Differential Equations Exercises Solutions

CHAPTER 1

Introduction

1.1. Problems

(1) What does it mean to be a solution to a differential equation?
- **Solution:** A solution is a function $y(t)$ that when you plug the function into both sides of the Differential Equation, you get equality.

(2) Check if the function $y(t) = t + 1$ a solution to the following differential equation:

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$

- **Solution:** The left hand side of the equation is

$$\begin{aligned} \text{LHS} &= \frac{d}{dt}(t+1) \\ &= 1 \end{aligned}$$

while the right hand side of the equation is

$$\begin{aligned} \text{RHS} &= \frac{y(t)^2 - 1}{t^2 + 2t} \\ &= \frac{(t+1)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t + 1 - 1}{t^2 + 2t} \\ &= \frac{t^2 + 2t}{t^2 + 2t} = 1. \end{aligned}$$

Since the $LHS = RHS$, then $y(t) = t + 1$ **is a solution** to this differential equation.

(3) Check if the function $y(x) = x + x \ln x$ solves the following Initial Value Problem (IVP):

$$x\frac{dy}{dx} = x + y, \quad y(1) = 3.$$

- **Solution:**
- We first check that $y(x)$ solves the differential equation. The left hand side of the equation is

$$\begin{aligned} \text{LHS} = x\frac{dy(x)}{dx} &= x\frac{d}{dx}(x + x\ln x) \\ &= x\left(1 + \ln x + x \cdot \frac{1}{x}\right), \text{ by product rule} \\ &= x(1 + \ln x + 1) \\ &= x + x\ln x + x \\ &= 2x + x\ln x \end{aligned}$$

while the right hand side of the equation is

$$\begin{aligned}\text{RHS} &= x + y(x)\\ &= x + (x + x\ln x)\\ &= 2x + x\ln x.\end{aligned}$$

Since the $LHS = RHS$, then $y(x) = x + x\ln x$ is a solution to this differential equation. YES, this is a particular solution to the Diff. Eq.

- Now we need to check if $y(x) = x + x\ln x$ solves the Initial condition.

$$\begin{aligned}y(1) &= 1 + 1\cdot\ln 1\\ &= 1 + 0\\ &= 1.\end{aligned}$$

- But unfortunately this does not solve the initial condition because the initial condition was $y(1) = 3$, hence it is **not a solution to the IVP.**

(4) Find the equilibrium solutions to the equation

$$\frac{dy}{dt} = y^2 + 2y$$

- **Solution:** By factoring the RHS we get

$$\frac{dy}{dt} = y\,(y+2)$$

and setting the RHS equal to zero we get that $y = 0, -2$ are the equilibrium(constant) solutions.

(5) Find the equilibrium solutions to the equation

$$\frac{dy}{dt} = y^4 t - 3y^3 t + 2y^2 t$$

- **Solution:** By factoring the RHS we get

$$\frac{dy}{dt} = y^2 t\,(y-1)\,(y-2)$$

and setting the RHS equal to zero we get that $y = 0, 1, 2$ are the equlibrium(constant) solutions. Note that $t = 0$, is NOT an equilibrium solution, because that wouldn't even make any sense since t is an independent variable. Solutions are functions $y(t)$. So when I write $y = 0, 1, 2$ are solutions, I mean the functions

$$y(t) = 0, y(t) = 1, y(t) = 2$$

are the solutions. If you plot these functions, they are constant horizontal lines.

(6) Classify the following equations as ODEs or PDEs.

(a) $\frac{dy}{dt} = 2yt$
- **Solution:** The solution to this equation would be a function $y = y(t)$. Since there is only one independent variable, then this is an ODE.

(b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$
- **Solution:** The solution to this equation would be a function $u = u(t, x)$. Since there are two independent variables, then this is a PDE. Note that the dependent variable is u, which is different than the standard y that we have been using. We can always change the letters of our variables. Please don't let that confuse you.

- *Fun Fact:* This particular PDE is a very famous PDE and is called the *heat equation*. It models the flow of heat in a medium over time.

(c) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
- **Solution:** The solution to this equation would be a function $u = u(t, x, y)$. Since there are three independent variables, then this is a PDE.
  - *Fun Fact:* This particular PDE is a very famous PDE and is called the *wave equation*. This PDE along with boundary conditions, describes the amplitude and phase of the wave.

(d) $x\frac{d^2y}{dx^2} = y\frac{dy}{dx} + x^2y$
- **Solution:** The solution to this equation would be a function $y = y(x)$. Since there is only one independent variable, then this is an ODE.

(e) $2y'' - y' + y = 0$
- **Solution:** The solution to this equation would be a function $y = y(t)$. Since there is only one independent variable, then this is an ODE.

(7) Classify the order of the following differential equations. Also classify if it is linear or nonlinear.

(a) $\frac{dy}{dt} = 2yt$
- **Solution:** This is a first order linear ODE.

(b) $y\frac{d^2y}{dt^2} = \cos t$
- **Solution:** This second order non-linear ODE. The $y\frac{d^2y}{dt^2}$ makes this nonlinear.

(c) $ty''' - y'' - 2y = 0$
- **Solution:** This is a third order linear ODE. Don't let the ty''' fool you into thinking it's non-linear. When we talk about linear, we're only looking for linear in y, and we can treat t's as constants.

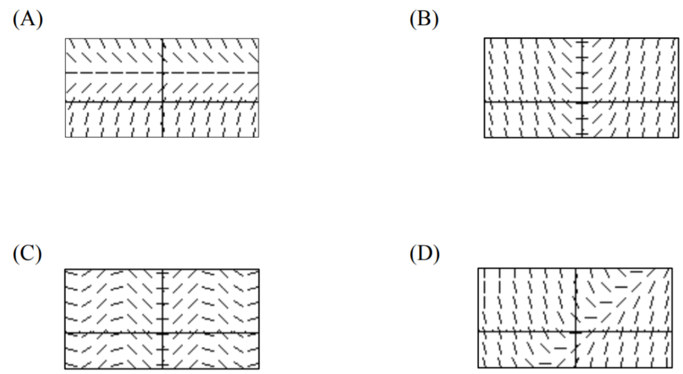
(d) $\frac{dy^6}{dt^6} - 2\frac{dy}{dt} + y = t^2$
- **Solution:** This is a sixth order linear ODE. Don't let the t^2 fool you into thinking it's non-linear. When we talk about linear, we're only looking for linear in y, not in t.

(e) $\cos y + y' = t$
- **Solution:** This is a first order non-linear ODE.

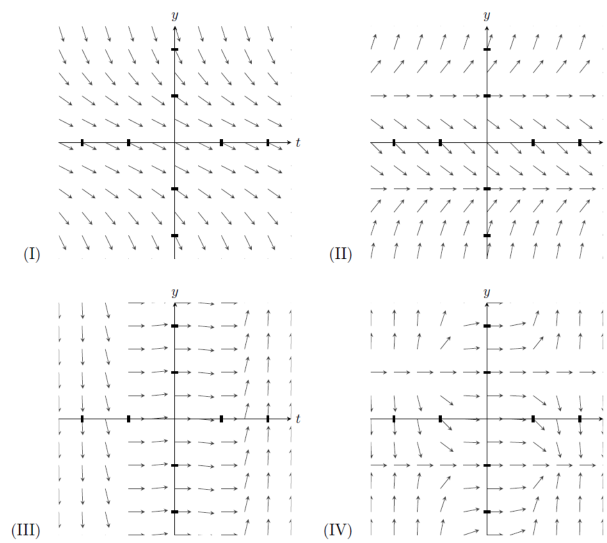
(f) $6y''' - y^2 = y^{(5)}$
- **Solution:** This is a fifth order non-linear ODE. The y^2 makes this non-linear. Note that y^2 means $y \cdot y$, while $y^{(2)}$ would mean second derivative.

(g) $\frac{d^2y}{dt^2} = \frac{y}{y+t}$
- **Solution:** This is a second order non-linear ODE.

1.2. Problems



(1) Match the following slope fields with their equations
 (a) $\frac{dy}{dt} = \sin t$
  - **Solution:** C
 (b) $\frac{dy}{dt} = t - y$
  - **Solution:** D
 (c) $\frac{dy}{dt} = 2 - y$
  - **Solution:** A
 (d) $\frac{dy}{dt} = t$
  - **Solution:** B

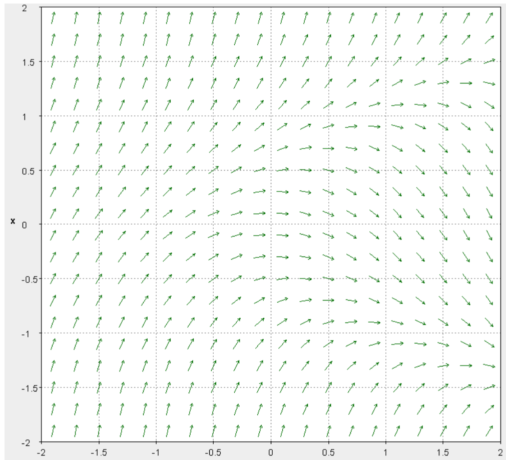


(2) Match the following slope fields with their equations
 (a) $\frac{dy}{dt} = t^4(y^2 - 1)$
  - **Solution:** IV
 (b) $\frac{dy}{dt} = t^3(t^2 - 1)$
  - **Solution:** III
 (c) $\frac{dy}{dt} = (y - 1)(y + 1)$
  - **Solution:** II
 (d) $\frac{dy}{dt} = -\sqrt{1 + y^4}$

- **Solution:** I

(3) Suppose the following ODE

$$\frac{dy}{dt} = y^2 - t$$

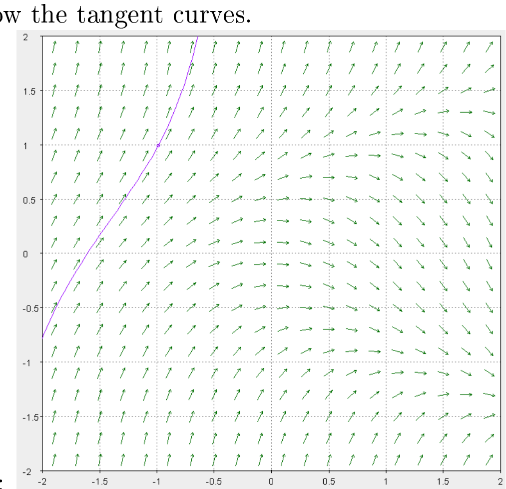


has the following Slope Field:

(a) Suppose $y(t)$ is a solution to this ODE and also you know that $y(-1) = 1$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t \to \infty} y(t) = ?$$

- **Solution:** If the solution goes through the point $y(-1) = 1$, then my prediction for the solution would have to follow the tangent curves.

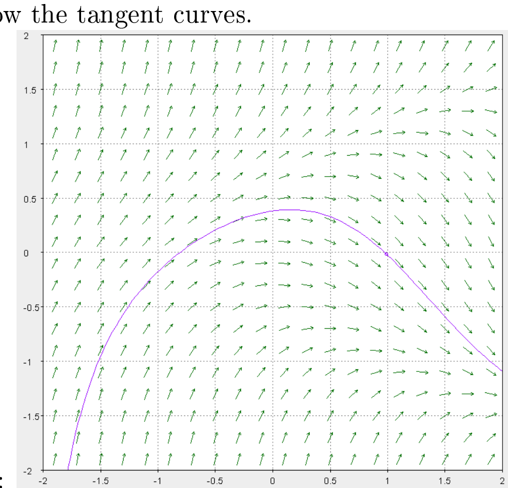


- It would probably look like:
- Hence by my sketch

$$\lim_{t \to \infty} y(t) = \infty.$$

(b) Suppose $y(t)$ is a solution to this ODE and also you know that $y(1) = 0$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t \to \infty} y(t) = ?$$

- **Solution:** If the solution goes through the point $y(1) = 0$, then my prediction for the solution would have to follow the tangent curves.
- It would probably look like: 

- Using the given information, we have could say that

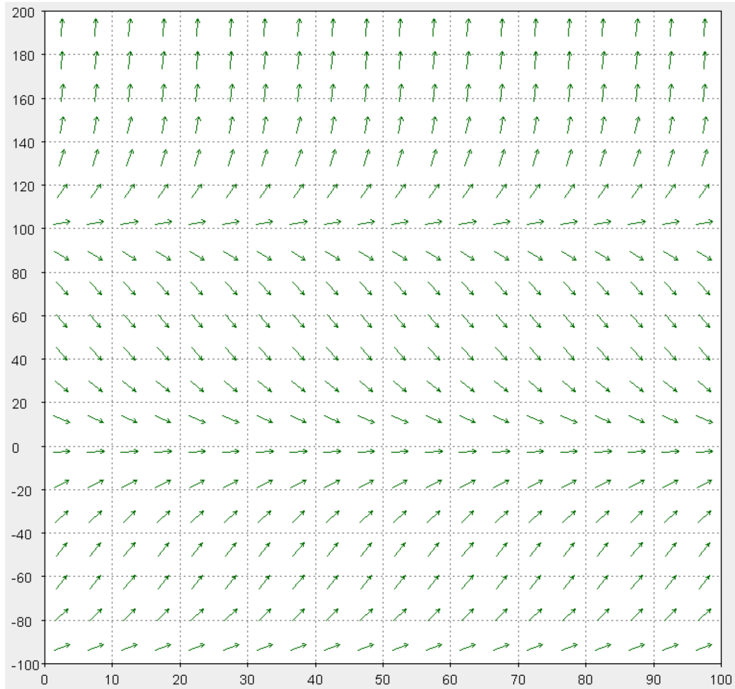
$$\lim_{t \to \infty} y(t) < -1.$$

- It might be too bold to say that the limit is $-\infty$. This is because what if the solution keeps going down but then eventually goes back up? Who knows? But given the current slope field, the only prediction we can make is that it is less than -1, because the tangents above the point $(2, -1)$ all point down.
- If you'd like to be more precise, it actually seems that the limit might be between -1.5 and -1.

(4) Let $P(t)$ represent the population of the Phan fish breed. Suppose you come up with the following differential equation that models $P(t)$:

$$\frac{dP}{dt} = P(P - 100)(P + 100)/100000$$

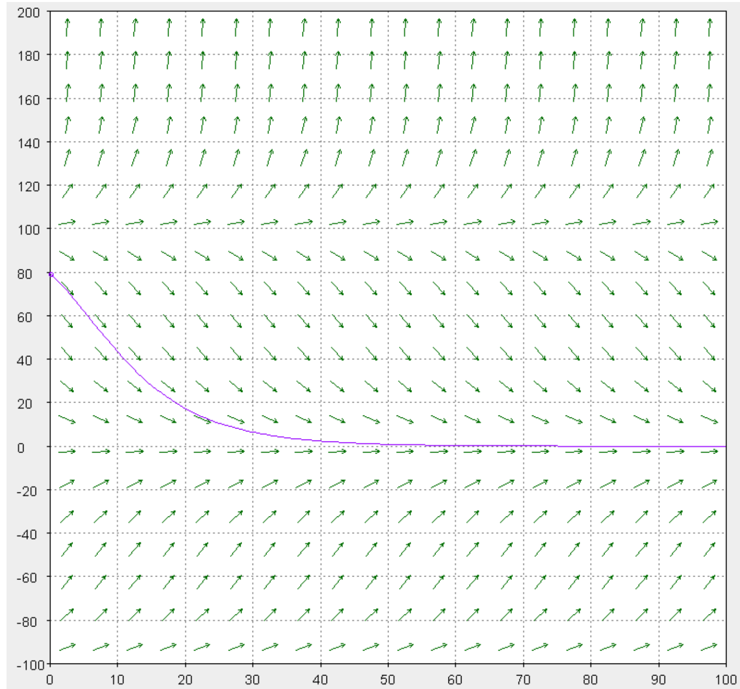
Its Slope Field is given by:



(a) Suppose that the population of the Phan fish is 80 at time $t = 0$. What is the long term behavior for the population of the Phan fish? Will it keep increasing/decreasing, stabilize to a certain number, or go extinct?

- **Solution:** If $P(0) = 80$ then my prediction for the solution would have to follow the tangent curves.

- It would probably look like:



- Hence by my sketch we can make a guess that

$$\lim_{t\to\infty} P(t) = 0,$$

hence we model that the population of the Phan fish will go extinct.

CHAPTER 2

First Order Differential Equations

2.1. Problems

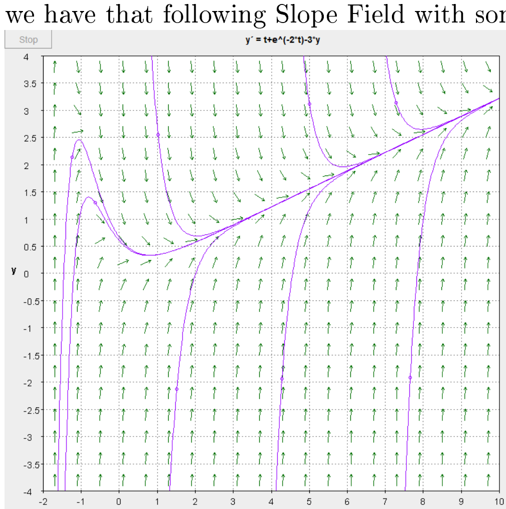
(1) Use a computer app to draw the direction field for the given differential equations. Use the direction field to describe the long term behavior of the solution for large t. (Meaning use the direction field to predict $\lim_{t\to\infty} y(t)$ for different starting points). Find the general solution of the given differential equations, and use it to determined how solutions behave as $t \to \infty$.

