MTH 197 DEFINITIONS

Definition 1. *A universal statement says that a certain property is true for all elements of a set.*

Definition 2. *An existential statement says some property is true for at least one object.*

Definition 3. *A conditional statement says that if something is true, then something else has to be true.*

Definition 4. *A universal conditional statement is both universal and conditional.*

Definition 5. *A universal existential statement is a two-part statement whose first part is universal and second part is existential.*

Definition 6. *An existential universal statement is a two-part statement whose first part is existential and second part is universal.*

Definition 7. *The set of integers is the set* $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Definition 8. *The set of rational numbers is*

$$\mathbb{Q} = \left\{ x = \frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

Definition 9. *An ordered pair is an object of the form* (a, b)*, where* a *and* b *are any objects.*

Definition 10. *Two ordered pairs are equal,* $(a, b) = (c, d)$*, provided that* $a = c$ *and* $b = d$.

Definition 11. *If* A *and* B *are sets, then the Cartesian product of* A *and* B*, denoted by* $A \times B$ *is the set of ordered pairs* (a, b)*, where* $a \in A$ *and* $b \in B$.

Definition 12. *Let A and B be sets. Then A is called a* <u>subset</u> *of B, written $A \subseteq B$, provided that every element of A is an element of B.*

Definition 13. *Let A and B be sets. A* <u>relation R from A to B</u> *is a subset of $A \times B$. Given an ordered pair (x, y) in R, we say "x is related to y" and write xRy.*

The set A is called the <u>domain</u> *of R and set B is called the* <u>co-domain</u>.

Definition 14. *A* <u>function F from A to B</u> *is a relation from A to B satisfying:*

(1) *For all $x \in A$, there exists $y \in B$ such that $(x, y) \in F$.*

(2) *For all $x \in A$ and $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.*

Definition 15. *A statement (or proposition) is a sentence that is true or false but not both.*

Definition 16. *Let p and q be statements.*

(1) *$\sim p$ is read "not p" and is called the negation of p. It is the statement defined by*

$$\sim p : \begin{cases} \text{TRUE} & \text{when } p \text{ is FALSE} \\ \text{FALSE} & \text{when } p \text{ is TRUE} \end{cases}$$

(2) *$p \wedge q$ is read "p and q" and is called the conjunction of p and q. It is the statement defined by*

$$p \wedge q \text{ is } \begin{cases} \text{TRUE} & \text{when } \textbf{both } p \text{ and } q \text{ are TRUE} \\ \text{FALSE} & \text{otherwise (i.e., at least one is FALSE)} \end{cases}$$

(3) *$p \vee q$ is read "p or q" and is called the disjunction of p and q. It is the statement defined by*

$$p \vee q \text{ is } \begin{cases} \text{TRUE} & \text{when } \textbf{at least one of } p \text{ and } q \text{ is TRUE} \\ \text{FALSE} & \text{otherwise (i.e., both } p \text{ and } q \text{ are FALSE)} \end{cases}$$

Definition 17. *A statement form (or propositional form) is is a well-formed expression made up of statement variables $(p, q, r, \ldots)$ and logical connectives $(\sim, \wedge, \vee, \ldots)$.*

Definition 18. *Two statement forms P and Q are logically equivalent, denoted $P \equiv Q$, if they have identical truth values for every possible assignment of truth values to their statement variables.*

Definition 19.

- *A tautology is a statement form that is always true, regardless of the truth values assigned to the its variables.*
- *A contradiction is one that is always false, regardless of the truth values assigned to its variables.*

Definition 20. *The <u>conditional</u> of* p *by* q *is "*p *implies* q*" and denoted by* $p \to q$*. It is the statement defined by*

$$p \to q \text{ is } \begin{cases} \textit{FALSE} & \text{when } p \text{ is } \textit{TRUE} \text{ and } q \text{ is } \textit{FALSE} \\ \textit{TRUE} & \text{otherwise} \end{cases}$$

Definition 21. *The <u>contrapositive</u> of* $p \to q$ *is* $\sim q \to \sim p$*.*

Definition 22. *The <u>converse</u> of* $p \to q$ *is* $q \to p$*.*

Definition 23. *The <u>biconditional</u> of* p *and* q *is written* $p \leftrightarrow q$ *(or* p *iff* q*, or* $p \Leftrightarrow q$*), read "*p *if and only if* q*," and it means* $p \to q$ *and* $q \to p$*.*

