbb{R}$, then

$$\begin{aligned} f(x) &= f\left(\frac{y-7}{3}\right) \\ &= 3\left(\frac{y-7}{3}\right) + 7 \\ &= y - 7 + 7 \\ &= y. \end{aligned}$$

Hence f is onto. □

(3) Can a function $f : \{0, 1\} \to \{0, 1, 2\}$ ever be onto? Prove or give an example.

- **Solution:**

- A function $f : \{0,1\} \to \{0,1,2\}$ can never be onto. The reason being that suppose

$$f(0) = x \in \{0,1,2\}$$
$$f(1) = y \in \{0,1,2\}.$$

then the image of f is the set

$$\text{Image of } f = \{x, y\}$$

Since there are three elements in $\{0,1,2\}$, then the image of f can never be the entire set of $\{0,1,2\}$. Hence f will almost miss a point, so that f can never be onto.

CHAPTER 8

Properties of Relations

Section 8.1 - Relations on Sets

- Do book problems

Section 8.2 - Reflexivity, Symmetry, and Transitivity.

(1) Define a relation R on $\mathbb{Z}$ by

$$mRn \text{ if and only if } 5 \mid (m-n).$$

Prove R is an equivalence relation by showing the following three parts.

(a) **Part(a):** Show R is reflexive.
- **Solution:**
- We will unravel these definitions to help us prove what we need to prove.

PROOF. We must show that $\forall m,n \in \mathbb{Z}, mRm$. This means, we must show that $5 \mid (m-m)$.
This is obvious since clearly $5 \mid 0$. Thus mRm, and it follows that R is reflexive. □

(b) **Part(b):** Show R is symmetric.
- **Solution:**

PROOF. We must show that " $\forall m \in \mathbb{Z}$ if mRn then nRm." This means, we must show that

$$\text{"if } 5 \mid (m-n) \text{ then } 5 \mid (n-m)\text{."}$$

In other words, this means

$$\text{"if } (m-n) = 5k, \text{for some } k \in \mathbb{Z} \text{ then } (n-m) = 5(\text{some integer})\text{"}$$

So let us suppose that $(m-n) = 5k$, then by multiplying by -1 we get

$$(n-m) = 5(-k).$$

Since $-k \in \mathbb{Z}$ then we showed that $5 \mid (n-m)$
Thus nRm, and it follows that R is symmetric. □

(c) **Part(c):** Show R is transitive.
- **Solution:**

PROOF. We must show that " $\forall x,y,z \in \mathbb{Z}$ if xRy and yRz then xRz." This means, we must show that

$$\text{"if } 5 \mid (x-y) \text{and } 5 \mid (y-z) \text{ then } 5 \mid (x-z)\text{."}$$

In other words, this means

$$\text{"if } (x-y) = 5k, \text{and } (y-z) = 5l \text{for some } k,l \in \mathbb{Z} \text{ then } (x-z) = 5(\text{some integer})\text{"}$$

So let us suppose that $(x-y) = 5k$ and $(y-z) = 5l$ then by substitution (and solving the two former equations for x and z) we have

$$\begin{aligned}(x-z) &= (5k+y) - (y-5l) \\ &= 5k + y - y + 5l \\ &= 5k + 5l \\ &= 5(k+l).\end{aligned}$$

Since $k + l \in \mathbb{Z}$ then we showed that $5 \mid (x-z)$
Thus if xRy and yRz then we just showed that xRz, and it follows that R is transitive. □

(2) Find all equivalence classes of the equivalence relation R on $A = \{a,b,c,d,1\}$

$$R = \left\{ \begin{array}{ccccc} (a,a) & ,(b,b) & ,(c,c) & ,(d,d) & (1,1) \\ (a,b) & ,(b,a) & & ,(d,1) & (1,d) \end{array} \right\}$$

- **Solution:**

- We have

$$
\begin{aligned}
[a] &= \{a, b\} = [b] \\
[c] &= \{c\} \\
[d] &= \{d, 1\} = [1] .
\end{aligned}
$$