(a) $y' + 3y = t + e^{-2t}$

- **Solution:**
- **Qualitative analysis:** Using the applet DField, and rewriting

$$\frac{dy}{dt} = t + e^{-2t} - 3y$$

we have that following Slope Field with some sketch of solution:

- 

- Using the slope Field we predict that all solutions satisfy

$$\lim_{t\to\infty} y(t) = +\infty.$$

- **Solve analytically:** We have $p(t) = 3$, and $g(t) = t + e^{-2t}$. The integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int 3dt} = e^{3t}.$$

Then the solution is given by

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right] \\ &= \frac{1}{e^{3t}}\left[\int e^{3t}\left(t+e^{-2t}\right)dt + C\right] \\ &= \frac{1}{e^{3t}}\left[\int \left(te^{3t}+e^{t}\right)dt + C\right] \\ &= \frac{1}{e^{3t}}\left[\frac{1}{3}te^{3t}-\frac{1}{9}e^{3t}+e^{t}+C\right] \\ &= \frac{1}{3}t-\frac{1}{9}+e^{-2t}+Ce^{-3t}. \end{aligned}$$

- Now that we have the exact solution we can indeed confirm that

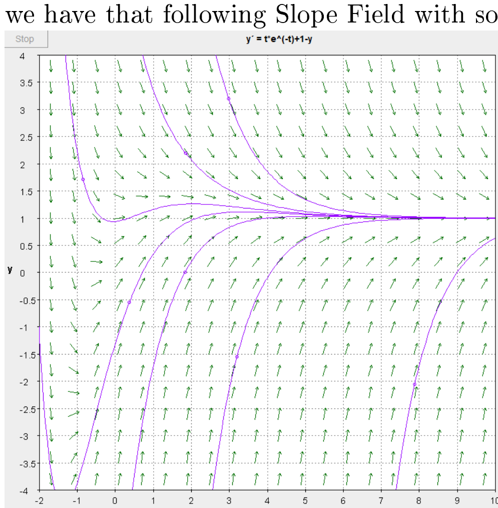
$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty}\left(\frac{1}{3}t-\frac{1}{9}+e^{-2t}+Ce^{-3t}\right) \\ &= \infty + 0 \\ &= +\infty. \end{aligned}$$

(b) $y' + y = te^{-t} + 1$

- **Solution:**
- **Qualitative analysis:** Using the applet DField, and rewriting

$$\frac{dy}{dt} = te^{-t} + 1 - y$$

we have that following Slope Field with some sketch of solution:

- 

- Using the slope Field we predict that all solutions satisfy

$$\lim_{t\to\infty} y(t) = 1.$$

- **Solve analytically:** We have $p(t) = 1$, and $g(t) = te^{-t} + 1$. The integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int 1dt} = e^{t}.$$

Then the solution is given by

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right] \\ &= \frac{1}{e^t}\left[\int e^t\left(te^{-t}+1\right)dt + C\right] \\ &= \frac{1}{e^t}\left[\int \left(t+e^t\right)dt + C\right] \\ &= \frac{1}{e^t}\left[\frac{t^2}{2}+e^t+C\right] \\ &= \frac{t^2}{2}e^{-t}+1+Ce^{-t} \end{aligned}$$

- Now that we have the exact solution we can indeed confirm that

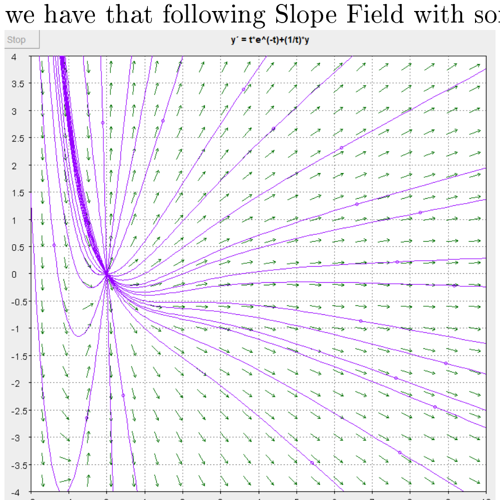
$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty}\left(\frac{t^2}{2}e^{-t}+1+Ce^{-t}\right) \\ &= 0+1+0 \\ &= 1. \end{aligned}$$

(c) $ty' - y = t^2e^{-t}$

- **Solution:**
- **Qualitative analysis:**Using the applet DField, and rewriting

$$\frac{dy}{dt} = te^{-t} + \frac{1}{t}y$$

we have that following Slope Field with some sketch of solution:

- 

- Using the slope Field we predict that all solutions satisfy

$$\lim_{t\to\infty} y(t) = +\infty \text{ or } \lim_{t\to\infty} y(t) = -\infty, \text{ or } \lim_{t\to\infty} y(t) = 0$$

depending on the starting point.

- **Solve analytically:** Don't forget to first rewrite it in the form $\frac{dy}{dt} + p(t)y = g(t)$:

$$y' - \frac{1}{t}y = te^{-t}$$

We have $p(t) = -\frac{1}{t}$, and $g(t) = te^{-t}$. The integrating factor is

$$\begin{aligned}\mu(t) &= e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} \\ &= e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}\end{aligned}$$

Then the solution is given by

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right] \\ &= \frac{1}{t^{-1}}\left[\int t^{-1}\left(te^{-t}\right)dt + C\right] \\ &= t\left[\int e^{-t}dt + C\right] \\ &= t\left[-e^{-t} + C\right] \\ &= -te^{-t} + Ct.\end{aligned}$$

- Now that we have the exact solution we can indeed confirm that

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty}\left(-te^{-t} + Ct\right) \\ &= \lim_{t\to\infty}\left(-te^{-t}\right) + \lim_{t\to\infty}(Ct) \\ &= 0 + C\lim_{t\to\infty} t.\end{aligned}$$

- Now note that if $C > 0$ then $\lim_{t\to\infty} y(t) = +\infty$, and if $C < 0$ then $\lim_{t\to\infty} y(t) = -\infty$, and finally if $C = 0$ then $\lim_{t\to\infty} y(t) = 0$.

(d) $2y' + y = 3t$

- **Solution:**
  - Try the qualitative analysis yourself. The Analytic soluton is $y(t) = 3t - 6 + Ce^{-t/2}$ and note that we always have

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{t\to\infty}\left(3t - 6 + Ce^{-t/2}\right) \\ &= \infty - 6 + 0 \\ &= +\infty.\end{aligned}$$

(2) Find the particular solution to given initial value problem.

(a) $y' - y = 2te^{2t}$, $y(0) = 1$

- **Solution:** Using integrating factors, one obtains the general solution

$$y(t) = 2te^{2t} - 2e^{2t} + Ce^{t}.$$

Using the initial condition we have that

$$1 = y(0) = 2 \cdot 0 \cdot e^{2\cdot 0} - 2e^{2\cdot 0} + Ce^{0} = -2 + C$$

hence $C = 3$, thus the particular solution to the IVP is

$$y(t) = 2te^{2t} - 2e^{2t} + 3e^{t}.$$

(b) $ty' + 2y = \sin t,\ y(\pi/2) = 1,\ t > 0$

- **Solution:** Using integrating factors, one obtains the general solution

$$y(t) = \frac{1}{t^2}\sin t - \frac{1}{t}\cos t + \frac{C}{t^2}.$$

Using the initial condition we have that

$$\begin{aligned} 1 = y(0) &= \frac{1}{\left(\frac{\pi}{2}\right)^2}\sin\frac{\pi}{2} - \frac{1}{\frac{\pi}{2}}\cos\frac{\pi}{2} + \frac{C}{\left(\frac{\pi}{2}\right)^2} \\ &= \frac{4}{\pi^2} + \frac{4}{\pi^2}C \end{aligned}$$

hence $C = \frac{\pi^2}{4} - 1$, thus the particular solution to the IVP is

$$y(t) = \frac{1}{t^2}\sin t - \frac{1}{t}\cos t + \frac{1}{t^2}\left(\frac{\pi^2}{4} - 1\right).$$

(3) Consider the following initial value problem:

$$ty' + (t+1)\,y = 2te^{-t}, \quad y(1) = a, \quad t > 0$$

where a is any real number.

(a) Find the particular solution that solves this IVP.

- **Solution:** Using the technique for Linear 1st order ODE, you should get:

$$y(t) = te^{-t} - e^{-t}t^{-1} + eae^{-t}t^{-1}.$$

2.2. Problems

(1) Find the general solutions for the following differential equations. Find the *explicit* solutions if you can. If you can't solve for y exactly, then leave it as an *implicit* solution:

(a) $y' = ky$ where k is a parameter.

- **Solution:** First note $y = 0$ is an equilibrium solution. Since k is a parameter then we want to keep track of it. Then rewrite $y' = \frac{dy}{dt}$ and separate variables

$$\begin{aligned}
\frac{dy}{dt} = ky &\iff \frac{1}{y}dy = kdt \\
&\iff \int \frac{1}{y}dy = \int kydt \\
&\iff \ln|y| = kt + c_1 \\
&\iff |y| = e^{kt+c_1}, \\
&\iff |y| = e^{c_1}e^{kt}, \text{rename } c_2 = e^{c_1} \\
&\iff |y| = c_2 e^{kt}.
\end{aligned}$$

Now since

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

then we can get rid of the absolute value by putting a $\pm$ on the RHS (we'll use this trick often)

$$\begin{aligned}
|y| = c_2 e^{kt} &\iff y = \pm c_2 e^{kt} \\
&\iff y = c_3 e^{kt}, \text{ rename } c_3 = \pm c_2
\end{aligned}$$

- Thus since $c_3$is our final constant I'll just rename it as C and get

$$\text{General solution : } y(t) = Ce^{kt}.$$

- Note that the equilibrium solution of $y = 0$ is included in our formula by setting $C = 0$. So this is indeed the most **general solution**.

(b) $y' = \dfrac{x^2}{y}$

- **Solution:** First note there are no equilibrium solutions. Then rewrite $y' = \frac{dy}{dx}$ and separate variable

$$\begin{aligned} \frac{dy}{dx} = \frac{x^2}{y} &\iff ydy = x^2 dx \\ &\iff \int ydy = \int x^2 dx \\ &\iff \frac{y^2}{2} = \frac{x^3}{3} + c_1 \\ &\iff y^2 = \frac{2x^3}{3} + 2c_1, \\ &\iff y^2 = \frac{2x^3}{3} + c_2, \text{rename } c_2 = 2c_1 \\ &\iff y = \pm\sqrt{\frac{2x^3}{3} + c_2} \end{aligned}$$

Thus since $c_2$is our final constant I'll just rename it as C and get

$$\text{General solution : } y(t) = \pm\sqrt{\frac{2x^3}{3} + C}.$$

(c) $\dfrac{dy}{dx} = \dfrac{3x^2 - 1}{3 + 2y}$

- **Solution:** First note there are no equilibrium solutions. Separate variable

$$\begin{aligned} \frac{dy}{dx} = \frac{3x^2 - 1}{3 + 2y} &\iff (3 + 2y)\, dy = \left(3x^2 - 1\right) dx \\ &\iff \int (3 + 2y)\, dy = \int \left(3x^2 - 1\right) dx \\ &\iff 3y + y^2 = x^3 - x + c_1 \end{aligned}$$

We need to solve for y. Since this is a quadratic in y then we can use the quadratic formula: Rewrite the above as

$$y^2 + 3y - x^3 + x - c_1 = 0,$$

rename $c_2 = -c_1$ and get

$$y^2 + 3y - x^3 + x + c_2 = 0$$

then just like in the notes, we can solve $ay^2 + by + c = 0$ with here a, b, c being

$$\begin{aligned} a &= 1 \\ b &= 3 \\ c &= -x^3 + x + c_2. \end{aligned}$$

and using the quadratic formula we get

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4\left(-x^3 + x + c_2\right)}}{2} \\ &= \frac{-3}{2} \pm \frac{1}{2}\sqrt{9 + 4x^3 - 4x - 4c_2} \end{aligned}$$

but notice that $9 - 4c_2$ is just a constant, so I'll rename that as $c_3 = 9 - 4c_2$ and get

$$\text{General solution}: y(t) = -\frac{3}{2} \pm \frac{1}{2}\sqrt{4x^3 - 4x + c_3}.$$

- Now this is a **perfectly good correct answer**, but if you can simplify even more (if you'd like to, but you don't have to on exams), by factoring out the 4 inside the square root, to get

$$\begin{aligned} y(t) &= -\frac{3}{2} \pm \frac{1}{2}\sqrt{4\left(x^3 - x + \frac{c_3}{4}\right)} \\ &= -\frac{3}{2} \pm \frac{2}{2}\sqrt{x^3 - x + \frac{c_3}{4}} \\ &= -\frac{3}{2} \pm \sqrt{x^3 - x + c_4}, \text{ renamed } c_4 = \frac{c_3}{4}. \end{aligned}$$

And hence an even more simplified version of the General solution is

$$\text{More simplified General solution}: y(t) = -\frac{3}{2} \pm \sqrt{x^3 - x + C}.$$

(d) $xy' = \dfrac{\left(1 - y^2\right)^{1/2}}{y}$

- **Solution:** Note that $y(t) = 1, -1.$ are equilibrium solutions. Then separate variables

$$\begin{aligned} x\frac{dy}{dx} = \frac{\left(1 - y^2\right)^{1/2}}{y} &\iff \frac{ydy}{\left(1 - y^2\right)^{1/2}} = \frac{1}{x}dx \\ &\iff \int \frac{y}{\left(1 - y^2\right)^{1/2}}dy = \int \frac{1}{x}dx \\ &\iff \int \frac{y}{\left(1 - y^2\right)^{1/2}}dy = \ln|x| + c \end{aligned}$$

To integrate the LHS we use u-substitution with $u = 1 - y^2$ and get $du = -2ydy$, or $-\frac{du}{2} = ydy$ so that

$$\begin{aligned} LHS &= \int \frac{y}{\left(1 - y^2\right)^{1/2}}dy = -\frac{1}{2}\int u^{-1/2}du \\ &= -\frac{1}{2}\frac{u^{1/2}}{1/2} = -u^{1/2} = -\sqrt{1 - y^2}. \end{aligned}$$

Thus

$$\begin{aligned} \text{LHS=RHS} &\iff -\sqrt{1-y^2} = \ln|x| + c \\ &\iff \sqrt{1-y^2} = -\ln|x| + c \\ &\iff 1-y^2 = (C - \ln|x|)^2 \\ &\iff y^2 = 1 - (C-\ln|x|)^2 \\ &\iff y = \pm\sqrt{1-(C-\ln|x|)^2} \end{aligned}$$

- Note that since the formula does not contain $y(x) = \pm\sqrt{1-(C-\ln|x|)^2}$ the equilibrium solutions $y = 1, -1$ then the general explicit solution is given by

$$\text{General solution : } \begin{cases} y(t) = \pm\sqrt{1-(C-\ln|x|)^2} \\ y(t) = -1 \\ y(t) = 1 \end{cases}$$

(e) $\dfrac{dy}{dx} = \dfrac{x^2}{1+y^2}$

- **Solution:** Note that there are no equilibrium solutions. Then rewrite $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ and separate variables

$$\begin{aligned} \frac{dy}{dx} = \frac{x^2}{1+y^2} &\iff \left(1+y^2\right) dy = x^2 dx \\ &\iff \int \left(1+y^2\right) dy = \int x^2 dx \\ &\iff y + \frac{y^3}{3} = \frac{x^3}{3} + c_1 \\ &\iff y + \frac{y^3}{3} - \frac{x^3}{3} + C = 0, \text{ where I let } C - c_1 \end{aligned}$$

- Recall we are trying to solve for y. And in general it is hard to solve for cubic (even though there is a "cubic formula", I don't expect you to know what it is). Hence we will leave the solution as an implicit solution:
- Thus

$$\text{General Implicit solution : } y + \frac{y^3}{3} - \frac{x^3}{3} + C = 0.$$

(f) $\dfrac{dy}{dx} = \dfrac{x}{\cos\left(y^2\right) y}$

- **Solution:** Note that there are no equilibrium solutions. Then separate variables

$$\begin{aligned}\frac{dy}{dx}=\frac{x}{\cos\left(y^2\right)y} &\iff \cos\left(y^2\right)ydy=\int xdx\\ &\iff \int\cos\left(y^2\right)ydy=\int xdx\\ &\iff \frac{1}{2}\sin\left(y^2\right)=\frac{x^2}{2}+c_1,\ \text{by u-substitution}\\ &\iff \sin\left(y^2\right)=x^2+c_2\end{aligned}$$

- Note that we can solve this exactly to get

$$\text{General Explicit solution}: y(t)=\pm\sqrt{\sin^{-1}\left(x^2+C\right)}.$$

- **What is the domain of this function?** We have to be careful here. Because the domain of $\sin^{-1}x$ is $[-1,1]$, while its range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ And since we can only put positive numbers in the square root function. Then the domain of $y(t)$ is all real numbers where

$$-1\leq x^2+C\leq 1 \text{ and } 0\leq\sin^{-1}\left(x^2+C\right)$$

and this only happens when

$$0\leq x^2+C\leq 1.$$

- In general, I won't expect you to know the domain and range of inverse trig functions. So the domains of the functions is whereever the equations are defined. Thus it might be easier to write solutions implicitely:

$$\text{General Implicit solution}: \text{ All functions satisfying} \sin\left(y^2\right)=x^2+C.$$

(2) Consider the ODE

$$\frac{dy}{dt}=\frac{4y}{t}.$$

(a) What kind of differential equation is this? Is it Linear? Is it separable?
- **Solution:** Note that since we can write the ODE as

$$\frac{dy}{dt}-\frac{4}{t}y=0$$

then it is linear. It is also separable!
- This means, any of the two methods would work.

(b) If the ODE is both Separable and Linear. Then use both methods to solve this equation. And check to make sure you get the same answer.
- **Solution:**
- Solving it as a Linear ODE: Since $\frac{dy}{dt}-\frac{4}{t}y=0$ then we can let $p(t)=-\frac{4}{t}$ and $g(t)=0$. The integrating factor is

$$\begin{aligned}\mu(t)&=e^{\int p(t)dt}=e^{\int-\frac{4}{t}dt}\\ &=e^{-4\ln t}=e^{\ln t^{-4}}=\frac{1}{t^4}\end{aligned}$$

Then the solution is given by

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right] \\ &= \frac{1}{t^{-4}}\left[\int \frac{1}{t^4}\cdot 0dt + C\right] \\ &= t^4\left[0 + C\right] \\ &= Ct^4. \end{aligned}$$

- Solving it by treating as a separable equation: Separating we get

$$\begin{aligned} \frac{dy}{dt} = \frac{4y}{t} &\iff \frac{dy}{y} = \frac{4}{t}dt \\ &\iff \int \frac{dy}{y} = \int \frac{4}{t}dt \\ &\iff \ln|y| = 4\ln|t| + c_1 \\ &\iff \ln|y| = \ln t^4 + c_1 \\ &\iff |y| = e^{\ln t^4 + c_1} \\ &\iff |y| = e^{c_1}e^{\ln t^4} \\ &\iff y = Ce^{\ln t^4}, \text{ by letting } C = \pm e^{c_1} \\ &\iff y = Ct^4. \end{aligned}$$

  And we get the same answer. Note that the equilibrium solution $y = 0$, is also included by letting $C = 0$.
- In general you can choose whichever method you prefer.

(3) Find the general solution to the following differential equation:

$$\frac{dy}{dt} = (y+1)(y-2).$$

(Hint: Use Partial fractions!)