Definition 24. *An argument form is* <u>*valid*</u> *if when the premises are all true, then the conclusion is true, no matter what statements are substituted for the statement variables in the premises. A valid argument form is called a* <u>*rule of inference*</u>*.*

Definition 25. *The following rule of inference is called* <u>*modus ponens*</u>

$$\begin{array}{ll} & \textit{If } p, \textit{ then } q. \\ & p \\ \therefore & q \end{array}$$

Definition 26. *The following rule of inference is called* <u>*modus tollens*</u>

$$\begin{array}{ll} & \textit{If } p, \textit{ then } q. \\ & \sim q \\ \therefore & \sim p \end{array}$$

Definition 27. *A* <u>*predicate*</u> *is a sentence that contains a finite number of variables and becomes a statement (T or F) when specific values are substituted for the variables. The* <u>*domain*</u> *of a predicate (variable) is the set of all values that may be substituted in place of the variable.*

Definition 28. *If $P(x)$ is a predicate with domain D, then the* <u>*truth set of $P(x)$*</u> *is the set of all elements of D that make $P(x)$ true. It is denoted by*

$$\{x \in D | P(x)\}.$$

Definition 29. *(takes over Definition 1) A* <u>*universal statement*</u> *is one of the form*

$$\forall x \in D, Q(x)$$

where $Q(x)$ is a predicate with domain D. It is defined to be true if $Q(x)$ is true for every x in D, and false if $Q(x)$ is false for at least one x in D.

Definition 30. *(takes over Definition 2) An* <u>*existential statement*</u> *is one of the form*

$$\exists x \in D \text{ such that } Q(x)$$

where $Q(x)$ is a predicate with domain D. It is defined to be true if $Q(x)$ is true for at least one x in D, and false if $Q(x)$ is false for every x in D.

Definition 31. *The* <u>*contrapositive*</u> *of the statement* $\forall x \in D, P(x) \to Q(x)$ *is* $\forall x \in D, \sim Q(x) \to \sim P(x)$ *and the* <u>*converse*</u> *is* $\forall x \in D, Q(x) \to P(x)$

Definition 32. *An integer* n *is* *even* *provided that* $n = 2k$*, for some* $k \in \mathbb{Z}$*.*

Definition 33. *An integer* n *is* *odd* *provided that* $n = 2k + 1$*, for some* $k \in \mathbb{Z}$*.*

Definition 34. *An integer* n *is* *prime* *provided that* $n > 1$ *and for all positive integers* r *and* s*, if* $n = rs$*, then* $r = n$ *or* $s = n$*.*

An integer n *is* *composite* *provided that* $n > 1$ *and* $n = rs$*, for some integers* r*,* s *with* $1 < r < n$ *and* $1 < s < n$*.*

Definition 35. *A real number r is rational provided that $r = \frac{a}{b}$, for some $a, b \in \mathbb{Z}$ s.t. $b \neq 0$. The set of rational numbers is defined by*

$$\mathbb{Q} = \left\{ x \in \mathbb{R} \mid x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Definition 36. *Let $n, d \in \mathbb{Z}$. Then n is divisible by d provided that $n = dk$ for some $k \in \mathbb{Z}$ and $d \neq 0$.*

Definition 37. *Given a non-negative integer n and a positive integer d,*

- *n div $d =$ the integer quotient of n divided by d,*
- *n mod $d =$ the integer remainder of n divided by d.*

Definition 38. *Given a real number x, the floor of x, denoted $\lfloor x \rfloor$, is the unique integer n such that $n \leq x < n+1$.*

Definition 39. *Given a real number x, the ceiling of x, denoted $\lceil x \rceil$, is the unique integer n such that $n-1 < x \leq n$.*

Definition 40. *Given* $n \in \mathbb{Z}^+$, *let* $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$. *This is read "n factorial." Also,* $0! = 1$.

This is a Theorem, not a definition. You are asked to learn the statement of this Theorem.

The Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n and let a be a fixed integer. Suppose the following two properties hold:

(1) $P(a)$ is true

(2) $\forall k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \geq a$.