Section 8.3 - Equivalence Relations

(1) Find the induced relation $R_{\mathcal{P}}$ for each partition $\mathcal{P}$ of the set $A = \{a, b, c, d, e\}$

(a) $\mathcal{P} = \{\{a, b, c\}, \{d, e\}\}$

- **Solution:**
- We have

$$R_{\mathcal{P}} = \left\{ \begin{array}{lllll} (a,a) & ,(b,b) & ,(c,c) & ,(d,d) & (e,e) \\ (a,b) & ,(b,a) & ,(a,c) & (d,e) & (e,d) \\ (c,a) & ,(b,c) & (c,b) & & \end{array} \right\}$$

(b) $\mathcal{P} = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}$

- **Solution:**
- We have

$$R_{\mathcal{P}} = \left\{ \begin{array}{lllll} (a,a) & ,(b,b) & ,(c,c) & ,(d,d) & (e,e) \\ (a,b) & ,(b,a) & & & \end{array} \right\}$$

CHAPTER 9

Probability (based on Lectures Notes)

Section 9.1 - Counting

(1) Suppose a License plate must consist of 7 numbers or letters. How many license plates are there if
  (a) there can only be letters?
    - **Solution:** $26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = (26)^7$
  (b) the first three places are numbers and the last four are letters?
    - **Solution:**$10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = (10)^3 \cdot (26)^4$
  (c) the first three places are numbers and the last four are letters, but there can not be any repetitions in the same license plate?
    - **Solution:**$10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 \cdot 23$

(2) A school of 50 students has awards for the top math, english, history and science student in the school
  (a) How many ways can these awards be given if each student can only win one award?
    - **Solution:**$50 \cdot 49 \cdot 48 \cdot 47$
  (b) How many ways can these awards be given if students can win multiple awards?
    - **Solution:**$50 \cdot 50 \cdot 50 \cdot 50 = (50)^4$

(3) An iPhone password can be made up of any 4 digit combination.
  (a) How many different passwords are possible?
    - **Solution:**$(10)^4$
  (b) How many are possible if all the digits are odd?
    - **Solution:**5^4
  (c) How many can be made in which all digits are different or all digits are the same?
    - **Solution:**$10 \cdot 9 \cdot 8 \cdot 7 + 10$

(4) An $n-$place Boolean function is a function of the form $f : \{0,1\}^n \to \{0,1\}$. How many $n-$place Boolean functions exist?
  - **Solution:** A $n-$place Boolean function is a function $f : \{0,1\}^n \to \{0,1\}$.
    - First let us find how many inputs there exists in $\{0,1\}^n$. We note that

$$\text{number of elements in } \{0,1\}^n = \underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n.$$

    - Now each input has 2 possible outputs, so

$$\underbrace{2}_{\text{possible outputs for 1st input}} \cdot \underbrace{2}_{\text{possible outputs for 2nd input}} \cdots \underbrace{2}_{\text{possible outputs for the } 2^n \text{ input}}$$

$$= 2^{2^n}.$$

(5) There is a class of 25 people made up of 11 guys and 14 girls.
  (a) How many ways are there to make a committee of 5 people?
    - **Solution:**$\binom{25}{5}$
  (b) How many ways are there to pick a committee of 5 of all girls?
    - **Solution:**$\binom{14}{5}$
  (c) How many ways are there to pick a committee of 3 girls and 2 guys?
    - **Solution:**$\binom{14}{3} \cdot \binom{11}{2}$

(6) If a student council contains 10 people, how many ways are there to elect a president, a vice president, and a 3 person prom committee from the group of 10 students?
- **Solution:** $10 \cdot 9 \cdot \binom{8}{3}$

(7) Suppose you are organizing your textbooks on a book shelf. You have three chemistry books, 5 math books, 2 history books and 3 english books.
  (a) How many ways can you order the textbooks if you must have math books first, english books second, chemistry third, and history fourth?
    - **Solution:** $5!3!3!2!$
  (b) How many ways can you order the books if each subject must be ordered together?
    - **Solution:** $4!\,(5!3!3!2!)$