- **Solution:** Note that $y(t) = -1, 2.$ are equilibrium solutions. Then rewrite $\frac{dy}{dt} = (y+1)(y-2)$ and separate variable

$$\begin{aligned} \frac{dy}{dt} = (y+1)(y-2) &\iff \frac{dy}{(y+1)(y-2)} = dt \\ &\iff \int \frac{dy}{(y+1)(y-2)} = \int dt \\ &\iff \int \frac{dy}{(y+1)(y-2)} = t + c_1. \end{aligned}$$

  to integrate $\int \frac{dy}{(y+1)(y-2)}$ we need to use partial fractions.
- Recall to do partial fractions we have

$$\frac{1}{(y+1)(y-2)} = \frac{A}{y+1} + \frac{B}{y-2}$$

multiply both sides by $(y+1)(y-2)$ and get

$$\begin{aligned} 1 &= A(y-2) + B(y+1) \\ LHS &= RHS \end{aligned}$$

rewrite the RHS by putting all y's together

$$1 = (A+B)\,y + (B-2A)$$

and rewrite the LHS and recall there is an imaginary $0 \cdot y$ and get

$$0 \cdot y + 1 = (A+B)\,y + (B-2A)$$

comparing coefficients we have that

$$\begin{aligned} 0 &= A + B \\ 1 &= B - 2A \end{aligned}$$

and solving this system we have that

$$A = -\frac{1}{3} \text{ and } B = \frac{1}{3}.$$

- Putting back into the ODE equation we have

$$\begin{aligned}
\int \left(-\frac{1}{3}\frac{1}{y+1} + \frac{1}{3}\frac{1}{y-2}\right) dy = t + c_1 &\iff \frac{1}{3}\ln|y-2| - \frac{1}{3}\ln|y+1| = t + c_1 \\
&\iff \ln|y-2|^{1/3} - \ln|y+1|^{1/3} = t + c_1 \\
&\iff \ln\frac{|y-2|^{1/3}}{|y+1|^{1/3}} = t + c_1 \\
&\iff \frac{|y-2|^{1/3}}{|y+1|^{1/3}} = e^{c_1}e^t, \text{ and then let } c_2 = e^{c_1} \\
&\iff \frac{|y-2|^{1/3}}{|y+1|^{1/3}} = c_2 e^t, \\
&\iff \frac{|y-2|}{|y+1|} = c_3 e^{3t}, \text{ where } c_3 = c_2^3 \\
&\iff \frac{(y-2)}{(y+1)} = c_4 e^{3t}, \text{ where } c_4 = \pm c_3 \\
&\iff (y-2) = c_4 e^{3t}(y+1) \\
&\iff y - 2 = c_4 e^{3t} y + c_4 e^{3t} \\
&\iff y\left(1 - c_4 e^{3t}\right) = c_4 e^{3t} + 2 \\
&\iff y = \frac{c_4 e^{3t} + 2}{1 - c_4 e^{3t}}
\end{aligned}$$

Since this formula already includes the Equilibrium solution $y(t) = 2$ when $C = 0$, then

$$\text{General Explicit solution}: \begin{cases} y(t) = \frac{Ce^{3t}+2}{1-Ce^{3t}} \\ y(t) = -1 \end{cases}.$$

2.3. Problems

(1) First check each if the following differential equations are homogeneous. Then find the general solutions for the following differential equations.

(a) $x^2\dfrac{dy}{dx} = -\left(y^2 - yx\right).$

- **Solution:** First we write it as $\frac{dy}{dx} = -\frac{\left(y^2-yx\right)}{x^2}$ and we try to get the RHS to look like $F\left(\frac{y}{x}\right)$. by doing some algebra note that
$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)$$
hence we can make the substitution $v = \frac{y}{x}$ on the RHS and use the fact that $\frac{dy}{dx} = x\frac{dv}{dx} + v$ on the LHS:
$$x\frac{dv}{dx} + v = -v^2 + v$$
and simplifying we get
$$x\frac{dv}{dx} = -v^2.$$
- This new equation is separable:
$$\begin{aligned} x\frac{dv}{dx} = -v^2 &\iff \int v^{-2}dv = -\int \frac{1}{x}dx \\ &\iff \int v^{-2}dv = -\ln|x| + C \\ &\iff -\frac{1}{v} = -\ln|x| + C \\ &\iff \frac{1}{v} = \ln|x| + C, \text{ note that I just renamed } -C \text{ by } C \text{ again} \\ &\iff v = \frac{1}{\ln|x| + C}. \end{aligned}$$
- Now that we've solve for v. We need to go back to y, using the substitution we had made $v = \frac{y}{x}$
$$\begin{aligned} v = \frac{1}{\ln|x| + C} &\iff \frac{y}{x} = \frac{1}{\ln|x| + C} \\ &\iff y = \frac{x}{\ln|x| + C}. \end{aligned}$$
- Thus $y(x) = \frac{x}{\ln|x|+C}$ is our final answer.
  - There is also an equilibrium solution of $y = 0$
- Thus the general explicit solution is given by
$$\begin{cases} y(x) = \frac{x}{\ln|x|+C} \\ y(x) = 0 \end{cases}$$

(b) $\dfrac{dy}{dx} = \dfrac{x + 3y + 2\frac{y^2}{x}}{3x + y}.$

- **Solution:** First we try to get the RHS to look like $F\left(\frac{y}{x}\right)$. By doing some algebra (divide by x everywhere) note that

$$\frac{dy}{dx} = \frac{1 + 3\frac{y}{x} + 2\left(\frac{y}{x}\right)^2}{3 + \frac{y}{x}}$$

hence we can make the substitution $v = \frac{y}{x}$ on the RHS and use the fact that $\frac{dy}{dx} = x\frac{dv}{dx} + v$ on the LHS:

$$x\frac{dv}{dx} + v = \frac{1 + 3v + 2v^2}{3 + v}$$

and simplifying we get

$$x\frac{dv}{dx} = \frac{1 + 3v + 2v^2}{3 + v} - v \iff x\frac{dv}{dx} = \frac{1 + 3v + 2v^2}{3 + v} - \frac{3v + v^2}{3 + v}$$
$$\iff x\frac{dv}{dx} = \frac{1 + v^2}{3 + v}$$

- This new equation is separable:

$$x\frac{dv}{dx} = \frac{1 + v^2}{3 + v} \iff \int \frac{3 + v}{v^2 + 1} dv = \int \frac{1}{x} dx$$
$$\iff \int \frac{3}{v^2 + 1} dv + \int \frac{v}{v^2 + 1} dv = \ln|x| + C$$
$$\iff 3\tan^{-1} v + \frac{1}{2}\ln\left|v^2 + 1\right| = \ln|x| + C$$

- We won't be able to solve for v in this equation. ($\tan^{-1}$ and ln don't mix) . But we still need to go back to y, using the substitution we had made $v = \frac{y}{x}$

$$3\tan^{-1} v + \frac{1}{2}\ln\left|v^2 + 1\right| = \ln|x| + C \iff 3\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left|\frac{y^2}{x^2} + 1\right| = \ln|x| + C$$

- Thus the **implicit general solution** is given by:

$$3\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left|\frac{y^2}{x^2} + 1\right| = \ln|x| + C$$

(c) $\frac{dy}{dx} = \frac{y}{x} - \frac{x^2 - y^2}{2xy}$.

- **Solution:** First we try to get the RHS to look like $F\left(\frac{y}{x}\right)$. By doing some algebra (divide the fraction by x^2 everywhere, because the leading in y in the numerator is y^2) note that

$$RHS = \frac{y}{x} - \frac{x^2 - y^2}{2xy} = \frac{y}{x} - \frac{\left(x^2 - y^2\right)/x^2}{(2xy)/x^2} = \frac{y}{x} - \frac{1 - \left(\frac{y}{x}\right)^2}{2\frac{y}{x}}$$

hence we can make the substitution $v = \frac{y}{x}$ on the RHS and use the fact that $\frac{dy}{dx} = x\frac{dv}{dx} + v$ on the LHS:

$$x\frac{dv}{dx} + v = v - \frac{1 - (v)^2}{2v}$$

and simplifying we get

$$x\frac{dv}{dx} = \frac{v^2-1}{2v}.$$

- Then we can separate variables:

$$\begin{aligned} x\frac{dv}{dx} = \frac{1-v^2}{2v} &\iff \frac{2v}{v^2-1}dv = \frac{dx}{x} \\ &\iff \int \frac{2v}{v^2-1}dv = \int \frac{dx}{x} \\ &\iff \ln\left|v^2-1\right| = \ln|x| + c \\ &\iff \left|v^2-1\right| = e^c e^{\ln|x|} \\ &\iff v^2 - 1 = c\,|x| \\ &\iff v^2 = 1 + c\,|x| \\ &\iff v = \pm\sqrt{1+c\,|x|} \end{aligned}$$

- Putting $v = \frac{y}{x}$ back in we get

$$\begin{aligned} v = \pm\sqrt{1+c\,|x|} &\iff \frac{y}{x} = \pm\sqrt{1+c\,|x|} \\ &\iff y = \pm x\sqrt{1+c\,|x|}. \end{aligned}$$

- General solution is

$$y(x) = \pm x\sqrt{1+c\,|x|}.$$

(2) Consider the following homogeneous equation:

$$\frac{dy}{dx} = \frac{y-x}{y+x}.$$

(a) Use the substitution $v = \frac{y}{x}$ to rewrite the equation only in terms of v and x.

- **Solution:** First we write it as $\frac{dy}{dx} = \frac{y-x}{y+x}$ and we try to get the RHS to look like $F\left(\frac{y}{x}\right)$. by doing some algebra (fivide everything by x) and note that

$$\frac{dy}{dx} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1}$$

hence we can make the substitution $v = \frac{y}{x}$ on the RHS and use the fact that $\frac{dy}{dx} = x\frac{dv}{dx} + v$ on the LHS:

$$x\frac{dv}{dx} + v = \frac{v-1}{v+1}$$

and simplifying we get

$$x\frac{dv}{dx} = \frac{v-1}{v+1} - v.$$

note that $\frac{v-1}{v+1} - v = \frac{v-1}{v+1} + \frac{-v^2-v}{v+1} = -\frac{v^2+1}{v+1}$ thus

$$x\frac{dv}{dx} = -\frac{v^2+1}{v+1}.$$

(b) Solve for the general solution.

- **Solution:** Using
$$x\frac{dv}{dx}=-\frac{v^2+1}{v+1}.$$
  - This new equation is separable:
$$x\frac{dv}{dx}=-\frac{v^2+1}{v+1} \iff \frac{v+1}{v^2+1}dv=-\frac{1}{x}dx$$
$$\iff \int\frac{v+1}{v^2+1}dv=-\int\frac{1}{x}dx.\ (\star)$$
We need to integrate $\int\frac{v+1}{v^2+1}dv$: Write
$$\int\frac{v+1}{v^2+1}dv=\int\frac{v}{v^2+1}dv+\int\frac{1}{v^2+1}dv$$
and do u-substitution on the first integral, and the second integral remember that its an inverse tan:
$$\int\frac{v+1}{v^2+1}dv=\frac{1}{2}\ln\left|v^2+1\right|+\tan^{-1}v.$$
  - Putting this back in $(\star)$ we get
$$\int\frac{v+1}{v^2+1}dv=-\int\frac{1}{x}dx \iff \frac{1}{2}\ln\left|v^2+1\right|+\tan^{-1}v=-\ln|x|+C$$
  - Now that we've solve for v. We need to go back to y, using the substitution we had made $v=\frac{y}{x}$
$$\frac{1}{2}\ln\left|\frac{y^2}{x^2}+1\right|+\tan^{-1}\left(\frac{y^2}{x^2}\right)=-\ln|x|+C$$
  - We can simplify even more by noting that $\frac{1}{2}\ln\left|\frac{y^2}{x^2}+1\right|=\frac{1}{2}\ln\left(x^{-2}\left(y^2+x^2\right)\right)=\frac{1}{2}\ln\left(x^{-2}\right)+\frac{1}{2}\ln\left(y^2+x^2\right)=-\ln|x|+\frac{1}{2}\ln\left(y^2+x^2\right)$and substitution this into the LHS we get
$$-\ln|x|+\frac{1}{2}\ln\left(y^2+x^2\right)+\tan^{-1}\left(\frac{y^2}{x^2}\right)=-\ln|x|+C$$
which we can cancel the $-\ln|x|$ in each side. And get
$$\frac{1}{2}\ln\left(y^2+x^2\right)+\tan^{-1}\left(\frac{y^2}{x^2}\right)=C.$$
  - We will leave this as the **implicit solution.**

(3) Consider the following homogeneous equation:
$$\frac{dy}{dx}=\frac{-y^2-yx}{x^2}.$$

(a) Use the substitution $v=\frac{y}{x}$ to rewrite the equation only in terms of v and x.
- **Solution:** First we write it as $\frac{dy}{dx}=\frac{-y^2-yx}{x^2}$ and we try to get the RHS to look like $F\left(\frac{y}{x}\right)$. By doing some algebra note that
$$\frac{dy}{dx}=-\left(\frac{y}{x}\right)^2-\left(\frac{y}{x}\right)$$

hence we can make the substitution $v = \frac{y}{x}$ on the RHS and use the fact that $\frac{dy}{dx} = x\frac{dv}{dx} + v$ on the LHS:

$$x\frac{dv}{dx} + v = -v^2 - v$$

and simplifying we get

$$x\frac{dv}{dx} = -v^2 - 2v.$$

(b) Solve for the general solution.

- **Solution:** Using

$$x\frac{dv}{dx} = -v^2 - 2v.$$

  – This new equation is separable:

$$\begin{aligned} x\frac{dv}{dx} = -v^2 - 2v &\iff \int \frac{dv}{v^2 + 2v} = -\int \frac{1}{x}dx \\ &\iff \int \frac{dv}{v\,(v+2)} = -\ln|x| + C, \quad (\star) \end{aligned}$$

  – We need to integrate the LHS of $\int \frac{dv}{v(v+2)}$ (use partial fractions) to get $\frac{1}{v(v+2)} = \frac{1}{2}\frac{1}{v} - \frac{1}{2}\frac{1}{v+2}$. Hence plugging this into the LHS of $(\star)$ we get

$$\begin{aligned} \int \frac{dv}{v\,(v+2)} = -\ln|x| + C &\iff \int \left(\frac{1}{2}\frac{1}{v} - \frac{1}{2}\frac{1}{v+2}\right) dv = -\ln|x| + c_1 \\ &\iff \frac{1}{2}\ln|v| - \frac{1}{2}\ln|v+2| = -\ln|x| + c_1 \\ &\iff \ln|v| - \ln|v+2| = -2\ln|x| + c_2 \end{aligned}$$

  – Taking e of everything we get

$$\begin{aligned} e^{\ln|v| - \ln|v+2|} = e^{-2\ln|x| + C} &\iff e^{\ln|v|}e^{-\ln|v+2|} = e^{c_2}e^{-2\ln|x|} \\ &\iff \frac{|v|}{|v+2|} = c_3 e^{\ln x^{-2}} \\ &\iff \frac{v}{v+2} = \frac{c_4}{x^2} \text{ where } c_4 = \pm c_3 \\ &\iff v = \frac{c_4}{x^2}v + 2\frac{c_4}{x^2} \\ &\iff v\left(1 - \frac{c_4}{x^2}\right) = +2\frac{c_4}{x^2} \\ &\iff v = \frac{2c_4}{x^2\left(1 - \frac{c_4}{x^2}\right)} \end{aligned}$$

  – We need to go back to y, using the substitution we had made $v = \frac{y}{x}$

$$\begin{aligned} \frac{y}{x} = \frac{2c_4}{x^2\left(1 - \frac{c_4}{x^2}\right)} &\iff y = \frac{2xc_4}{x^2\left(1 - \frac{c_4}{x^2}\right)} \\ &\iff y = \frac{2xc_4}{x^2 - c_4}. \end{aligned}$$

– and dividing out everyting by $c_4$we get

$$y=\frac{2x}{\frac{1}{c_4}x^2-1}$$

and finally renaming $C=\frac{1}{c_4}$ we get the simplest version:

$$y(x)=\frac{2x}{Cx^2-1}$$

(4) Using the given substitution. Solve the differential equation:

(a) Rewrite $\frac{dy}{dx}+xy=x^2y^2$ using the substitution $u=\frac{1}{y}$, only in terms of u,x.

- **Solution:** Using $u=\frac{1}{y}$ then solving for y we get

$$y=\frac{1}{u}.$$

- Then differentiating

$$\frac{dy}{dx}=-u^{-2}\frac{du}{dx}.$$

and substitution this into LHS and RHS of the ODE we get

$$-\frac{1}{u^2}\frac{du}{dx}+x\frac{1}{u}=x^2\frac{1}{u^2}$$

hence

$$\frac{du}{dx}-xu=-x^2.$$

(b) Rewrite $\frac{dy}{dx}+y=\frac{x}{y^2}$ using the substitution $u=y^3$, only in terms of u,x.

- **Solution:** Using $u=y^3$ then solving for y we get

$$y=u^{1/3}.$$

- Then differentiating

$$\frac{dy}{dx}=\frac{1}{3}u^{-2/3}\frac{du}{dx}.$$

and substitution this into LHS and RHS of the ODE we get

$$\frac{1}{3}u^{-2/3}\frac{du}{dx}+u^{1/3}=\frac{x}{\left(u^{1/3}\right)^2}$$

hence multiplying everyting by $u^{2/3}$ we get

$$\frac{1}{3}\frac{du}{dx}+u=x.$$

2.4. Problems

(1) Initially, a tank contains 100 L of water with 10 kg of sugar in solution. Water containing sugar flows into the tank at the rate of 2 L/min, and the well-stirred mixture in the tank flows out at the rate of 5 L/min. The concentration $c(t)$ of sugar in the incoming water varies as $c(t) = 2 + \cos(3t)$ kg/L. Let $Q(t)$ be the amount of sugar (in kilograms) in the tank at time t (in minutes). Write the Initial Value Problem that $Q(t)$ satisfies?

- **Solution:**

Step1: Define variables

Let $Q(t)$ =amount of sugar at time t. Let $Q(0) = 10$ kg.

Step2: Find Rate in/ Rate out

Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \begin{pmatrix} \text{concentrarion} \\ \text{of stuff coming in} \end{pmatrix} \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate}.$$

Using the information from the problem we have

$$\begin{aligned} \text{Rate in} &= \left(c(t)\frac{\text{kg}}{\text{L}}\right)\left(2\frac{\text{L}}{\text{min}}\right) \\ & \quad \text{-sugar water solution} \\ &= 2\left(2+\cos(3t)\right)\frac{\text{kg}}{\text{min}}. \end{aligned}$$

and

$$\begin{aligned} \text{Rate out} &= \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate} \\ &= \left(\frac{Q(t)}{w(t)}\frac{kg}{L}\right) \times 5\frac{\text{L}}{\text{min}}. \end{aligned}$$

where

$$\begin{aligned} w(t) &= \text{water at time t} \\ &= 100\text{L} + \left(2\frac{\text{L}}{\text{min}} - 5\frac{\text{L}}{\text{min}}\right)t \\ &= 100 - 3t. \end{aligned}$$

hence

$$\begin{aligned} \text{Rate out} &= \left(\frac{Q(t)}{w(t)}\frac{kg}{L}\right) \times 5\frac{\text{L}}{\text{min}}. \\ &= \frac{5Q(t)}{100-3t}. \end{aligned}$$

Step 3: Write the IVP

Always recall that for mixing problems we have

$$\begin{aligned}\frac{dQ}{dt} &= \text{Rate in} - \text{Rate out},\\ \frac{dQ}{dt} &= 2\left(2+\cos(3t)\right) - \frac{5Q}{100-3t}.\end{aligned}$$

and the initial condition is

$$Q(0) = 10.$$

(2) Initially, a tank contains 500 L (liters) of pure water. Water containing 0.3kg of salt per liter is entering at a rate of 2 L/min, and the mixture is allowed to flow out of the tank at a rate of 1 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms (kg). What is the IVP that $Q(t)$ satisfies?