Definition 41. *(takes over Definition 12) Let A and B be sets. Then A is a subset of B, denoted $A \subseteq B$, provided that*

$$\forall x (x \in A \to x \in B)$$

Definition 42. *If A and B are sets, then $\underline{A = B}$ provided that $A \subseteq B$ and $B \subseteq A$.*

Definition 43. *A is a proper subset of B provided that $A \subseteq B$ and $A \neq B$.*

Definition 44. *Let A and B are sets. Then:*

(1) *A intersect B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$*

(2) *A union B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$*

(3) *A minus B is the set $A - B = \{x \mid x \in A \text{ and } x \notin B\}$*

Definition 45. *The empty set, denoted $\emptyset$, is the set with no elements.*

Definition 46. *Let A be a set. The power set of A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A, i.e.,*

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}.$$

Note: this means, $X \in \mathcal{P}(A) \leftrightarrow X \subseteq A$

Definition 47. *Given sets $A_1, A_2, \ldots$ and a nonnegative integer n, define:*

(1) $\displaystyle\bigcap_{i=1}^{n} A_i = A_1 \cap \ldots \cap A_n = \{x \mid x \in A_i, \text{ for all } i = 1, 2, \ldots, n\}$

(2) $\displaystyle\bigcap_{i=1}^{\infty} A_i = \{x \mid x \in A_i, \text{ for all } i \geq 1\}$

(3) $\displaystyle\bigcup_{i=1}^{n} A_i = A_1 \cup \ldots \cup A_n = \{x \mid x \in A_i, \text{ for at least one } i = 1, 2, \ldots, n\}$

(4) $\displaystyle\bigcup_{i=1}^{\infty} A_i = \{x \mid x \in A_i, \text{ for at least one } i \geq 1\}$

Definition 48. *Suppose f and g are functions from X to Y. Then f equals g, written $f = g$, provided that $f(x) = g(x)$, for all $x \in X$.*

Definition 49. *Given a set X, the identity function on X is the function $I_X \colon X \to X$ defined by $I_X(x) = x$.*

Definition 50. *An (n-place) Boolean function is a function $f \colon \{0, 1\}^n \to \{0, 1\}$.*

Definition 51. *Let $f\colon X \to Y$. Then f is one-to-one provided that*

$$\forall x_1, x_2 \in X, f(x_1) = f(x_2) \to x_1 = x_2$$

or equivalently,

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \to f(x_1) \neq f(x_2)$$

Definition 52. *Let $f\colon X \to Y$. Then f is onto provided that*

$$\forall y \in Y,\ \exists x \in X \text{ s.t. } y = f(x)$$

Definition 53. *Let $f\colon X \to Y$. Then f is a bijection or one-to-one correspondence provided that f is one-to-one and onto.*

Definition 54. *Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined by* $R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}$.

Definition 55. *A relation on a set A is a relation from A to A.*

Definition 56. *Suppose R is a relation on a set A. Then*

(r) *R is reflexive iff* $\forall x \in A,\ x\,R\,x$.

(s) *R is symmetric iff* $\forall x, y \in A,\ x\,R\,y \to y\,R\,x$.

(t) *R is transitive iff* $\forall x, y, z \in A, x\,R\,y$ and $y\,R\,z \to x\,R\,z$.

Definition 57. *A binary relation R on a set A is an* <u>*equivalence relation*</u> *provided that it is reflexive, symmetric, and transitive.*

Definition 58. *Let R be an equivalence relation on A and $a \in A$. Then the* <u>*equivalence class*</u> *of a is denoted by $[a]$ and defined by $[a] = \{x \in A \mid x \, R \, a\}$.*

Definition 59. *Two sets A and B are called disjoint if $A \cap B = \emptyset$.*

Definition 60. *A finite or infinite collection $\mathcal{P}$ of nonempty subsets A_i of a set A is a partition of A provided that*

(1) *$\forall x \in A$, there is some A_i in $\mathcal{P}$ such that $x \in A_i$.*

(2) *For all A_i and A_j in $\mathbb{P}$, if $A_i \neq A_j$, then A_i and A_j are disjoint.*

Definition 61. *Let $\mathcal{P}$ be a partition of A. The relation $R_{\mathcal{P}}$ on A induced by $\mathcal{P}$ is defined by*

$$(x, y) \in R_{\mathcal{P}} \quad \textit{iff} \quad \exists A_i \in \mathcal{P} \textit{ s.t. } x \textit{ and } y \textit{ are both in } A_i.$$

Definition 62. *Let R be a relation on a set X. Then R is* <u>*antisymmetric*</u> *provided that* $\forall x, y \in X$, *if* $x\,R\,y$ *and* $y\,R\,x$, *then* $x = y$.