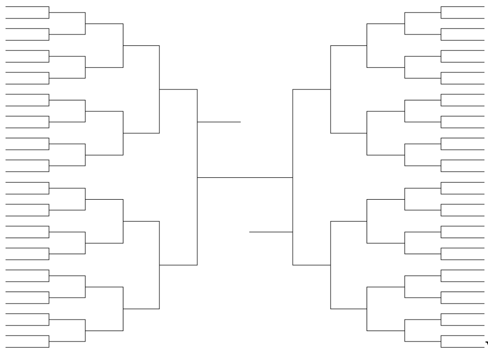
(8) You buy a Powerball lottery ticket. You choose 5 numbers between 1 and 59 (picked on white balls) and one number between 1 and 35 (picked on a red ball).How many ways can you
  (a) win the jackpot (guess all the numbers correctly)?
    - **Solution:** 1
  (b) match all the white balls but not the red ball?
    - **Solution:** $\binom{5}{5} \cdot 34 = 1 \cdot 34 = 34$
  (c) match 3 white balls and the red ball?
    - **Solution:** $\binom{5}{3} \cdot \binom{54}{2} \cdot 1$

(9) A couple wants to invite their friends to be in their wedding party. The groom has 8 possible groomsmen and the bride has 11 possible bridesmaids. The wedding party will consist of 5 groomsman and 5 bridesmaids.
  (a) How many wedding party's are possible?
    - **Solution:** $\binom{8}{5} \cdot \binom{11}{5}$
  (b) Suppose that two of the possible groomsmen are feuding and will only accept an invitation if the other one is not going. How many wedding party's are possible?
    - **Solution:** $\binom{6}{5} \cdot \binom{11}{5} + \binom{2}{1} \cdot \binom{6}{4} \cdot \binom{11}{5}$
  (c) Suppose that two of the possible bridesmaids are feuding and will only accept an invitation if the other one is not going. How many wedding party's are possible?
    - **Solution:** $\binom{8}{5} \cdot \binom{9}{5} + \binom{8}{5} \cdot \binom{2}{1} \cdot \binom{9}{4}$
  (d) Suppose that one possible groosman and one possible woman refuse to serve together. How many wedding party's are possible?
    - **Solution:** $\binom{7}{5} \cdot \binom{10}{5} + 1 \cdot \binom{7}{4} \cdot \binom{10}{5} + \binom{7}{5} \cdot 1 \cdot \binom{10}{4}$

(10) There are 52 cards in a standard deck of playing cards. The poker hand is consists of five cards. How many poker hands are there?
- **Solution:** $\binom{52}{5}$

(11) There are 30 people in a communications class. Each student must have a one-on-one conversation with each student in the class for a project. How many total one-on-one convesations will there be?
- **Solution:** $\binom{30}{2}$

(12) Suppose a college basketball tournament consists of 64 teams playing head to head in a knock-out style tournament. There are 6 rounds, the round of 64, round of 32, round of 16, round of 8, the final four teams, and the finals. Suppose you are filling out a bracket such as this

which specifies which teams will win each game in each round. How many possible brackets can you make?

- **Solution:** First notice that the 64 teams play 63 total games: 32 games in the first round, 16 in the second round, 8 in the 3rd round, 4 in the regional finals, 2 in the final four, and then the national championship game.That is, 32+16+8+4+2+1= 63. Since there are 63 games to be played, and you have two choices at each stage in your bracket, there are 2^{63} different ways to fill out the bracket. That is

$$2^{63} = 9,223,372,036,854,775,808.$$

(13) You have eight distinct pieces of food. You want to choose three for breakfast, two for lunch, and three for dinner. How many ways to do that?