- **Solution:**

Step1: Define variables

Let $Q(t)$ =amount of salt at time t. Let $Q(0) = 0$ kg since the tank only contains pure water initially.

Step2: Find Rate in/ Rate out

Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \left(\begin{array}{c}\text{concentrarion}\\ \text{of stuff coming in}\end{array}\right) \times \text{Rate.}$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \left(\begin{array}{c}\text{concentrarion}\\ \text{of stuff going out}\end{array}\right) \times \text{Rate.}$$

Using the information from the problem we have

$$\begin{aligned}\text{Rate in} &= \underbrace{\left(.3\frac{\text{kg}}{\text{L}}\right)\left(2\frac{\text{L}}{\text{min}}\right)}_{\text{-salt water solution}}\\ &= .6\frac{\text{kg}}{\text{min}}.\end{aligned}$$

and

$$\begin{aligned}\text{Rate out} &= \left(\begin{array}{c}\text{concentrarion}\\ \text{of stuff going out}\end{array}\right) \times \text{Rate}\\ &= \left(\frac{Q(t)}{w(t)}\frac{kg}{L}\right) \times 1\frac{\text{L}}{\text{min}}.\end{aligned}$$

where

$$\begin{aligned}w(t) &= \text{water at time t}\\ &= 500\text{L} + \left(2\frac{\text{L}}{\text{min}} - 1\frac{\text{L}}{\text{min}}\right)t\\ &= 500 + t.\end{aligned}$$

hence

$$\begin{aligned} \text{Rate out} &= \left(\frac{Q(t)}{w(t)}\frac{kg}{L}\right) \times 1\frac{\text{L}}{\text{min}}. \\ &= \frac{Q(t)}{500+t.}. \end{aligned}$$

Step 3: Write the IVP
Always recall that for mixing problems we have

$$\begin{aligned} \frac{dQ}{dt} &= \text{Rate in} - \text{Rate out}, \\ \frac{dQ}{dt} &= .6 - \frac{Q}{500+t}. \end{aligned}$$

and the initial condition is

$$Q(0) = 0.$$

(3) Initially, a tank contains 400 L of water with 10 kg of salt in solution. Water containing 0.1 kg of salt per liter (L) is entering at a rate of 1 L/min, and the mixture is allowed to flow out of the tank at a rate of 2 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms. What is the IVP that $Q(t)$ satisfies?

- **Solution:**

Step1: Define variables
Let $Q(t)$ =amount of salt at time t. Let $Q(0) = 10$ kg.
Step2: Find Rate in/ Rate out
Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \left(\begin{array}{c}\text{concentrarion} \\ \text{of stuff coming in}\end{array}\right) \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \left(\begin{array}{c}\text{concentrarion} \\ \text{of stuff going out}\end{array}\right) \times \text{Rate}.$$

Using the information from the problem we have

$$\begin{aligned} \text{Rate in} &= \underbrace{\left(.1\frac{\text{kg}}{\text{L}}\right)\left(1\frac{\text{L}}{\text{min}}\right)}_{\text{-salt water solution}} \\ &= .1\frac{\text{kg}}{\text{min}}. \end{aligned}$$

and

$$\begin{aligned} \text{Rate out} &= \left(\begin{array}{c}\text{concentrarion} \\ \text{of stuff going out}\end{array}\right) \times \text{Rate} \\ &= \left(\frac{Q(t)}{w(t)}\frac{kg}{L}\right) \times 2\frac{\text{L}}{\text{min}}. \end{aligned}$$

where

$$\begin{aligned} w(t) &= \text{water at time t} \\ &= 400\text{L} + \left(1\frac{\text{L}}{\text{min}} - 2\frac{\text{L}}{\text{min}}\right)t \\ &= 400 - t. \end{aligned}$$

hence

$$\begin{aligned} \text{Rate out} &= \left(\frac{Q(t)}{w(t)}\frac{kg}{L}\right) \times 2\frac{\text{L}}{\text{min}}. \\ &= \frac{2Q(t)}{400 - t.}. \end{aligned}$$

Step 3: Write the IVP
Always recall that for mixing problems we have

$$\begin{aligned} \frac{dQ}{dt} &= \text{Rate in} - \text{Rate out}, \\ \frac{dQ}{dt} &= .1 - \frac{2Q}{400 - t}. \end{aligned}$$

and the initial condition is

$$Q(0) = 10.$$

(4) Consider a pond that initially constains 10 million gal of pure water. Water containing a polluted chemical flows into the pond at the rate of 6 million gal/year, and the mixture in the pond flows out at the rate of 5 million gal/year. The concentration $\gamma(t)$ of chemical in the incoming water varies as $\gamma(t) = 2 + \sin 2t$ grams/gal. Let $Q(t)$ be the amount of chemical at time t measured by millions of grams. What is the IVP that $Q(t)$ satisfies?

- **Solution:**

Step1: Define variables

Let $Q(t)$ =amount of chemical at time t. Let $Q(0) = 0$ grams, since initially the pond has only pure water.

Step2: Find Rate in/ Rate out
Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \left(\begin{array}{c}\text{concentrarion} \\ \text{of stuff coming in}\end{array}\right) \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \left(\begin{array}{c}\text{concentrarion} \\ \text{of stuff going out}\end{array}\right) \times \text{Rate}.$$

Using the information from the problem we have

$$\begin{aligned} \text{Rate in} &= \underbrace{\left(\gamma(t)\frac{\text{grams}}{\text{gal}}\right)}_{\text{chemical}} \underbrace{\left(6\frac{\text{gal}}{\text{year}}\right)}_{\text{solution}} \\ &= 12 + 6\sin 2t\frac{\text{grams}}{\text{year}}. \end{aligned}$$

and

$$\begin{aligned}\text{Rate out} &= \begin{pmatrix}\text{concentrarion} \\ \text{of stuff going out}\end{pmatrix} \times \text{Rate} \\ &= \left(\frac{Q(t)}{w(t)} \frac{\text{grams}}{\text{gal}}\right) \times 5 \frac{\text{gal}}{\text{year}}.\end{aligned}$$

where

$$\begin{aligned}w(t) &= \text{water at time t} \\ &= 10 \text{ gallons} + \left(6 \frac{\text{gallons}}{\text{min}} - 5 \frac{\text{gallons}}{\text{min}}\right) t \\ &= 10 + t.\end{aligned}$$

hence

$$\begin{aligned}\text{Rate out} &= \left(\frac{Q(t)}{w(t)} \frac{kg}{L}\right) \times 5 \frac{\text{L}}{\text{min}}. \\ &= \frac{5Q(t)}{10+t.}.\end{aligned}$$

Step 3: Write the IVP
Always recall that for mixing problems we have

$$\begin{aligned}\frac{dQ}{dt} &= \text{Rate in} - \text{Rate out}, \\ \frac{dQ}{dt} &= 12 + 6\sin 2t - \frac{5Q}{10+t.}.\end{aligned}$$

and the initial condition is

$$Q(0) = 0.$$

(5) A tank contains 200 gal of liquid. Initially, the tank contains pure water. At time $t = 0$, brine containing 3 lb/gal of salt begins to pour into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to drain away at the same rate. How many minutes must elapse before there are 100 lb of salt in the tank?

- **Solution:**

Step1: Define variables
Let $y(t)$ =amount of salt at time t. Let $y(0) = 0$, since initially the tank has only pure water.
Step2: Find Rate in/ Rate out
Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \begin{pmatrix}\text{concentrarion} \\ \text{of stuff coming in}\end{pmatrix} \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \begin{pmatrix}\text{concentrarion} \\ \text{of stuff going out}\end{pmatrix} \times \text{Rate}.$$

Using the information from the problem we have

$$\begin{aligned}\text{Rate in} &= \underbrace{\left(3\frac{\text{lb}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right)}_{\text{brine solution}} \\ &= 6\frac{\text{lb}}{\text{min}}.\end{aligned}$$

and

$$\begin{aligned}\text{Rate out} &= \begin{pmatrix}\text{concentrarion} \\ \text{of stuff going out}\end{pmatrix} \times \text{Rate} \\ &= \left(\frac{y(t)}{w(t)}\frac{\text{lb}}{\text{gal}}\right) \times 2\frac{\text{gal}}{\text{min}}.\end{aligned}$$

where since it's draining the same rate that it is leaving, then the amounut of solution in the tank is constant:

$$\begin{aligned}w(t) &= \text{water at time t} \\ &= 200 \text{ gallons} + \left(2\frac{\text{gallons}}{\text{min}} - 2\frac{\text{gallons}}{\text{min}}\right) t \\ &= 200\end{aligned}$$

hence

$$\begin{aligned}\text{Rate out} &= \left(\frac{y(t)}{w(t)}\frac{\text{lb}}{\text{gal}}\right) \times 2\frac{\text{gal}}{\text{min}}. \\ &= \frac{2y(t)}{200} = \frac{1}{100}y.\end{aligned}$$

Step 3: Write the IVP
Always recall that for mixing problems we have

$$\begin{aligned}\frac{dy}{dt} &= \text{Rate in} - \text{Rate out}, \\ \frac{dy}{dt} &= 6 - \frac{1}{100}y.\end{aligned}$$

and the initial condition is

$$y(0) = 0.$$

Step 4: Solve IVP as a linear equation (it's also separable) and get

$$y(t) = 600 - 600e^{-t/100}.$$

But the questions for what time it takes until there is 100 lbs of salt in the tank, Then set

$$y(t) = 100$$

and solve for t. That is, solve

$$100 = 600 - 600e^{-t/100}$$

and get

$$\begin{aligned}\text{time it takes to fill tank to hundred lbs} &= 100 \ln \frac{6}{5} \\ &\approx 18.23 \text{ minutes}\end{aligned}$$

(6) A huge tank initially contains 10 gallons (gal) of water with 6 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the well-stirred mixture is allowed to flow out of the tank at a rate of 2 gal/min. What is the amount of the salt in the tank after 10 min?

- **Solution:**

Step1: Define variables

Let $y(t)$ =amount of salt at time t. Let $y(0) = 6$ lbs, since initially the tank has only pure water.

Step2: Find Rate in/ Rate out

Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \begin{pmatrix} \text{concentrarion} \\ \text{of stuff coming in} \end{pmatrix} \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate}.$$

Using the information from the problem we have

$$\begin{aligned} \text{Rate in} &= \underbrace{\left(1\frac{\text{lb}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right)}_{\text{brine solution}} \\ &= 3\frac{\text{lb}}{\text{min}}. \end{aligned}$$

and

$$\begin{aligned} \text{Rate out} &= \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate} \\ &= \left(\frac{y(t)}{w(t)}\frac{\text{lb}}{\text{gal}}\right) \times 2\frac{\text{gal}}{\text{min}}. \end{aligned}$$

where

$$\begin{aligned} w(t) &= \text{water at time t} \\ &= 10 \text{ gallons} + \left(3\frac{\text{gallons}}{\text{min}} - 2\frac{\text{gallons}}{\text{min}}\right)t \\ &= 10 + t \end{aligned}$$

hence

$$\begin{aligned} \text{Rate out} &= \left(\frac{y(t)}{w(t)}\frac{\text{lb}}{\text{gal}}\right) \times 2\frac{\text{gal}}{\text{min}}. \\ &= \frac{2y(t)}{10+t} = \end{aligned}$$

Step 3: Write the IVP

Always recall that for mixing problems we have

$$\begin{aligned} \frac{dy}{dt} &= \text{Rate in} - \text{Rate out}, \\ \frac{dy}{dt} &= 3 - \frac{2}{10+t}y. \end{aligned}$$

and the initial condition is

$$y(0) = 6.$$

Step 4: Solve IVP as a linear equation (it's also separable) and get that rewriting

$$\frac{dy}{dt} + \frac{2}{10+t}y = 3$$

then

$$\begin{aligned}\mu(t) &= e^{\int \frac{2}{10+t}dt} = e^{2\ln(10+t)} = (10+t)^2 \\ y(t) &= \frac{1}{\mu(t)}\left[\int g(t)\mu(t)dt + C\right] \\ &= \frac{1}{(10+t)^2}\left[3\int (10+t)^2\, dt + C\right] \\ &= \frac{1}{(10+t)^2}\left[(10+t)^3 + C\right]\end{aligned}$$

and solving for C we get

$$\begin{aligned}6 &= \frac{10^3 + C}{10^2} \implies C = -400 \\ y(t) &= \frac{(10+t)^3 - 400}{(10+t)^2}\end{aligned}$$

But the questions for the amount of salt at time $t = 10$, hence the answer is

$$y(10) = \frac{(20)^3 - 400}{(20)^2} = 19.$$

(7) Initially a tank holds 40 gallons of water with 10 lb of salt in solution. A salt solution containing $\frac{1}{2}$b of salt per gallon runs into the tank at the rate of 4 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $y(t)$ be the amount of salt in the tank after t minutes. Then what is $y(20)$.

- **Solution:**
- Just like before we can set up the following IVP

$$\frac{dy}{dt} = 2 - \frac{2y}{40+2t}, \qquad y(0) = 10$$

and then solving for $y(t)$ and plugging $t = 20$ we get

$$y(20) = 35.$$

2.5. Problems

(1) A detective is called to the scene of a crime where a dead body has just been found.

- She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 80°F. The detective checks the programmable thermostat and finds that the room has been kept at a constant 68°F for the past 3 days.
- After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5° F. This last temperature reading was taken exactly one hour after the first one.
- The next day the detective is asked by another investigator, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6°) prior to death, what is her answer to this question? Newton's Law of Cooling can be used to determine a victim's time of death.
- **Solution:** One needs to solve the following IVP: Let $T(t)$ be the temperature of the victim, then
$$\frac{dT}{dt} = k(T - 68), \quad T(0) = 98.6$$
and need to use the information
$$T(t_c) = 80,$$
$$T(t_c + 1) = 78.5$$
to solve for k and t_c.
- First solving for $T(t)$ we get
$$T(t) = 68 + (98.6 - 68)\,e^{kt}$$
$$= 68 + 30.6e^{kt}.$$
- Then using
$$80 = 68 + 30.6e^{kt_c},$$
$$78.5 = 68 + 30.6e^{k(t_c+1)}$$
- Solving the first equation for k we get
$$k = \frac{1}{t_c}\ln\frac{12}{30.6}$$
and plugging this into second equation we get
$$78.5 = 68 + 30.6e^{\frac{1}{t_c}\ln\frac{12}{30.6}(t_c+1)}$$
and hence
$$t_c \approx 7.01 \text{ hours.}$$
- This means the murder occured 7 hours and .6 minutes ago. That is, the muder occured arround 3 : 23 pm.

2.6. Problems

(1) What is the largest open interval in which the solution to the IVPs in part (a) and part (b) are guaranteed to exist by the Existence and Uniqueness Theorem?

(a) The IVP given by:

$$\begin{cases} \left(t^2+t-2\right) y' + e^t y = \frac{(t-4)}{(t-6)} \\ y(-3) = -1. \end{cases}$$

- **Solution:**
- First rewrite

$$y' + \frac{e^t}{(t+2)(t-1)} y = \frac{(t-4)}{(t-6)(t+2)(t-1)}$$

- Where $p(t) = \frac{e^t}{(t+2)(t-1)}$ and $g(t) = \frac{(t-4)}{(t-6)(t+2)(t-1)}$.
- To find the largest open interval, we simply need to check the largest open interval containing the initial value t_0 in which both $p(t)$ and $g(t)$ are continuous
- To do this we look for the bad points (non-continuous points) of p and g
- The function $p(t) = \frac{e^t}{(t+2)(t-1)}$is continuous whenever $t+2 \neq 0$ and when $t-1 \neq 0$
  - Thus we must have $t \neq -2, 1$.
- The function $g(t) = \frac{(t-4)}{(t-6)(t+2)(t-1)}$is continuous whenever $(t-6)(t+2)(t-1) \neq 0$. (Note that $t = 4$is NOT a problem since it's in the numerator)
  - Thus we must have $t \neq -2, 1, 6$.
- Both functions are simultaneously continuous (draw a number line to help you find out when p, g are **both** continuous) on

$$(-\infty, -2) \cup (-2, 1) \cup (1, 6) \cup (6, \infty)$$

since $t_0 = -3$ falls inside $(-\infty, -2)$ then the solution to this IVP must have a domain as large as

$$I = (-\infty, -2),$$

as guaranteed by the theorem.

(b) The IVP given by:

$$\begin{cases} \left(t^2+t-2\right) y' + e^t y = \frac{(t-4)}{(t-6)} \\ y(5) = 47. \end{cases}$$

- **Solution:**
- Note that this is the same equation as in part (a), so we know that both p, g are continuous on

$$(-\infty, -2) \cup (-2, 1) \cup (1, 6) \cup (6, \infty)$$

- Since the new initial point $t_0 = 5$ falls inside $(1, 6)$ then the solution to this IVP must have a domain as large as

$$I = (1, 6),$$

by the theorem.

(2) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-3) y' + y = \frac{(t-3)\cdot\ln(t-1)}{t-10} \\ y(6) = -7. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

- **Solution:**
- To apply the Existence and Uniqueness Theorem we first need to rewrite this Linear equation in the form

$$y' + p(t)y = g(t)$$

and we get

$$y' + \frac{1}{(t-3)}y = \frac{\ln(t-1)}{t-10}$$

- Where $p(t) = \frac{1}{(t-3)}$ and $g(t) = \frac{\ln(t-1)}{t-10}$.
- To find the largest open interval, we simply need to check the largest open interval containing the initial value t_0 in which both $p(t)$ and $g(t)$ are continuous
- To do this we look for the bad points (non-continuous points) of p and g
- The function $p(t) = \frac{1}{(t-3)}$is continuous whenever $t \neq 3$.
  - Thus we must have $t \neq 3$.
- The function $g(t) = \frac{t\ln(t-1)}{t-10}$is continuous whenever $t-1 > 0$ and when $t-10 \neq 0$
  - Meaning when $t > 1$ and $t \neq 10$.
- Both functions are simultaneously continuous (draw a number line to help you find out when p, g are **both** continuous) on

$$(1,3) \cup (3,10) \cup (10,\infty)$$

since $t_0 = 6$ falls inside $(3,10)$ then the solution to this IVP must have a domain as large as

$$I = (3,10),$$

by the theorem.