Definition 63. *Let R be a relation on X. Then R is a* <u>*partial order relation*</u> *provided that R is reflexive, antisymmetric, and transitive.*

Definition 64. *A set X together with the partial order R on X is called a* <u>*partially ordered set*</u> *or* <u>*poset*</u>.

Definition 65. *A* <u>*total order relation*</u> *on X is a partial order R such that* $\forall x, y \in X$, *either* xRy *or* yRx.

Definition 66. *Let $(X, \leq)$ be a poset. A subset $C \subseteq X$ is called a* <u>*chain*</u> *provided that* $\forall x, y \in C$, $x \leq y$ or $y \leq x$. *The* <u>*length*</u> *of C is one less than the number of elements in C.*

Definition 67. *Let $(X, \leq)$ be a poset. Then $a \in X$ is called a:*

(1) <u>*maximal element*</u> *iff* $x \leq a$ *or x and a are not comparable, for all* $x \in X$

(2) <u>*greatest element*</u> *iff* $x \leq a$, *for all* $x \in X$

(3) <u>*minimal element*</u> *iff* $a \leq x$ *or x and a are not comparable, for all* $x \in X$

(4) <u>*least element*</u> *iff* $a \leq x$, *for all* $x \in X$

Definition 68. *Suppose $\leq$ and $\leq'$ are partial orders on a set X. Then $\leq'$ is a* <u>*refinement*</u> *of $\leq$ provided that* $x \leq y \to x \leq' y$, *for all* $x, y \in X$.

Definition 69. *(**Basic Counting Principle**) Suppose 2 experiments are to be performed.*

If one experiement can result in m possibilities

Second experiment can result in n possibilities

Then together there are mn possibilities

Definition 70. *With n objects, a **permutation** is an arrangement/ordering of n objects. There are*

$$n(n-1)\cdots 3\cdot 2\cdot 1 = n!$$

*different **permutations** of the n objects.*

Definition 71. *If $r \leq n$, then*

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

*and we say "n choose r", represents the number of possible **combinations** of objects taken r at a time.*

Definition 72. *A **sample space** S is the set of all possible outcomes of a random experiment. An element $x \in S$ is called an **outcome**. An **event** E is a subset of S.*

Definition 73. *A **probability** $\mathbb{P}$ is a function $\mathbb{P}: S \to \mathbb{R}$ where the input is a set/event such that*

Axiom 1: $0 \leq \mathbb{P}(E) \leq 1$ *for all events E.*

Axiom 2: $\mathbb{P}(S) = 1$.

Axiom 3: *(disjoint property) If the events $E_1, E_2, \ldots$ are pairwise disjoint/mutually exclusive then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

Mutually exclusive means that $E_i \cap E_j = \emptyset$ when $i \neq j$.

Definition 74. *If S is a finite sample space of equally likely outcomes and $E \subseteq S$, then the **probability** of E is given by*

$$\mathbb{P}(E) = \frac{N(E)}{N(S)}.$$

Definition 75. *We say E and F are **independent events** if*

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F).$$

Definition 76. *If $\mathbb{P}(F) > 0$, we define the **conditional probability of** E **given** F, by*

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Now, $\mathbb{P}(E \mid F)$ is read "the probability of E given F."

Definition 77. *If $F_1, \ldots, F_n$ are mutually exclusive (disjoint) events such that they make up everything, $S = \bigcup_{i=1}^n F_i$, then the **Law of Total Probability** says*

$$\mathbb{P}(E) \;=\; \sum_{i=1}^{n} \mathbb{P}(E \mid F_i)\,\mathbb{P}(F_i).$$

***Bayes's Formula** says that, for any j,*

$$\mathbb{P}(F_j \mid E) = \frac{\mathbb{P}(E \mid F_j)\,\mathbb{P}(F_j)}{\sum_{i=1}^n \mathbb{P}(E \mid F_i)\,\mathbb{P}(F_i)}.$$