- **Solution:** $\binom{8}{3,2,3} = \frac{8!}{3!2!3!}$.
  - Or $\binom{8}{3} \cdot \binom{5}{2} \cdot \binom{3}{3} = \frac{8!}{3!2!3!}$

Section 9.2 - Introduction to Probability

(1) Suppose a box contains 3 balls : 1 red, 1 green, and 1 blue
  (a) Consider an experiment that consists of randomly selecting 1 ball from the box and then replacing it in the box and drawing a second ball from the box. List all possible outcomes in the sample space.
    - **Solution:** Since every ball can be drawn first and every marble can be drawn second, there are $3 \cdot 3 = 9$ possibilities:
    - The sample space is

$$S = \{RR, RG, RB, GR, GG, GB, BR, BG, BB\}$$

    (we let the first letter the color of the first ball and the second letter be the color of the second ball).
  (b) Consider an experiment that consists of randomly selecting 1 ball from the box and then drawing a second ball from the box without replacing the first. List all possible outcomes.
    - **Solution:** In this case, the color of the second ball cannot match the color of the first, so there are 6 possibilities: RG, RB, GR, GB, BR, and BG. Hence the sample space is

$$S = \{RG, RB, GR, GB, BR, BG\}.$$

(2) Suppose that A and B are mutually exclusive (disjoint) events for which $P(A) = .3$ and $P(B) = .5$.
  (a) What is the probability that A occurs but B does not? (i.e. find $P(A \cap B^c)$)
    - **Solution:** Since A and B are mutually exclusive, the only way A can occur is when B does not. This means that $P(A \cap B^c) = P(A) = .3$.
  (b) What is the probability that neither A nor B occurs? (i.e. find $P(A^c \cap B^c)$)
    - **Solution:** Since $A \cap B = \emptyset$. Axiom 3 tell us that $P(A \cup B) = P(A) + P(B) = .8$. Since we want $P(A^c \cap B^c)$, we use DeMorgan's law to see that this is $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) = .2$.

(3) Forty percent of college students from a certain college are members of neither an academic club nor a greek organization. Fifty percent are members of academic clubs while thirty percent are members of a greek organization. Suppose a student is chosen at random, what is the probability that this students is a member
  (a) of an academic club or a greek organization?
    - **Solution:**
    - Try to fill in a Venn Diagram.
    - Or, using the properties that $\mathbb{P}(C) = 1 - \mathbb{P}(C^c)$ for any event C, we have

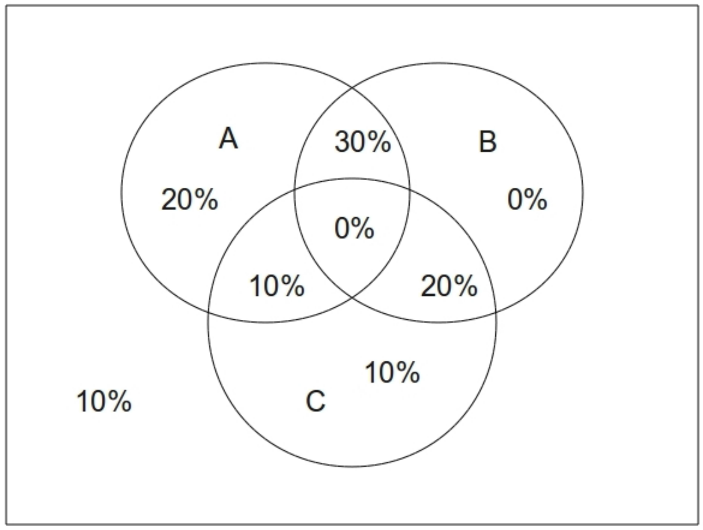
$$\begin{aligned}\mathbb{P}(A \cup B) &= 1 - \mathbb{P}((A \cup B)^c) \\ &= 1 - .4 \\ &= .6\end{aligned}$$

  (b) of an academic club and a greek organization?
    - **Solution:**
    - Try to fill in a Venn Diagram.
    - Or, using the property that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, we have

$$.6 = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = .5 + .3 - \mathbb{P}(A \cap B)$$

    Thus $\mathbb{P}(A \cap B) = .2$.