(3) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-1)\,y' + \sqrt{t+2}y = \frac{3}{t-3} \\ y(2) = -5. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

- **Solution:**
- First rewrite

$$y' + \frac{\sqrt{t+2}}{(t-1)}y = \frac{3}{(t-3)(t-1)}$$

- Where $p(t) = \frac{\sqrt{t+2}}{(t-1)}$ and $g(t) = \frac{3}{(t-3)(t-1)}$.
- To find the largest open interval, we simply need to check the largest open interval containing the initial value t_0 in which both $p(t)$ and $g(t)$ are continuous
- To do this we look for the bad points (non-continuous points) of p and g
- The function $p(t) = \frac{\sqrt{t+2}}{(t-1)}$is continuous whenever $t-1 \neq 0$ and when $t+2 \geq 0$
  - Thus we must have $t \neq 1$ and $t \geq -2$.
- The function $g(t) = \frac{3}{(t-3)(t-1)}$is continuous whenever $t-3 \neq 0$ and $t-1 \neq 0$
  - Meaning when $t \neq 1, 3$.
- Both functions are simultaneously continuous (draw a number line to help you find out when p, g are **both** continuous) on

$$(-2,1) \cup (1,3) \cup (3,\infty)$$

since $t_0 = 2$ falls inside $(1,3)$ then the solution to this IVP must have a domain as large as

$$I = (1,3),$$

by the theorem.

(4) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} t^2y' + \ln|t-4|\, y = \frac{t-1}{\sin t} \\ y(5) = 9. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

- **Solution:**
- First rewrite

$$y' + \frac{\ln|t-4|}{t^2}y = \frac{t-1}{t^2 \sin t}$$

- The function $p(t) = \frac{\ln|t-4|}{t^2}$ is continuous when $t \neq 0$ and $t - 4 \neq 0$.
  - Thus we must have $t \neq 0, 4$.
- The function $g(t) = \frac{t-1}{t^2 \sin t}$ is continuous when $t \neq 0$ and when $t \neq \pm n\pi$ for any interger n.
  - So continuous whenever $t \neq 0$ and $t \neq \ldots, -3\pi - 2\pi, -\pi, 0, \pi, 2\pi, 3\pi \ldots$
- Note that the problem point 4 is in between π and 2π, that is; $\pi < 4 < 2\pi$!
- Both functions are simultaneously continuous (draw number lines to help you find out when p, g are **both** continuous) on

$$\cdots \cup (-2\pi, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 4) \cup (4, 2\pi) \cup \cdots$$

since $t_0 = 5$ falls inside $(4, 2\pi)$ then the solution to this IVP must have a domain as large as

$$I = (4, 2\pi),$$

by the theorem.

(5) Consider the IVP below

$$\frac{dy}{dt} = y^{1/5}, \quad y(0) = 0.$$

(a) Is this a Linear or nonlinear equation? Can you use Theorem 1 from Section 2.6?

- **Solution:**
- This is a nonlinear equation, due to the $y^{1/5}$.
- Theorem 1 from Section 2.6 only applies to Linear equations, thus we can't use Theorem 1 for this IVP.

(a) Using Theorem 2 from Section 2.6 (the general theorem), can you guarantee that there is a unique solution to this IVP? Why?

- **Solution:**
- To apply Theorem 2, we need the right hand side equation

$$f(t,y) = y^{1/5}$$

to be continuous and we need

$$\frac{\partial f}{\partial y} = \frac{1}{5y^{4/5}}$$

to be continous around the point $(t_0, y_0) = (0,0)$. But since $\frac{1}{5y^{4/5}}$ is not continuous when $y_0 = 0$, then we cannot guarantee uniqueness of the solution.

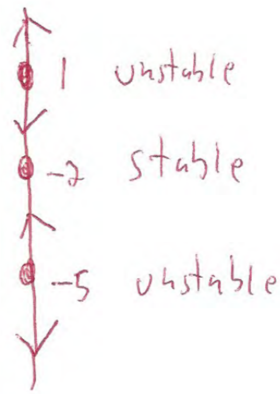
2.7. Problems

(1) Consider the following differential equation:

$$\frac{dy}{dt} = (y+2)(y-1)(y+5)$$

(a) Draw a Phase Line. Classify the Equilibrium solutions.

- **Solution:**



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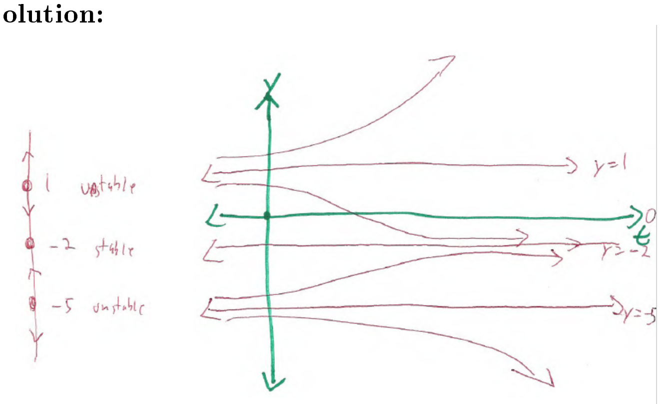
(b) Draw all possible sketch of solutions of this differential equation.

- **Solution:**



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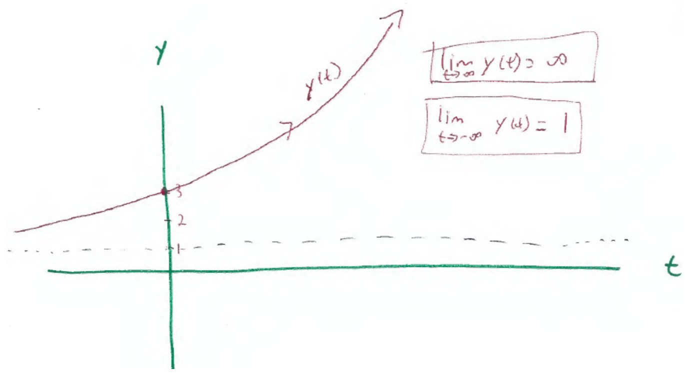
(c) Consider the IVP

$$\frac{dy}{dt} = (y+2)(y-1)(y+5), \quad y(0) = 3.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

- **Solution:**



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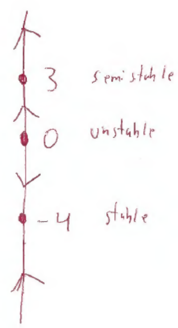
(2) Consider the following differential equation:

$$\frac{dy}{dt} = y\,(y-3)^2\,(y+4)$$

(a) Draw a Phase Line. Classify the Equilibrium solutions.

- **Solution:**



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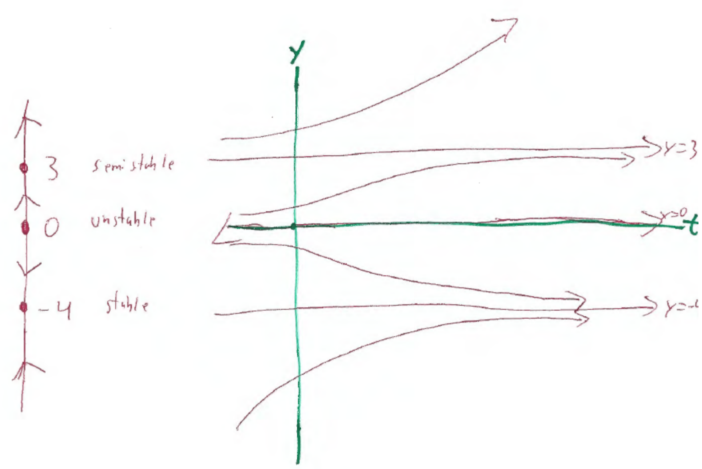
(b) Draw all possible sketch of solutions of this differential equation.

- **Solution:**



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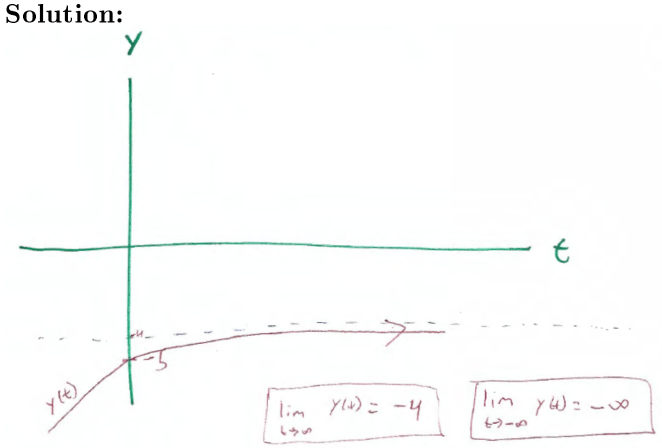
(c) Consider the IVP

$$\frac{dy}{dt} = y\left(y-3\right)^2\left(y+4\right), \quad y(0) = -5.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

- **Solution:**



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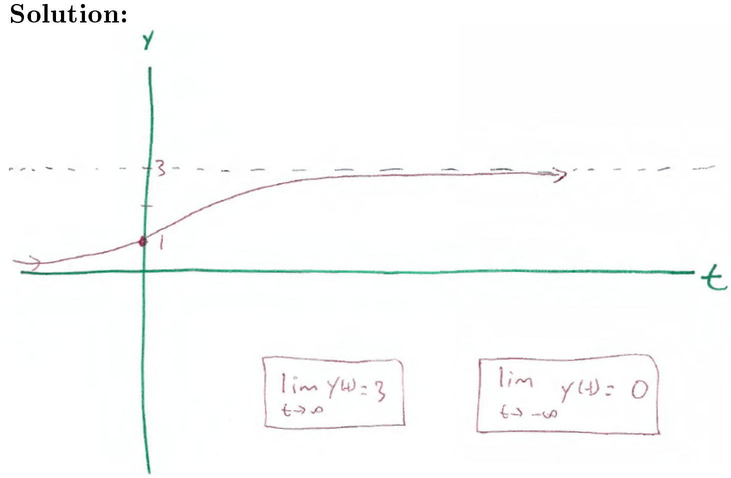
(d) Consider the IVP

$$\frac{dy}{dt} = y\left(y-3\right)^2\left(y+4\right), \quad y(0) = 1.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

- **Solution:**



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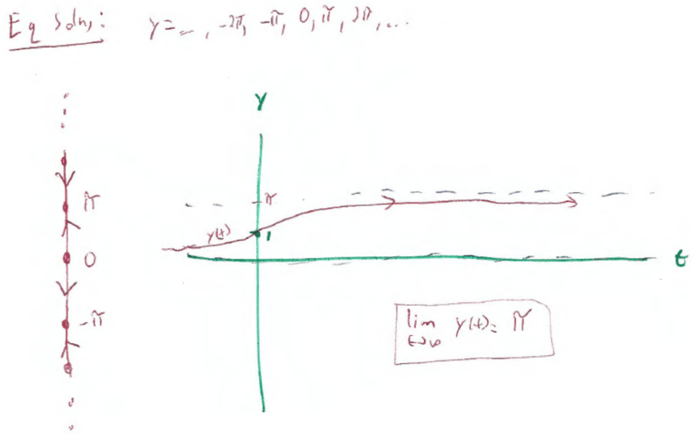
(3) Let $y(t)$ be the unique solution to the IVP given by

$$\frac{dy}{dt} = y^2 \sin y, \quad y(0) = 1.$$

Draw a Phase Line for the ODE to find out $\lim_{t\to\infty} y(t)$ for the unique solution of the IVP above.

- **Solution:**



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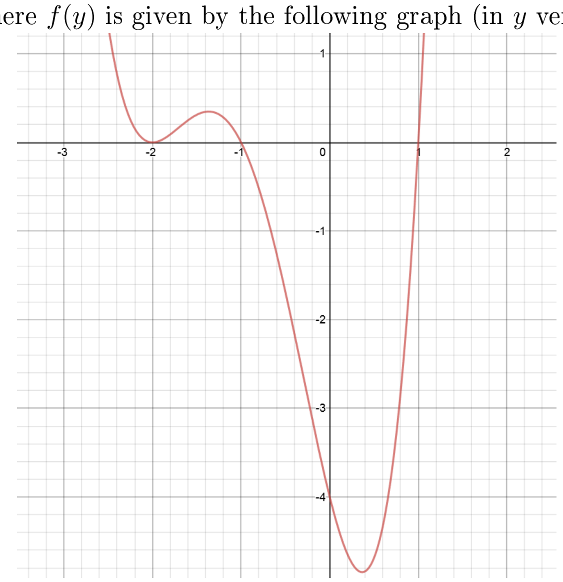
(4) Consider the differential equation

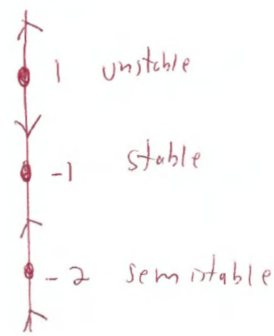
$$\frac{dy}{dt} = f(y)$$

where $f(y)$ is given by the following graph (in y versus $f(y)$):



(a) Draw the Phase Line and classify the Equilibrium solutions.

• **Solution:**


1 unstable
-1 stable
-2 semistable

•

2.8. Problems

(1) Determine whether each of the following equations are exact. If it is exact, find the general implicit solution in the form $\psi(x,y)=C$.

(a) $(2x+3)+(2y-2)\,y'=0$

- **Solution:**
- Note that $M(x,y)=2x+3$ and $N(x,y)=2y-2$ and we compute

$$M_y = 0$$
$$N_x = 0$$

and since $M_y = N_x$ then this equation is exact!

- To solve we compute find the implicit general solution $\psi(x,y)=C$.
- By the Theorem from Sec 2.8, we know that we have that

$$\begin{aligned}\psi_x = M \qquad \Longrightarrow \psi &= \int M(x,y)dx \\ &= \int (2x+3)\,dx \\ &= x^2+3x+h(y)\end{aligned}$$

while

$$\begin{aligned}\psi_y = N \qquad \Longrightarrow \psi &= \int N(x,y)dy \\ &= \int (2y-2)\,dy \\ &= y^2-2y+g(x)\end{aligned}$$

- Thus we collect everything that we have missing from the two versions of ψ (without overcounting) and get

$$\psi(x,y)=x^2+3x+y^2-2y$$

- Thus the **general implicit solution** is given by

$$x^2+3x+y^2-2y=C.$$

(b) $(2x+4y)+(2x-2y)\,y'=0$

- **Solution:**
- Note that $M(x,y)=2x+4y$ and $N(x,y)=2x-2y$ and we compute

$$M_y = 4$$
$$N_x = 2$$

and since $M_y \neq N_x$ then this equation is NOT exact! Hence we can't solve it using the methods from this section.

(c) $\left(3x^2-2xy+2\right)dx+\left(6y^2-x^2+3\right)dy=0$

- **Solution:**
- Note that $M(x,y)=3x^2-2xy+2$ and $N(x,y)=6y^2-x^2+3$ and we compute

$$M_y = -2x$$
$$N_x = -2x$$

and since $M_y = N_x$ then this equation is exact!

- To solve we compute find the implicit general solution $\psi(x,y) = C$.
- By the Theorem from Sec 2.8, we know that we have that

$$\begin{aligned}\psi_x = M \qquad \Longrightarrow \psi &= \int M(x,y)dx \\ &= \int \left(3x^2 - 2xy + 2\right) dx \\ &= x^3 - x^2y + 2x + h(y)\end{aligned}$$

while

$$\begin{aligned}\psi_y = N \qquad \Longrightarrow \psi &= \int N(x,y)dy \\ &= \int \left(6y^2 - x^2 + 3\right) dy \\ &= 2y^3 - x^2y + 3y + g(x)\end{aligned}$$

- Thus we collect everything that we have missing from the two versions of ψ (without overcounting) and get

$$\psi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y$$

- Thus the **general implicit solution** is given by

$$x^3 - x^2y + 2x + 2y^3 + 3y = C.$$

(d) $\left(2xy^2 + 2y\right) + \left(2x^2y + 2x\right) y' = 0$

- **Solution:**
- Note that $M(x,y) = 2xy^2 + 2y$ and $N(x,y) = 2x^2y + 2x$ and we compute

$$\begin{aligned}M_y &= 4xy + 2 \\ N_x &= 4xy + 2\end{aligned}$$

and since $M_y = N_x$ then this equation is exact!

- To solve we compute find the implicit general solution $\psi(x,y) = C$.
- By the Theorem from Sec 2.8, we know that we have that

$$\begin{aligned}\psi_x = M \qquad \Longrightarrow \psi &= \int M(x,y)dx \\ &= \int \left(2xy^2 + 2y\right) dx \\ &= x^2y^2 + 2xy + h(y)\end{aligned}$$

while

$$\begin{aligned}\psi_y = N \qquad \Longrightarrow \psi &= \int N(x,y)dy \\ &= \int \left(2x^2y + 2x\right) dy \\ &= x^2y^2 + 2xy + g(x)\end{aligned}$$

- Thus we collect everything that we have missing from the two versions of ψ (without overcounting) and get

$$\psi(x,y) = x^2y^2 + 2xy$$

- Thus the **general implicit solution** is given by

$$x^2y^2 + 2xy = C.$$

(e) $\dfrac{dy}{dx} = -\dfrac{ax+by}{bx+cy}$
- **Solution:**
- First we need to rewrite this in the form $Mdx + Ndy = 0$:

$$(ax+by)\,dx + (bx+cy)\,dy = 0$$

- Note that $M(x,y) = ax+by$ and $N(x,y) = bx+cy$ and we compute

$$M_y = b$$
$$N_x = b$$

and since $M_y = N_x$ then this equation is exact!
- To solve we compute find the implicit general solution $\psi(x,y) = C$.
- By the Theorem from Sec 2.8, we know that we have that

$$\psi_x = M \quad \implies \psi = \int M(x,y)dx$$
$$= \int (ax+by)\,dx$$
$$= a\frac{x^2}{2} + bxy + h(y)$$

while

$$\psi_y = N \quad \implies \psi = \int N(x,y)dy$$
$$= \int (bx+cy)\,dy$$
$$= bxy + c\frac{y^2}{2} + g(x)$$

- Thus we collect everything that we have missing from the two versions of ψ (without overcounting) and get

$$\psi(x,y) = a\frac{x^2}{2} + bxy + c\frac{y^2}{2}$$

- Thus the **general implicit solution** is given by

$$a\frac{x^2}{2} + bxy + c\frac{y^2}{2} = C.$$

(f) $(e^x \sin y + 3y)\,dx - (3x - e^x \sin y)\,dy = 0$
- **Solution:**

- Note that $M(x,y) = e^x \sin y + 3y$ and $N(x,y) = -3x + e^x \sin y$ and we compute

$$M_y = e^x \cos y + 3$$
$$N_x = -3 + e^x \sin y$$

and since $M_y \neq N_x$ then this equation is NOT exact!

(g) $\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2)\, dy = 0, x > 0$

- **Solution:**
- Note that $M(x,y) = \frac{y}{x} + 6x$ and $N(x,y) = \ln x - 2$ and we compute

$$M_y = \frac{1}{x}$$
$$N_x = \frac{1}{x}$$

and since $M_y = N_x$ then this equation is exact!
- To solve we compute find the implicit general solution $\psi(x,y) = C$.
- By the Theorem from Sec 2.8, we know that we have that

$$\begin{aligned} \psi_x = M \quad \implies \psi &= \int M(x,y)dx \\ &= \int \left(\frac{y}{x} + 6x\right) dx \\ &= y \ln x + 3x^2 + h(y) \end{aligned}$$

while

$$\begin{aligned} \psi_y = N \quad \implies \psi &= \int N(x,y)dy \\ &= \int (\ln x - 2)\, dy \\ &= y \ln x - 2y + g(x) \end{aligned}$$

- Thus we collect everything that we have missing from the two versions of ψ (without overcounting) and get

$$\psi(x,y) = y \ln x + 3x^2 - 2y$$

- Thus the **general implicit solution** is given by

$$y \ln x + 3x^2 - 2y = C.$$

(2) Find the implicit particular solution to the initial value problem

$$\left(9x^2 + y - 1\right) dx - (4y - x)\, dy = 0, \quad y(1) = 0.$$

- **Solution:**
- Note that $M(x,y) = 9x^2 + y - 1$ and $N(x,y) = -4y + x$ and we compute

$$M_y = 1$$
$$N_x = 1$$

and since $M_y = N_x$ then this equation is exact!
- To solve we compute find the implicit general solution $\psi(x,y) = C$.

- By the Theorem from Sec 2.8, we know that we have that

$$\psi_x = M \implies \psi = \int M(x,y)dx = \int \left(9x^2 + y - 1\right) dx = 3x^3 + xy - x + h(y)$$

while

$$\psi_y = N \implies \psi = \int N(x,y)dy = \int (-4y + x)\, dy = -2y^2 + xy + g(x)$$

- Thus we collect everything that we have missing from the two versions of ψ (without over-counting) and get

$$\psi(x,y) = 3x^3 + xy - x - 2y^2$$

- Thus the **general implicit solution** is given by

$$3x^3 + xy - x - 2y^2 = C.$$

- To find the value of C, we simply use the initial condition $y(1) = 0$ to get

$$3 + 0 - 1 - 0 = C \implies C = 2$$

to get

$$3x^3 + xy - x - 2y^2 = 2.$$

(3) Find the values of b for which the given equation is exact.

$$\left(ye^{2xy} + x\right) dx + bxe^{2xy} dy = 0.$$

- **Solution:**
- Note that $M(x,y) = ye^{2xy} + x$ and $N(x,y) = bxe^{2xy}$ and we compute

$$M_y = e^{2xy} + 2xye^{2xy}$$
$$N_x = be^{2xy} + 2ybxe^{2xy}$$

and for this ODE to be exact we need

$$M_y = N_x$$

hence

$$e^{2xy} + 2xye^{2xy} = be^{2xy} + 2ybxe^{2xy}$$

which are equal only when

$$b = 1.$$

2.9. Problems

(1) Find the approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$ and 0.4 using Euler's Method with $h = 0.1$.

$$\frac{dy}{dt} = t + y, \quad y(0) = 1.$$

- **Solution:**
- We make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\,h$	$f(t_k, y_k) = t_k + y_k$
0	0	1	$0 + 1 = \mathbf{1}$
1	0.1	$y_1 = 1 + 1 \cdot (.1) = \mathbf{1.1}$	$f(t_1, y_1) = 0.1 + 1.1 = 1.2$
2	0.2	$y_2 = 1.1 + (1.20) \cdot (.1) = \mathbf{1.22}$	$f(t_2, y_2) = 0.2 + 1.22 = 1.42$
3	0.3	$y_3 = 1.22 + (1.42) \cdot (.1) = \mathbf{1.362}$	$f(t_3, y_3) = 0.3 + 1.362 = 1.662$
4	0.4	$y_4 = 1.362 + (1.662) \cdot (.1) = \mathbf{1.5282}$	

- Hence

$$y(0.1) \approx \mathbf{1.1}$$
$$y(0.2) \approx 1.22$$
$$y(0.3) \approx 1.362$$
$$y(0.4) \approx 1.5282.$$

(2) Find the approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$ and 0.4 using Euler's Method with $h = 0.05$.

$$\frac{dy}{dt} = t + y^2, \quad y(0) = 1.$$

- **Solution:**
- We make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\,h$	$f(t_k, y_k) = t_k + y_k^2$
0	0	1	1
1	0.05	$y_1 = 1.05$	$f(t_1, y_1) = \mathbf{461/400}$
2	0.1	$y_2 = \mathbf{8861/8000} = \mathbf{1.107625}$	$f(t_1, y_1) = \mathbf{12963660/9770377}$
3	0.15	$y_3 = \mathbf{4292730/3656603}$	$f(t_1, y_1)\,\mathbf{8658620/5665903}$
4	0.2	11178230/8939891	1545168/876223
5	0.25	6566530/4905709	3681136/1802965
6	0.3	1099469/763184	84140/35421
7	0.35	7333465/4702731	3211634/1154539
8	0.4	1398466/823357 = 1.69849287733	

Hence

$$y(0.1) \approx 1.10762$$
$$y(0.2) \approx 1.25037$$
$$y(0.3) \approx 1.44063$$
$$y(0.4) \approx 1.6984$$

(3) Find the approximate value of $y(2)$ using Euler's Method with $h = 0.5$ for the solution of the following IVP

$$\frac{dy}{dt} = y(3 - ty), \quad y(0) = 0.5.$$

- **Solution:**
- We make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\,h$	$f(t_k, y_k) = y_k(3 - t_k y_k)$
0	0	0.5	1.5
1	0.5	$y_1 = \mathbf{5/4 = 1.25}$	$f(t_1, y_1) = \mathbf{95/32}$
2	1.0	$y_2 = 175/64 = 2.734375$	$f(t_2, y_2) = \mathbf{2975/4096}$
3	1.5	$y_3 = 25375/8192$	$f(t_3, y_3) = -21280010/4172981$
4	2.0	$5435048/9921647 \approx 0.54779$	

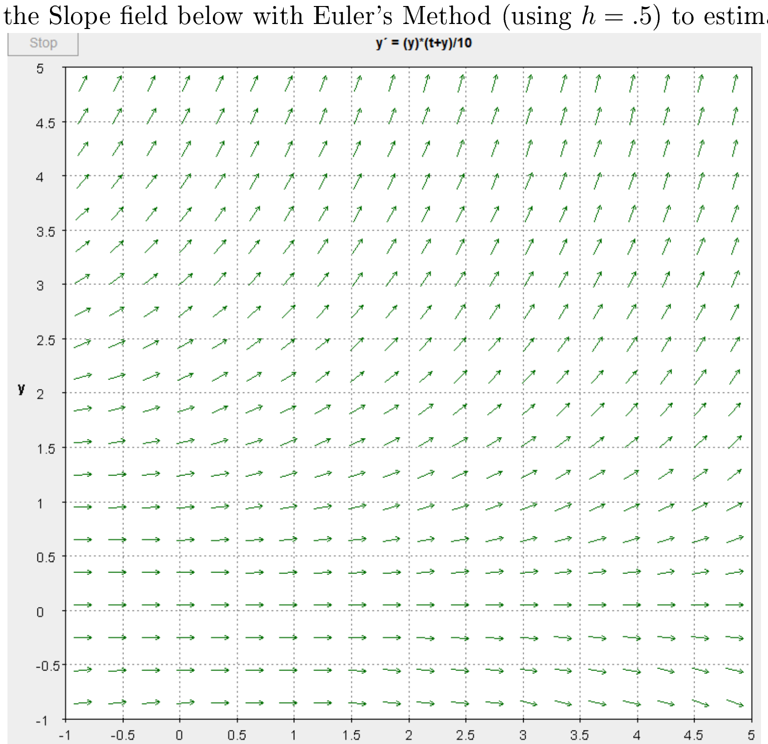
Hence

$$y(2) \approx 0.54779.$$

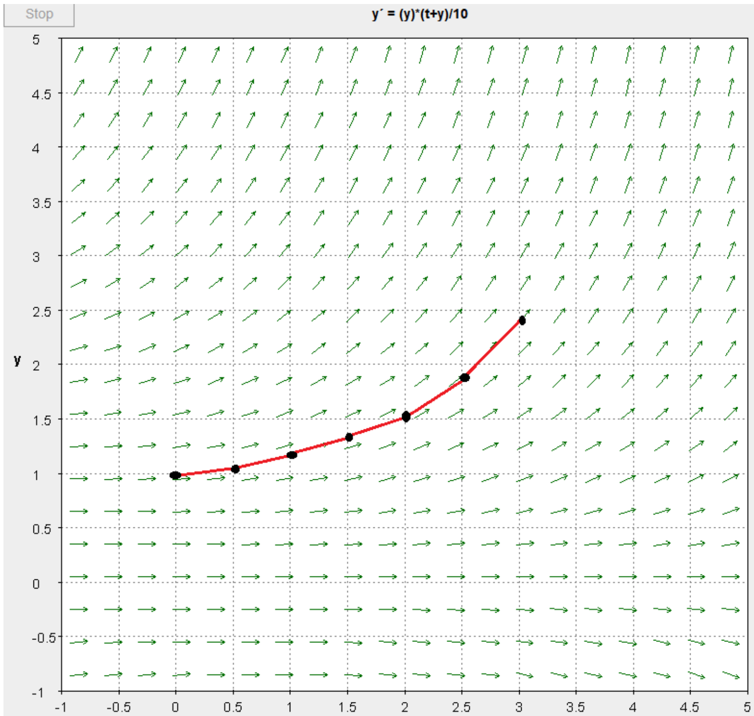
(4) Consider the solution $y(t)$ to the IVP:

$$\frac{dy}{dt} = y(t + y)/10, \quad y(0) = 1.$$

Use the Slope field below with Euler's Method (using $h = .5$) to estimate the value of $y(3)$:

(a) 

- **Solution:**

- 

- Hence from the curve sketch we have that

$$y(3) \approx 2.4$$

CHAPTER 3

Second Order Linear Equations

3.1. Problems

No Homework

3.2. Problems

(1) Check if the following functions are solutions to the given EQ?
  (a) Check directly if $y_1 = 2e^{5t}$ is a solution or not to $y'' - 6y' + 5y = 0$?
   - **Solution:**
   - To check $y_1 = 2e^{5t}$ is a solution to the ODE above we first plug y_1 into the LHS and set it Equal to the RHS.
   - But before we do, let's first start taking some derivatives

$$\begin{aligned} y_1 &= 2e^{5t} \\ y_1' &= 10e^{5t} \\ y_1'' &= 50e^{5t} \end{aligned}$$

   and now we can plug this into the LHS:

$$\begin{aligned} \text{LHS} &= y_1'' - 6y_1' + 5y_1 \\ &= \left(50e^{5t}\right) - 6\left(10e^{5t}\right) + 5\left(2e^{5t}\right) \\ &= 50e^{5t} - 60e^{5t} + 10e^{5t} \\ &= (50 - 60 + 10)\, e^{5t} \\ &= 0, \end{aligned}$$

   Now since the RHS of the equation is already

$$RHS = 0$$

   then since

$$LHS = RHS$$

   then y_1 must be a solution.
  (b) Check directly if $y_2 = 2e^t$ is a solution or not to $y'' - 6y' + 5y = t$?
   - **Solution:**
   - To check $y_2 = 2e^t$ is a solution to the ODE above we first plug y_2 into the LHS and set it Equal to the RHS.

- But before we do, let's first start taking some derivatives

$$y_2 = 2e^t$$
$$y_2' = 2e^t$$
$$y_2'' = 2e^t$$

and now we can plug this into the LHS:

$$\begin{aligned} \text{LHS} &= y_1'' - 6y_1' + 5y_1 \\ &= \left(2e^t\right) - 6\left(2e^t\right) + 5\left(2e^t\right) \\ &= 2e^{5t} - 12e^{5t} + 10e^{5t} \\ &= (2 - 12 + 10)\, e^{5t} \\ &= 0, \end{aligned}$$

Now since the RHS of the equation is

$$RHS = t$$

then since

$$LHS \neq RHS$$

then y_2 **IS NOT** a solution.

(2) Recall from the *Lecture Notes*, that if $y(t) = e^{rt}$ is a solution to the ODE given by

$$ay'' + by' + cy = 0$$

for constant a, b, c where $a \neq 0$, then the exponent r in front the t must be a solution to *the characteristic EQ* $ar^2 + br + c = 0$.

(a) By yourself, rederive that if $y(t) = Ae^{rt}$ is a **solution** to the equation above then the number r must satisfy the *characteristic EQ* $ar^2 + br + c = 0$ or $A = 0$. (**Hint:** How do we check something is a solution? Well you just plug it to the LHS and RHS and check if they are equal!)

- **Solution:**
- Again, how do we check something is a solution? We plug in $y(t) = Ae^{rt}$ into the LHS and RHS and set them equal to each other.
- Let's first start taking some derivative:

$$y(t) = Ae^{rt}$$
$$y'(t) = Are^{rt}$$
$$y''(t) = Ar^2e^{rt},$$

- Now plug in y into the LHS

$$\begin{aligned} \text{LHS} &= ay'' + by' + cy \\ &= a\left(Ar^2e^{rt}\right) + b\left(Are^{rt}\right) + c\left(Ae^{rt}\right) \\ &= aAr^2e^{rt} + bAre^{rt} + cAe^{rt} \\ &= \left(ar^2 + br + c\right) Ae^{rt} \end{aligned}$$

- Now the RHS is

$$RHS = 0.$$

- Thus if y is really a solution then $LHS = RHS$:

$$\begin{aligned} LHS = RHS &\iff \left(ar^2 + br + c\right) Ae^{rt} = 0 \\ &\iff \left(ar^2 + br + c\right) = 0 \end{aligned}$$

  where I divided both sides by $Ae^{rt} \neq 0$.
- Thus is order for $y(t) = Ae^{rt}$ to be a solution to the ODE, then r must satisfy the equation

$$ar^2 + br + c = 0,$$

  which from class is called the characteristic equation.

(3) Use the method given in Section 3.2 to find the general solution to

$$y'' + 5y' - 6y = 0$$

- **Solution:**
- From Section 3.2, in order to the solutions to Linear constant coefficient ODEs, we
  - 1) Solve the characteristic EQ: $ar^2 + br + c = 0$ and say r_1, r_2 are distinct and real!
  - 2) Then it was given to us that the **general solution** is given by $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
- First we solve the characteristic EQ:

$$r^2 + 5r - 6 = 0 \iff (r + 6)(r - 1) = 0$$

  so that $r_1 = 1$ and $r_2 = -6$.
- Then the **general solution** is given by

$$y(t) = c_1 e^t + c_2 e^{-6t}.$$

(4) Use the method given in Section 3.2 to find the general solution to

$$y'' - 7y' = 0$$

- **Solution:**
- From Section 3.1, in order to the solutions to Linear constant coefficient ODEs, we
  - 1) Solve the characteristic EQ: $ar^2 + br + c = 0$ and say r_1, r_2 are distinct and real!
  - 2) Then it was given to us that the **general solution** is given by $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
- First we solve the characteristic EQ:

$$r^2 - 7r = 0 \iff r(r - 7) = 0$$

  so that $r_1 = 0$ and $r_2 = 7$.
- Then the **general solution** is given by

$$\begin{aligned} y(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ &= c_1 e^{0t} + c_2 e^{7t} \\ &= c_1 + c_2 e^{7t}. \end{aligned}$$

(5) Use the method given in Section 3.2 to find the particular solution to the IVP

$$y'' + y' - 20y = 0, \quad y(0) = 18, y'(0) = 9$$

- **Solution:**
- First we find the general solution:
  - From Section 3.2, in order to the solutions to Linear constant coefficient ODEs, we
    - * 1) Solve the characteristic EQ: $ar^2 + br + c = 0$ and say r_1, r_2 are distinct and real!

    * 2) Then it was given to us that the **general solution** is given by $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

- First we solve the characteristic EQ:
$$r^2 + r - 20 = 0 \iff (r+5)(r-4) = 0$$
so that $r_1 = 4$ and $r_2 = -5$.
- Then the **general solution** is given by
$$y(t) = c_1 e^{4t} + c_2 e^{-5t}.$$
- To find the values of c_1, c_2, we use the initial conditions $y(0) = 18, y'(0) = 9$:
  - But first take a derivative:
$$y(t) = c_1 e^{4t} + c_2 e^{-5t}$$
$$y'(t) = 4c_1 e^{4t} - 5c_2 e^{-5t}$$
  - Solve for c_1, c_2:
$$18 = y(0) = c_1 + c_2$$
$$9 = 4c_1 - 5c_2$$
and solving this system we get
$$c_1 = 11, c_2 = 7$$
hence the particular solution is
$$y(t) = 11e^{4t} + 7e^{-5t}.$$

3.3. Problems

(1) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-3)\,y'' + \sin t y' + y = \frac{\ln(t-1)}{t-10} \\ y(15) = -7, y'(15) = 10 \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

- **Solution:**
- To apply the Existence and Uniqueness Theorem we first need to rewrite this Linear equation in the form

$$y'' + p(t)y' + q(t)y = g(t)$$

and we get

$$y'' + \frac{\sin t}{(t-3)}y' + \frac{1}{(t-3)}y = \frac{\ln(t-1)}{(t-3)(t-10)}$$

- Where $p(t) = \frac{\sin t}{(t-3)}$, $q(t) = \frac{1}{(t-3)}$and $g(t) = \frac{\ln(t-1)}{(t-3)(t-10)}$.
- To find the largest open interval, we simply need to check the largest open interval containing the initial value t_0 in which both $p(t), q(t)$ and $g(t)$ are simultaneously continuous
- To do this we look for the bad points (non-continuous points) of p, q and g
- The function $p(t) = \frac{\sin t}{(t-3)}$is continuous whenever $t \neq 3$.
  - Thus we must have $t \neq 3$.
- The function $q(t) = \frac{1}{(t-3)}$is continuous whenever $t \neq 3$.
  - Thus we must have $t \neq 3$.
- The function $g(t) = \frac{\ln(t-1)}{(t-3)(t-10)}$is continuous whenever $t - 1 > 0$ and when $t - 10 \neq 0$ and $t - 3 \neq 0$
  - Meaning when $t > 1$ and $t \neq 3, 10$.
- All functions are simultaneously continuous (draw a number line to help you find out when p, q, g are **all** continuous) on

$$(1,3) \cup (3,10) \cup (10,\infty)$$

since $t_0 = 15$ falls inside $(10,\infty)$ then the solution to this IVP must have a domain as large as

$$I = (10,\infty),$$

by the theorem.