(4) In City, 60% of the households subscribe to newspaper A, 50% to newspaper B, 40% to newspaper C, 30% to A and B, 20% to B and C, and 10% to A and C, but none subscribe to all three. (Hint: Draw a Venn diagram)
  (a) What percentage subscribe to exactly one newspaper?
    - **Solution:** We use these percentages to produce the Venn diagram below:

- 

- This tells us that 30% of households subscribe to exactly one paper.

(b) What percentage subscribe to at most one newspaper?
- **Solution:** The Venn diagram tells us that 100% - (10% + 20% + 30%) = 40% of households subscribe to at most one paper.

Section 9.3 - Computing Probabilities

(1) A pair of fair dice is rolled. What is the probability that the first die lands on a strictly higher value than the second die.

- **Solution:** Simple inspection we can see that the only possibilities

$$\begin{array}{rc} (6,1)\,(6,2)\,(6,3)\,(6,4)\,(6,5) & 5 \text{ possibilities} \\ (5,1)\,(5,2)\,(5,3)\,(5,4) & 4 \text{ possibilities} \\ (4,1)\,,(4,2),(4,3) & 3 \text{ possibilities} \\ (3,1)\,,(3,2) & 2 \text{ possibilities} \\ (2,1) & 1 \text{ possibility} \\ & = 15 \text{ total} \end{array}$$

Thus the probability is $\frac{15}{36}$.

(2) Nine balls are randomly withdrawn from an urn that contains 10 blue, 12 red, and 15 green balls. What is the probability that

(a) 2 blue, 5 red, and 2 green balls are withdrawn

- **Solution:** $\dfrac{\binom{10}{2}\binom{12}{5}\binom{15}{2}}{\binom{37}{9}}$

(b) at least 2 blue balls are withdrawn.

- **Solution:** We have

$$\begin{aligned} \mathbb{P}(\text{at least 2 Blue}) &= 1-\mathbb{P}(\text{ at most one Blue}) \\ &= 1-(\mathbb{P}(0 \text{ blue})+\mathbb{P}(1 \text{ blue})) \\ &= 1-\frac{\binom{27}{9}}{\binom{37}{9}}-\frac{\binom{10}{1}\binom{27}{8}}{\binom{37}{9}}. \end{aligned}$$

(3) Suppose 4 valedictorians (from different high schools) were all accepted to the 8 Ivy League universities. What is the probability that they each choose to go to a different Ivy League university?

- **Solution:** $\frac{8\cdot 7\cdot 6\cdot 5}{8^4}$

(4) There are 8 students in a class. What is the probability that at least two students share a common birthday month?

- **Solution:** When computing probability of an "at least" event, it is easier to take a compliment. Let

$$A=\{\text{at least 2 students share a common birthday month}\}.$$

then the compliment of A, would be

$$A^c=\{\text{at most one student shares a comomn birthday month}\}.$$

Then

$$\begin{aligned} \mathbb{P}(A) &= 1-\mathbb{P}(A^c) \\ &= 1-\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{12^8} \end{aligned}$$

Section 9.4 - Independent Events and Conditional Probability

(1) Let A and B be two *independent* events with $P(A) = .4$ and $P(A \cup B) = .64$. What is $P(B)$?
- **Solution:** Using independence we have
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)$$
and substituting what is given we get
$$.64 = .4 + \mathbb{P}(B) - .4\mathbb{P}(B).$$
Solving for $\mathbb{P}(B)$ we have $\mathbb{P}(B) = .4$.

(2) Two dice are rolled. Let $S_3 = \{\text{sum of two dice equals 3}\}$, $S_7 = \{\text{sum of two dice equals 7 }\}$, and $A_1 = \{\text{at least one of the dice shows a 1}\}$.
   (a) What is $\mathbb{P}(S_3 \mid A_1)$?
   - **Solution:** Note that the sample space is $S = \{(i,j) \mid i,j = 1,2,3,4,5,6\}$ with each outcome equally likely. Then

$$\begin{aligned} S_3 &= \{(1,2),(2,1)\} \\ S_7 &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\ A_1 &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\} \end{aligned}$$

   Then
$$\mathbb{P}(S_3 \mid A_1) = \frac{\mathbb{P}(S_3 \cap A_1)}{\mathbb{P}(A_1)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

   (b) What is $\mathbb{P}(S_7 \mid A_1)$?
   - **Solution:**
$$\mathbb{P}(S_7 \mid A_1) = \frac{\mathbb{P}(S_7 \cap A_1)}{\mathbb{P}(A_1)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

   (c) Are S_3 and A_1 indepedent? What about S_7 and A_1?
   - **Solution:**Note that $\mathbb{P}(A) = 2/36 \neq \mathbb{P}(A \mid C)$, so they are not independent. Similarly, $\mathbb{P}(B) = 6/36 \neq \mathbb{P}(B \mid C)$, so they are not independent.

(3) Suppose you roll two standard, fair, 6-sided dice. What is the probability that the sum is at least 9 given that you rolled at least one 6?
- **Solution:** Let E be the event
$$E = \{\text{there is at least one 6}\}$$
and let F be the event
$$F = \{\text{the sum is at least 9}\}.$$
We want to calculate $\mathbb{P}(F \mid E)$.
- Begin by noting that there are 36 possible rolls of these two dice and all of them are equally likely.
- We can see that 11 different rolls of these two dice will result in at least one 6, so $\mathbb{P}(E) = \frac{11}{36}$.
- There are 7 different rolls that will result in at least one 6 and a sum of at least 9 .
- They are $\{(6,3),(6,4),(6,5),(6,6),(3,6),(4,6),(5,6)\}$, so $\mathbb{P}(E \cap F) = \frac{7}{36}$. This tells us that
$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(E)} = \frac{7/36}{11/36} = \frac{7}{11}.$$

Section 9.5- Bayes's Formula

(1) Suppose Phanuel is a young bachelor. Phanuel goes to a bar 7 nights a week: 3 of the nights at bar A, 2 of the nights at bar B, and 2 of the nights at bar C. After asking, he'll get a girl's number 99 percent of the time at bar A, 96 percent of the time at bar B , and only 85 percent of the time at bar C.

(a) On a random night of the week, what is the probability that he gets a number?

- **Solution:**
  - Let $A = \{\text{at Bar A}\}$, $B = \{\text{at Bar B}\}$ and $C = \{\text{at Bar C}\}$,

$$\mathbb{P}(A) = \frac{3}{7}, \ \mathbb{P}(B) = \frac{2}{7}, \ \ \mathbb{P}(C) = \frac{2}{7}.$$

Let $N = \{\text{Gets a number}\}$then

$$\begin{aligned}\mathbb{P}(N) &= \mathbb{P}(N \mid A)\,\mathbb{P}(A) + \mathbb{P}(N \mid B)\,\mathbb{P}(B) + \mathbb{P}(N \mid C)\,\mathbb{P}(C) \\ &= (.99)\,\frac{3}{7} + (.96)\,\frac{2}{7} + (.85)\,\frac{2}{7} \\ &= .9414\end{aligned}$$

(b) Given that he does get a number, what is the probability that it was at bar A?

- **Solution:**
  - Using Bayes's theorem,

$$\begin{aligned}\mathbb{P}(A \mid N) &= \frac{\mathbb{P}(N \mid A)\,\mathbb{P}(A)}{\mathbb{P}(N)} \\ &= \frac{(.99)\,\frac{3}{7}}{.9414} \\ &\approx .45.\end{aligned}$$

(2) Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available.