(2) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} t^2y'' + e^ty' + (t-1)\,y = \sqrt{t+2} \\ y(-1) = 1, y'(-1) = 5 \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

- **Solution:**
- To apply the Existence and Uniqueness Theorem we first need to rewrite this Linear equation in the form

$$y'' + p(t)y' + q(t)y = g(t)$$

and we get

$$y'' + \frac{e^t}{t^2}y' + \frac{(t-1)}{t^2}y = \frac{\sqrt{t+2}}{t^2}$$

- Where $p(t) = \frac{e^t}{t^2}$, $q(t) = \frac{(t-1)}{t^2}$and $g(t) = \frac{\sqrt{t+2}}{t^2}$.
- To find the largest open interval, we simply need to check the largest open interval containing the initial value t_0 in which both $p(t), q(t)$ and $g(t)$ are simultaneously continuous
- To do this we look for the bad points (non-continuous points) of p, q and g
- The function $p(t) = \frac{e^t}{t^2}$is continuous whenever $t^2 \neq 0$.
  - Thus we must have $t \neq 0$.
- The function $q(t) = \frac{(t-1)}{t^2}$is continuous whenever $t^2 \neq 0$.
  - Thus we must have $t \neq 0$.
- The function $g(t) = \frac{\sqrt{t+2}}{t^2}$is continuous whenever $t + 2 > 0$ and when $t^2 \neq 0$.
  - Meaning when $t > -2$ and $t \neq 0$.
- All functions are simultaneously continuous (draw a number line to help you find out when p, q, g are **all** continuous) on

$$(-2, 0) \cup (0, \infty)$$

since $t_0 = -1$ falls inside $(-2, 0)$ then the solution to this IVP must have a domain as large as

$$I = (-2, 0),$$

by the theorem.

(3) Consider the equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p, q are continuous in some interval I. What are the 2 things you have to do by the General Solution Theorem in order to find the general solution to the ODE above

- **Solution:**
- The general solution theorem says that if $y'' + p(t)y' + q(t)y = 0$ is homogeneous 2nd order ODE, Then the **roadmap** to finding the general solution is:
- **1)** Find y_1 and y_2 that are solution to the ODE above,
- **2)** Check that the wronskian $W(y_1 y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ is NOT ZERO for at least one point in the interval I
- Then the **general solution** is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

(4) Consider the equation

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

(a) Is the function $y_1(t) = t^{\frac{1}{2}}$ a solution to this ODE?

- **Solution:**
- We take derivatives

$$y_1(t) = t^{\frac{1}{2}}$$
$$y_2'(t) = \frac{1}{2} t^{-\frac{1}{2}}$$
$$y_3'(t) = -\frac{1}{4} t^{-\frac{3}{2}}$$

then plug into the LHS

$$\begin{aligned} LHS &= 2t^2 y_1'' + 3t y_1' - y_1 \\ &= 2t^2\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - \left(t^{\frac{1}{2}}\right) \\ &= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

since $LHS = 0$ then yes!

(b) Is the function $y_2(t) = t^{-1}$ a solution to this ODE?

- **Solution:**
- We take derivatives

$$\begin{aligned} y_2(t) &= t^{-1} \\ y_2'(t) &= -t^{-2} \\ y_2''(t) &= -t^{-2}. \end{aligned}$$

then plug into the LHS

$$\begin{aligned} LHS &= 2t^2 y_2'' + 3t y_2' - y_2 \\ &= 2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - \left(t^{-1}\right) \\ &= 4t^{-1} - 3t^{-1} - t^{-1} \\ &= 0. \end{aligned}$$

since $LHS = 0$ then yes!

(c) Use the General Solution Theorem to show that

$$y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{-1}$$

gives the general solution to the ODE above.

- **Solution:**
- The General Solution Theorem says
- **1)** Find y_1 and y_2 that are solution to the ODE above.
  - Which we already found in parts (a) and (b) that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ are solutions.
- **2)** Check that the wronskian $W(y_1 y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$ is NOT ZERO for at least one point in the interval I
  - We compute this:

$$W(y_1, y_2) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2} \neq 0$$

- Then the **general solution** is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

and plugging y_1, y_2in we have

$$y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{-1}$$

3.4. Problems

(1) Find the general solution of the following 2nd Order Linear ODEs with constant coefficients.

(a) $y'' + 16y = 0$

- **Solution:**
- The characteristic equation is given by

$$r^2 + 16 = 0$$

and the roots are $r_{1,2} = \pm 4i$.
- For complex solutions, $r = \alpha + i\beta$ then general solution is given by

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

hence the general solution is given by

$$\begin{aligned} y(t) &= c_1 e^{0t} \cos(4t) + c_2 e^{0t} \sin(4t) \\ &= c_1 \cos(4t) + c_2 \sin(4t) \end{aligned}$$

(b) $y'' - 4y' + 9y = 0$

- **Solution:**
- The characteristic equation is given by

$$r^2 - 4r + 9 = 0$$

- We can always use the **quadratic fomula:**

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 9}}{2} \\ &= \frac{4 \pm \sqrt{16 - 36}}{2} \\ &= 2 \pm \frac{1}{2}\sqrt{-20} \\ &= 2 \pm \frac{1}{2}\sqrt{-4 \cdot 5} \\ &= 2 \pm \frac{2\sqrt{5}}{2} \cdot i \\ &= 2 \pm \sqrt{5}i \end{aligned}$$

- For complex solutions, $r = \alpha + i\beta$ then general solution is given by

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

hence the **general solution** is given by

$$y(t) = c_1 e^{2t} \cos\left(\sqrt{5}t\right) + c_2 e^{2t} \sin\left(\sqrt{5}t\right).$$

(c) $y'' - 4y' + 29y = 0$

- **Solution:**

- The characteristic equation is given by
$$r^2 - 4r + 29 = 0$$
and once can always use the **quadratic formula** and arrive at $r = 2 \pm 5i$!
  - **Another way** is to complete the square for a $r^2 + br + c = 0$ and the trick is to use $(\frac{b}{2})^2$ to **complete the square.**
  - Hence $b = -4$ so we'll use $(\frac{b}{2})^2 = \left(\frac{-4}{2}\right)^2 = 4$ to split 25 into pieces
  - That is,
$$\begin{aligned} r^2 - 4r + 25 = 0 &\iff r^2 - 4r + \mathbf{4} + 25 = 0 \\ &\iff (r-2)^2 + 25 = 0 \\ &\iff (r-2)^2 = -25 \\ &\iff r - 2 = \pm\sqrt{-25} \\ &\iff r - 2 = \pm 5i \\ &\iff r = 2 \pm 5i. \end{aligned}$$
- For complex solutions, $r = \alpha + i\beta$ then general solution is given by
$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$$
hence the **general solution** is given by
$$y(t) = c_1 e^{2t}\cos(5t) + c_2 e^{2t}\sin(5t).$$

(2) Find the particular solution to the following IVP:
$$y'' - 8y' + 17y = 0, \quad y(0) = -4, y'(0) = -1.$$

- **Solution:**
- The characteristic equation is given by
$$r^2 - 8r + 18 = 0$$
and the roots are $r_{1,2} = 4 \pm i$.
- The general solution and its derivative is given by
$$\begin{aligned} y(t) &= c_1 e^{4t}\cos(t) + c_2 e^{4t}\sin(t), \\ y'(t) &= 4c_1 e^{4t}\cos(t) - c_1 e^{4t}\sin(t) + 4c_2 e^{4t}\sin(t) + c_2 e^{4t}\cos(t) \end{aligned}$$
amd using initial conditions we have
$$\begin{aligned} -4 &= y(0) = c_1 \\ -1 &= y'(0) = 4c_1 + c_2 \end{aligned}$$
and hence $c_1 = -4$ and $c_2 = 15$, hence the particular solution to the IVP is given by
$$y(t) = -4e^{4t}\cos(t) + 15e^{4t}\sin t.$$

3.5. Problems

3.5.1. Part 1; Repeated roots.

(1) Find the general solution of the following 2nd Order Linear ODEs with constant coefficients.
  (a) $y'' + 14y' + 49y = 0$
    - **Solution:**
    - The characteristic equation is given by
$$r^2 + 14r + 49 = 0 \iff (r+7)^2 = 0$$
      and the roots are $r_{1,2} = -7, -7$ real and repeated
    - For repeated real roots, then general solution is given by
$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$
      hence the **general solution** is given by
$$y(t) = c_1 e^{-7t} + c_2 t e^{-7t}.$$
  (b) $y'' - 18y' + 81y = 0$
    - **Solution:**
    - The characteristic equation is given by
$$r^2 - 18r + 81 = 0 \iff (r-9)^2 = 0$$
      and the roots are $r_{1,2} = 9, 9$ real and repeated
    - For repeated real roots, then general solution is given by
$$y(t) = c_1 e^{9t} + c_2 t e^{9t}$$
      hence the **general solution** is given by
$$y(t) = c_1 e^{9t} + c_2 t e^{9t}.$$

(2) Find the particular solution to the following IVP:
$$y'' - 4y' + 4y = 0, \quad y(0) = 12, y'(0) = -3.$$
  - **Solution:**
  - The characteristic equation is given by
$$r^2 - 4r + 4 = 0$$
    and the roots are $r_{1,2} = 2, 2$, real and repeated
  - The general solution and its derivative is given by
$$y(t) = c_1 e^{2t} + c_2 t e^{2t}$$
$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}$$
    amd using initial conditions we have
$$12 = y(0) = c_1$$
$$-3 = y'(0) = 2c_1 + c_2$$
    and hence $c_1 = 12$ and $c_2 = -27$, hence the particular solution to the IVP is given by
$$y(t) = 12e^{2t} - 27te^{2t}.$$

3.5.2. Part 2; the method of reduction of order.

(1) Suppose you know that $y_1(t) = t$ is a solution to

$$t^2 y'' - 3ty' + 3y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1 y_1 + c_2 y_2$ the general solution of this ODE.

- **Solution:**
- **Step 1:** Use method of reduction of order to make guess $y_2(t) = v(t)y_1(t)$ and take its derivatives.

$$\begin{aligned} y_2 &= vt \\ y_2' &= v't + v \\ y_2'' &= v''t + v' + v' = v''t + 2v' \end{aligned}$$

- **Step 2:** Plug y_2 into LHS and simplify as much as possible.

$$\begin{aligned} \text{LHS} &= t^2 y_2'' - 3ty_2' + 3y_2 \\ &= t^2 \left(v''t + 2v'\right) - 3t\left(v't + v\right) + 3\left(vt\right) \\ &= t^3 v'' + 2t^2 v' - 3t^2 v' - 3tv + 3tv \\ &= t^3 v'' - t^2 v'. \end{aligned}$$

- **Step 3:** Set LHS equal to zero and obtain an equation of the form $a(t)v'' + b(t)v' = 0$. Namely

$$t^3 v'' - t^2 v' = 0.$$

Solve for v by making the substitution $w = v'$ (and use $w' = v''$) to get

$$t^3 w' - t^2 w = 0.$$

and you can solve this as a 1st Order Linear, or 1st Order separable. I'll use that fact that its separable:

$$\begin{aligned} t^3 w' - t^2 w = 0 &\iff t^3 \frac{dw}{dt} = t^2 w \\ &\iff \int \frac{dw}{w} = \int \frac{dt}{t} \\ &\iff \ln w = \ln t + k_1 \\ &\iff w = k_2 t \end{aligned}$$

and plugging $w = v'$ back in we get

$$\begin{aligned} w = k_2 t &\iff v' = k_2 t \\ &\iff v = \frac{k_2}{2} t^2 + k_3. \end{aligned}$$

Choosing $k_2 = 2$ and $k_3 = 0$ we get the simplest nontrivial v to be

$$v = t^2.$$

- **Step 4:** Plug v back into y_2:

$$y_2 = vt = t^2 t = t^3.$$

Hence the general solution to this equation is given by

$$y(t) = c_1 t + c_2 t^3.$$

(2) Suppose you know that $y_1(t) = t^{-1}$ is a solution to

$$2t^2y'' + ty' - 3y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1y_1 + c_2y_2$ the general solution of this ODE.

- **Solution:**
- **Step 1:** Use method of reduction of order to make guess $y_2(t) = v(t)y_1(t)$ and take its derivatives.

$$\begin{aligned} y_2 &= vt^{-1} \\ y_2' &= v't^{-1} - vt^{-2} \\ y_2'' &= v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3} \\ &= v''t^{-1} - 2v't^{-2} + 2vt^{-3} \end{aligned}$$

- **Step 2:** Plug y_2 into LHS and simplify as much as possible.

$$\begin{aligned} \text{LHS} &= 2t^2y_2'' + ty_2' - 3y_2 \\ &= 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + t\left(v't^{-1} - vt^{-2}\right) - 3\left(vt^{-1}\right) \\ &= v''2t - 4v' + 4t^{-1}v + v' - vt^{-1} - 3vt^{-1} \\ &= 2tv'' - 3v'. \end{aligned}$$

- **Step 3:** Set LHS equal to zero and obtain an equation of the form $a(t)v'' + b(t)v' = 0$. Namely

$$2tv'' - 3v' = 0.$$

Solve for v by making the substitution $w = v'$ (and use $w' = v''$) to get

$$2tw' - 3w = 0.$$

and you can solve this as a 1st Order Linear, or 1st Order separable. I'll use that fact that its separable:

$$\begin{aligned} 2tw' - 3w = 0 &\iff 2t\frac{dw}{dt} = 3w \\ &\iff \int \frac{dw}{w} = \frac{3}{2}\int \frac{dt}{t} \\ &\iff \ln w = \frac{3}{2}\ln t + k_1 \\ &\iff w = e^{k_1}e^{\frac{3}{2}\ln t} \\ &\iff w = k_2t^{3/2} \end{aligned}$$

and plugging $w = v'$ back in we get

$$\begin{aligned} w = k_2t^{3/2} &\iff v' = k_2t^{3/2} \\ &\iff v = k_2\frac{2}{5}t^{5/2} + k_3. \end{aligned}$$

Choosing $k_2 = 5/2$ and $k_3 = 0$ we get the simplest nontrivial v to be

$$v = t^{5/2}.$$

- **Step 4:** Plug v back into y_2:

$$y_2 = t^{5/2}t^{-1} = t^{5/2}t^{-1} = t^{3/2}.$$

Hence the general solution to this equation is given by

$$y(t) = c_1 t^{-1} + c_2 t^{3/2}.$$

(3) Suppose you know that $y_1(t) = t$ is a solution to

$$t^2 y'' + 2ty' - 2y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1 y_1 + c_2 y_2$ the general solution of this ODE.

- **Solution:**
- **Step 1:** Use method of reduction of order to make guess $y_2(t) = v(t)y_1(t)$ and take its derivatives.

$$\begin{aligned} y_2 &= vt \\ y_2' &= v't + v \\ y_2'' &= v''t + v' + v' = v''t + 2v' \end{aligned}$$

- **Step 2:** Plug y_2 into LHS and simplify as much as possible.

$$\begin{aligned} \text{LHS} &= t^2 y_2'' + 2ty_2' - 2y_2 \\ &= t^2 \left(v''t + 2v'\right) + 2t\left(v't + v\right) - 2\left(vt\right) \\ &= v''t^3 + 2t^2 v' + v'2t^2 + v2t - 2vt \\ &= v''t^3 + 4t^2 v'. \end{aligned}$$

- **Step 3:** Set LHS equal to zero and obtain an equation of the form $a(t)v'' + b(t)v' = 0$. Namely

$$v''t^3 + 4t^2 v' = 0.$$

Solve for v by making the substitution $w = v'$ (and use $w' = v''$) to get

$$t^3 w' + 4t^2 w = 0.$$

and you can solve this as a 1st Order Linear, or 1st Order separable. I'll use that fact that its separable:

$$\begin{aligned} t^3 w' + 4t^2 w = 0 &\iff t^3 \frac{dw}{dt} = -4t^2 w \\ &\iff \frac{dw}{w} = -\frac{4}{t}dt \\ &\iff \ln w = -4\ln t + k_1 \\ &\iff w = e^{-4\ln t + k_1} \\ &\iff w = e^{k_1}e^{-4\ln t} \\ &\iff w = k_2 t^{-4} \end{aligned}$$

and plugging $w = v'$ back in we get

$$\begin{aligned} w = k_2 t^{-4} &\iff v' = k_2 t^{-4} \\ &\iff v = \frac{k_2}{-3}t^{-3} + k_3 \end{aligned}$$

Choosing $k_2 = -3$ and $k_3 = 0$ we get the simplest nontrivial v to be

$$v = t^{-3}.$$

- **Step 4:** Plug v back into y_2:

$$y_2 = vt = t^{-3}t = t^{-2}.$$

Hence the general solution to this equation is given by

$$y(t) = c_1 t + c_2 t^{-2}.$$

(4) Suppose you know that $y_1(t) = t^2$ is a solution to

$$t^2 y'' - 3ty' + 4y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1 y_1 + c_2 y_2$ the general solution of this ODE.

- **Solution:**
- **Step 1:** Use method of reduction of order to make guess $y_2(t) = v(t)y_1(t)$ and take its derivatives.

$$\begin{aligned} y_2 &= vt^2 \\ y_2' &= v't^2 + 2vt \\ y_2'' &= v''t^2 + 2v't + 2v't + 2v \\ &= v''t^2 + 4v't + 2v \end{aligned}$$

- **Step 2:** Plug y_2 into LHS and simplify as much as possible.

$$\begin{aligned} \text{LHS} &= t^2 y_2'' - 3ty_2' + 4y_2 \\ &= t^2\left(v''t^2 + 4v't + 2v\right) - 3t\left(v't^2 + 2vt\right) + 4\left(vt^2\right) \\ &= v''t^4 + 4v't^3 + 2vt^2 - 3v't^3 - 6vt^2 + 4vt^2 \\ &= v''t^4 + v't^3 \end{aligned}$$

- **Step 3:** Set LHS equal to zero and obtain an equation of the form $a(t)v'' + b(t)v' = 0$. Namely

$$v''t^4 + v't^3 = 0.$$

Solve for v by making the substitution $w = v'$ (and use $w' = v''$) to get

$$t^4 w' + t^3 w = 0.$$

and you can solve this as a 1st Order Linear, or 1st Order separable. I'll use that fact that its separable:

$$\begin{aligned} t^4 w' + t^3 w = 0 &\iff t^4 w' + t^3 w \\ &\iff \frac{dw}{w} = -\frac{1}{t}dt \\ &\iff \ln w = -\ln t + k_1 \\ &\iff w = e^{-\ln t + k_1} \\ &\iff w = e^{k_1} e^{-\ln t} \\ &\iff w = k_2 t^{-1} \end{aligned}$$

and plugging $w = v'$ back in we get

$$\begin{aligned} w = k_2 t^{-1} &\iff v' = k_2 t^{-1} \\ &\iff v = k_2 \ln t + k_3 \end{aligned}$$

Choosing $k_2 = 1$ and $k_3 = 0$ we get the simplest nontrivial v to be

$$v = \ln t.$$

- **Step 4:** Plug v back into y_2:

$$y_2 = vt^2 = t^2 \ln t.$$

Hence the general solution to this equation is given by

$$y(t) = c_1 t^2 + c_2 t^2 \ln t.$$

3.6. Problems

(1) Consider the following non-homogeneous 2nd order ODE:

$$y'' + y' - 2y = e^{3t}.$$

(a) Find the General Solution

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' + y' - 2y = 0$$

but we learned that we must solve the characteristic polynomial $r^2 + r - 2 = (r+2)(r-1) = 0$ and get $r = -2, 1$ so that the solution is

$$y_h(t) = c_1 e^{-2t} + c_2 e^t.$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    - Since the $RHS = e^{3t}$, we let $y_p(t) = Ae^{3t}$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since Ae^{3t} is not already part of $y_h = e^{-2t} + c_2 e^t$, then we made the correct guess.
  - We want to find what the value of A is. We need to plug this into the LHS. So we start by taking derivatives:

$$y_p = Ae^{3t},$$
$$y_p' = 3Ae^{3t}$$
$$y_p'' = 9Ae^{3t}$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned} \text{LHS} = y_p'' + y_p' - 2y_p &= \left(9Ae^{3t}\right) + \left(3Ae^{3t}\right) - 2\left(Ae^{3t}\right) \\ &= 10Ae^{3t} \end{aligned}$$

- Setting $LHS = RHS$we have

$$\text{LHS} = \text{RHS}$$
$$10Ae^{3t} = e^{3t}$$

so that $A = \frac{1}{10}$.