(a) What is the chance that a purchased bulb will work for longer than 5000 hours? (Hint: Use Law of Total Probability, the numerator of Bayes's Formula)

- **Solution:** Let H be the event "works over 5000 hours". Let X be the event comes from factory X" and Y be the event "comes fom factory Y". Then by the Law of Total Probability

$$\begin{aligned}\mathbb{P}(H) &= \mathbb{P}(H \mid X)\,\mathbb{P}(X) + \mathbb{P}(H \mid Y)\,\mathbb{P}(Y) \\ &= (.99)\,(.6) + (.95)\,(.4) \\ &= .974.\end{aligned}$$

(b) Given that a lightbulb works for more than 5000 hours, what is the probability that it came from factory Y? (Hint: Bayes's formula, or just recall the definition of conditional probability)

- **Solution:** By Part (a) we have

$$\begin{aligned}\mathbb{P}(Y \mid H) &= \frac{\mathbb{P}(H \mid Y)\,\mathbb{P}(Y)}{\mathbb{P}(H)} \\ &= \frac{(.95)\,(.4)}{.974} \approx .39.\end{aligned}$$

(c) Given that a lightbulb work does not work for more than 5000 hours, what is the probability that it came from factory X?

- **Solution:** We again use the result from Part (a)

$$\begin{aligned}\mathbb{P}(X \mid H^c) &= \frac{\mathbb{P}(H^c \mid X)\,\mathbb{P}(X)}{\mathbb{P}(H^c)} = \frac{\mathbb{P}(H^c \mid X)\,\mathbb{P}(X)}{1-\mathbb{P}(H)} \\ &= \frac{(1-.99)(.6)}{1-.974} = \frac{(.01)(.6)}{.026} \\ &\approx .23\end{aligned}$$

(3) A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from Machine A?
- **Solution:** Let $D = \{\text{Bolt is defective}\}$, $A = \{\text{bolt is form machine } A\}$, $B = \{\text{bolt is from machine } C\}$. Then by Baye's theorem

$$\begin{aligned}\mathbb{P}(A \mid D) &= \frac{\mathbb{P}(D \mid A)\,\mathbb{P}(A)}{\mathbb{P}(D \mid A)\,\mathbb{P}(A) + \mathbb{P}(D \mid B)\,\mathbb{P}(B) + \mathbb{P}(D \mid C)\,\mathbb{P}(C)} \\ &= \frac{(.05)(.25)}{(.05)(.25) + (.04)(.35) + (.02)(.4)} \\ &= .362.\end{aligned}$$

(4) A multiple choice exam has 4 choices for each question. A student has studied enough so that the probability they will know the answer to a question is 0.5, the probability that they will be able to eliminate one choice is 0.25, otherwise all 4 choices seem equally plausible. If they know the answer they will get the question right. If not they have to guess from the 3 or 4 choices. As the teacher you want the test to measure what the student knows. If the student answers a question correctly what's the probability they knew the answer?
- **Solution:** Let C be the event the students the problem correct and K the event the students knows the answer. Using Bayes' theorem we have

$$\begin{aligned}&P(K \mid C) \\ &= \frac{P(C \mid K)P(K)}{P(C)} \\ &= \frac{P(C \mid K)P(K)}{P(C \mid K)P(K) + P(C \mid \text{Eliminates})P(\text{Eliminates}) + P(C \mid \text{Guess})P(\text{Guess})} \\ &= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{24}{31} \approx .774 = 77.4\%.\end{aligned}$$

(5) A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not actually present. One percent of the population actually has the disease. Calculate the probability that a person actually has the disease given that the test indicates the presence of the disease.
- **Solution:** Let $+$ signiffy a positive test result, and D means dissease is present. Then

$$\begin{aligned}\mathbb{P}(D \mid +) &= \frac{\mathbb{P}(+ \mid D)\,P(D)}{P(+ \mid D)\,P(D) + P(+ \mid D^c)P(D^c)} \\ &= \frac{(.95)(.01)}{(.95)(.01) + (.005)(.99)} \\ &= .657.\end{aligned}$$