- **Step4:** Plug A back in and get $y_p(t) = \frac{1}{10}e^{3t}$ and a **general solution** of

$$y(t) = y_h + y_p.$$
$$y(t) = c_1 e^{-2t} + c_2 e^t + \frac{1}{10}e^{3t}.$$

(b) Find the particular solution to the IVP:

$$y'' + y' - 2y = e^{3t}, \quad y(0) = \frac{1}{10}, y'(0) = \frac{13}{10}.$$

- **Solution:**

- By Part (a), we know the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 e^t + \frac{1}{10} e^{3t},$$
$$y'(t) = -2c_1 e^{-2t} + c_2 e^t + \frac{3}{10} e^{3t}.$$

Using the initial conditions we have

$$\frac{1}{10} = y(0) = c_1 + c_2 + \frac{1}{10}$$
$$\frac{13}{10} = y'(0) = -2c_1 + c_2 + \frac{3}{10}$$

the equations reduce to

$$0 = c_1 + c_2$$
$$1 = -2c_1 + c_2$$

and you get $c_1 = -\frac{1}{3}$ and $c_2 = \frac{1}{3}$ so that the particular solution to the IVP is

$$y(t) = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^t + \frac{1}{10} e^{3t}.$$

(2) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 2y' + 2y = e^{2t}.$$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' - 2y' + 2y = 0$$

but we learned that we must solve the characteristic polynomial $r^2 - 2r + 2 = 0$ and get $r = 1 \pm i$ so that the solution is

$$y_h(t) = c_1 e^t \cos t + c_2 e^t \sin t.$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    * Since the $RHS = e^{2t}$, we let $y_p(t) = Ae^{2t}$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since Ae^{2t} is not already part of $y_h(t) = c_1 e^t \cos t + c_2 e^t \sin t$, then we made the correct guess.
  - We want to find what the value of A is. We need to plug this into the LHS. So we start by taking derivatives:

$$y_p = Ae^{2t},$$
$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned} \text{LHS} = y_p'' - 2y_p' + 2y_p &= \left(4Ae^{2t}\right) - 2\left(2Ae^{2t}\right) + 2\left(Ae^{2t}\right) \\ &= 4Ae^{2t} - 4Ae^{2t} + 2Ae^{2t} \\ &= 2Ae^{2t} \end{aligned}$$

- Setting $LHS = RHS$we have

$$\begin{aligned} \text{LHS} &= \text{RHS} \\ 2Ae^{2t} &= e^{2t} \end{aligned}$$

so that $A = \frac{1}{2}$.
- **Step4:** Plug A back in and get $y_p(t) = \frac{1}{2}e^{2t}$ and a **general solution** of

$$\begin{aligned} y(t) &= y_h + y_p. \\ y(t) &= c_1 e^t \cos t + c_2 e^t \sin t + \frac{1}{2} e^{2t}. \end{aligned}$$

(3) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 4y' + 3y = 4e^{3t}.$$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' - 4y' + 3y = 0$$

but we learned that we must solve the characteristic polynomial $r^2 - 4r + 3 = (r-1)(r-3) = 0$ and get $r = 1, 3$ so that the solution is

$$y_h(t) = c_1 e^t + c_2 e^{3t}.$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    - Since the $RHS = 4e^{3t}$, we let $y_p(t) = Ae^{3t}$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since Ae^{3t} **IS already part** of $y_h(t) = c_1 e^t + c_2 e^{3t}$, then we need to guess.
    - 2nd Guess (when second guessing, multiply by t): $y_p(t) = Ate^{3t}$.
  - 3rd Guess? To make sure we don't need to third guess. We check that there are no repeats with y_h. Since Ate^{3t} **IS NOT already part** of $y_h(t) = c_1 e^t + c_2 e^{3t}$, then we made the right guess here.
  - We want to find what the value of A is. We need to plug this into the LHS. So we start by taking derivatives:

$$\begin{aligned} y_p &= Ate^{3t}, \\ y_p' &= Ae^{3t} + 3Ate^{3t} \\ y_p'' &= 3Ae^{3t} + 3Ae^{3t} + 9Ate^{3t} \\ &= 6Ae^{3t} + 9Ate^{3t} \end{aligned}$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned} \text{LHS} = y_p'' - 4y_p' + 3y_p &= \left(6Ae^{3t} + 9Ate^{3t}\right) - 4\left(Ae^{3t} + 3Ate^{3t}\right) \\ &\quad +3\left(Ate^{3t}\right) \\ &= 6Ae^{3t} + 9Ate^{3t} - 4Ae^{3t} - 12Ate^{3t} + 3Ate^{3t} \\ &= 2Ae^{3t} \end{aligned}$$

- Setting $LHS = RHS$we have

$$\text{LHS} = \text{RHS}$$
$$2Ae^{3t} = 4e^{3t}$$

so that $A = 2$.
- **Step4:** Plug A back in and get $y_p(t) = Ate^{3t} = 2te^{3t}$ and a **general solution** of

$$y(t) = y_h + y_p.$$
$$y(t) = c_1e^t + c_2e^{3t} + 2te^{3t}.$$

(4) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 2y' + y = e^t.$$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' - 2y' + y = 0$$

but we learned that we must solve the characteristic polynomial $r^2-2r+1 = (r-1)(r-1) = 0$ and get $r = 1, 1$ (repeated real) so that the solution is

$$y_h(t) = c_1e^t + c_2te^t.$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    * Since the $RHS = e^t$, we let $y_p(t) = Ae^t$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since Ae^t **IS already part** of $y_h(t) = c_1e^t + c_2te^t$, then we need to second guess.
    * 2nd Guess (when second guessing, multiply by t): $y_p(t) = Ate^t$.
  - 3rd Guess? To make sure we don't need to third guess. We check that there are no repeats with y_h. Since Ate^t **IS already part** of $y_h(t) = c_1e^t + c_2te^t$, nhen we need to guess again
    * 3rd Guess (when second guessing, multiply previous guess by t): $y_p(t) = At^2e^t$.
  - We want to find what the value of A is. We need to plug this into the LHS. So we start by taking derivatives:

$$\begin{aligned} y_p &= At^2e^t, \\ y_p' &= 2Ate^t + At^2e^t \\ y_p'' &= 2Ae^t + 2Ate^t + 2Ate^t + At^2e^t \\ &= 2Ae^t + 4Ate^t + At^2e^t \end{aligned}$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned} \text{LHS} = y_p'' - 2y_p' + y_p &= \left(2Ae^t + 4Ate^t + At^2e^t\right) - 2\left(2Ate^t + At^2e^t\right) \\ &\quad + \left(At^2e^t\right) \\ &= 2Ae^t + 4Ate^t + At^2e^t - 4Ate^t - 2At^2e^t \\ &\quad + At^2e^t \\ &= 2Ae^t \end{aligned}$$

- Setting $LHS = RHS$we have

$$\text{LHS} = \text{RHS}$$
$$2Ae^t = e^t$$

so that $A = \frac{1}{2}$.
- **Step4:** Plug A back in and get $y_p(t) = At^2e^t = \frac{1}{2}t^2e^t$ and a **general solution** of

$$y(t) = y_h + y_p.$$
$$y(t) = c_1e^t + c_2te^t + \frac{1}{2}t^2e^t.$$

(5) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + y' - 6y = 52\cos(2t).$$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' + y' - 6y = 0$$

but we learned that we must solve the characteristic polynomial $r^2+r-6 = (r+3)(r-2) = 0$ and get $r = -3, 2$ so that the solution is

$$y_h(t) = c_1e^{-3t} + c_2e^{2t}$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    - Since the $RHS = 52\cos(2t)$, we let $y_p(t) = A\cos(2t) + B\sin(2t)$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since $A\cos(2t)+B\sin(2t)$ is **not** already part of $y_h(t) = c_1e^{-3t}+c_2e^{2t}$, then we made the correct guess.
  - We want to find what the value of A, B is. We need to plug this into the LHS. So we start by taking derivatives:

$$y_p = A\cos(2t) + B\sin(2t),$$
$$y_p' = -2A\sin(2t) + 2B\cos(2t)$$
$$y_p'' = -4A\cos(2t) - 4B\sin(2t)$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned}\text{LHS} = y_p'' + y_p' - 6y_p &= (-4A\cos(2t) - 4B\sin(2t)) \\ &\quad + (-2A\sin(2t) + 2B\cos(2t)) \\ &\quad -6(A\cos(2t) + B\sin(2t)) \\ &= (-10A + 2B)\cos(2t) + (-2A - 10B)\sin(2t)\end{aligned}$$

- Setting $LHS = RHS$we have

$$\text{LHS} = \text{RHS}$$
$$(-10A + 2B)\cos(2t) + (-2A - 10B)\sin(2t) = 52\cos(2t) + 0\cdot\sin(2t)$$

so that you have to solve

$$52 = -10A + 2B$$
$$0 = -2A - 10B$$

and solving this system we get

$$A = -5, B = 1.$$

- **Step4:** Plug A, B back in and get $y_p(t) = -5\cos(2t) + \sin(2t)$ and a **general solution** of

$$y(t) = y_h + y_p.$$
$$y(t) = c_1 e^{-3t} + c_2 e^{2t} + -5\cos(2t) + \sin(2t).$$

(6) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + 2y' + 3y = \sin(t).$$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' + 2y' + 3y = 0$$

and get

$$y_h(t) = c_1 e^{-t}\sin\left(\sqrt{2}t\right) + c_2 e^{-t}\cos\left(\sqrt{2}t\right)$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    - Since the $RHS = \sin(t)$, we let $y_p(t) = A\cos(t) + B\sin(t)$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since $A\cos(t) + B\sin(t)$ is **not** already part of $y_h(t) = c_1 e^{-t}\sin(\sqrt{2}t) + c_2 e^{-t}\cos(\sqrt{2}t)$, then we made the correct guess.
  - We want to find what the value of A, B is. We need to plug this into the LHS. So we start by taking derivatives:

$$y_p = A\cos(t) + B\sin(t),$$
$$y_p' = -A\sin(t) + B\cos(t)$$
$$y_p'' = -A\cos(t) - B\sin(t)$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned}\text{LHS} = y_p'' + 2y_p' + 3y_p &= \text{DO WORK}\\ &= -A\cos(t) - B\sin(t)\\ &\quad -2A\sin(t) + 2B\cos(t)\\ &\quad +3A\cos(t) + 3B\sin(t)\\ &= (2A + 2B)A\cos(t) + (-2A + 2B)B\sin(t)\end{aligned}$$

- Setting $LHS = RHS$we have

$$\text{LHS} = \text{RHS}$$
$$\text{FROM STEP 2} = 0\cos(t) + \sin(t)$$
$$(2A + 2B)A\cos(t) + (-2A + 2B)B\sin(t) = 0\cos(t) + \sin(t)$$

set up a system of equations

$$2A + 2B = 0$$
$$-2A + 2B = 1$$

and solving this system we get

$$A = -1/4, B = 1/4.$$

- **Step4:** Plug A, B back in and get $y_p(t) = -\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)$ and a **general solution** of

$$y(t) = y_h + y_p.$$
$$y(t) = c_1 e^{-t}\sin\left(\sqrt{2}t\right) + c_2 e^{-t}\cos\left(\sqrt{2}t\right) - \frac{1}{4}\cos(t) + \frac{1}{4}\sin(t).$$

(7) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + 9y = 27t^2.$$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' + 9y = 0$$

but we learned that we must solve the characteristic polynomial $r^2 + 9 = 0$ and get $r = \pm 3i$ so that the solution is

$$y_h(t) = c_1\cos(3t) + c_2\sin(3t)$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient.
  - 1st Guess: (always based on the general form of the RHS= $g(t)$)
    * Since the $RHS = 27t^2$, we let $y_p(t) = At^2 + Bt + C$.
  - 2nd Guess? To make sure we don't need to second guess. We check that there are no repeats with y_h. Since $At^2 + Bt + C$ is **not** already part of $y_h(t) = c_1\cos(3t) + c_2\sin(3t)$, then we made the correct guess.
  - We want to find what the value of A, B is. We need to plug this into the LHS. So we start by taking derivatives:

$$y_p = At^2 + Bt + C,$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get. Plug y_p into the LHS:

$$\begin{aligned}\text{LHS} = y_p'' + 9y_p &= (2A) \\ &\quad +9\left(At^2 + Bt + C\right) \\ &= 9At^2 + 9Bt + (9C + 2A)\end{aligned}$$

- Setting $LHS = RHS$we have

$$\text{LHS} = \text{RHS}$$
$$9At^2 + 9Bt + (9C + 2A) = 27t^2 + 0t + 0$$

so that you have to solve

$$9A = 27$$
$$9B = 0$$
$$9C + 2A = 0$$

and solving this system we get

$$A = 3, B = 0, C = -\frac{2}{3}.$$

- **Step4:** Plug A, B back in and get $y_p(t) = 3t^2 - \frac{2}{3}$ and a **general solution** of

$$y(t) = y_h + y_p.$$
$$y(t) = c_1 \cos(3t) + c_2 \sin(3t) + 3t^2 - \frac{2}{3}.$$

(8) For the following ODEs. Use the method of undertermined coefficients (MOUC) to make the correct guess for the y_p. You DO NOT have to solve for the coefficients, $A, B, C \ldots$. Simply make the correct guess for the y_p.

(a) $y'' - 2y' + y = te^t$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' - 2y' + y = 0$$

but we learned that we must solve the characteristic polynomial $r^2 - 2r + 1 = (r-1)(r-1) = 0$ and get $r = 1, 1$ (repeated real) so that the solution is

$$y_h(t) = c_1 e^t + c_2 te^t.$$

- **Step2:**
- 1st Guess:(based on RHS) $y_p = (At + B) e^t$
- 2nd Guess: (based on if there are repeats with y_h) $y_p = \left(At^2 + Bt\right) e^t$
- 3rd Guess: (based on if there are repeats with y_h) $y_p = \left(At^3 + Bt^2\right) e^t$,
  - Since there no repeats with y_h then this is the final guess.

(b) $y'' + y' - 2y = t^2 e^t$

- **Solution:**
- **Step1:** Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,

$$y'' + y' - 2y = 0$$

but we learned that we must solve the characteristic polynomial $r^2 + r - 2 = (r+2)(r-1) = 0$ and get $r = -2, 1$ so that the solution is

$$y_h(t) = c_1 e^{-2t} + c_2 e^t.$$

- **Step2:**
- 1st Guess:(based on RHS) $y_p = \left(At^2 + Bt + C\right) e^t$
- 2nd Guess: (based on if there are repeats with y_h) $y_p = \left(At^3 + Bt^2 + Ct\right) e^t$
  - Since there no repeats with y_h then this is the final guess.

(c) $y'' + y' = t^2 + \cos t$

- **Solution:**

- <u>**Step1:**</u> Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,
$$y'' + y' = 0$$
but we learned that we must solve the characteristic polynomial $r^2 + r = r(r+1) = 0$ and get $r = 0, -1$ so that the solution is
$$\begin{aligned} y_h(t) &= c_1 e^{0t} + c_2 e^{-t}. \\ &= c_1 + c_2 e^{-t} \end{aligned}$$
- <u>**Step2:**</u>
- <u>1st Guess:</u>(based on RHS) $y_p = \left(At^2 + Bt + C\right) + (D\cos t + E\sin t)$
- <u>2nd Guess:</u> (based on if there are repeats with y_h) $y_p = y_p = \left(At^3 + Bt^2 + Ct\right) + (D\cos t + E\sin t)$
  - Since there no repeats with y_h then this is the final guess.

(d) $y'' + y' - 6y = e^{5t} + \sin(3t)$
- **Solution:**
- <u>**Step1:**</u> Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,
$$y'' + y' - 6y = 0$$
but we learned that we must solve the characteristic polynomial $r^2 + r - 6 = (r+3)(r-2) = 0$ and get $r = -3, 2$ so that the solution is
$$y_h(t) = c_1 e^{-3t} + c_2 e^{2t}$$
- <u>**Step2:**</u>
- <u>1st Guess:</u>(based on RHS) $y_p = Ae^{5t} + (B\cos 3t + C\sin 3t)$
  - Since there no repeats with y_h then this is the final guess.

(e) $y'' + y' - 2y = te^t + t^2$
- **Solution:**
- <u>**Step1:**</u> Find $y_h(t)$, which is simply the general solution to the **homogeneous EQ**,
$$y'' + y' - 2y = 0$$
but we learned that we must solve the characteristic polynomial $r^2 + r - 2 = (r+2)(r-1) = 0$ and get $r = -2, 1$ so that the solution is
$$y_h(t) = c_1 e^{-2t} + c_2 e^{t}.$$
- <u>**Step2:**</u>
- <u>1st Guess:</u>(based on RHS) $y_p = (At + B)e^t + \left(Ct^2 + Dt + E\right)$
- <u>2nd Guess:</u> (based on if there are repeats with y_h) $y_p = \left(At^2 + Bt\right)e^t + \left(Ct^2 + Dt + E\right)$
  - Since there no repeats with y_h then this is the final guess.

3.7. Problems

(1) A mass weighing 8 lb stretches a spring $\frac{1}{2}$ feet. The mass is pulled down an additional 1 feet. and then set in motion with an upward velocity of 2 ft/sec. Assume that there is no damping force and that the downward direction is the positive direction. The gravity constant g is 32 $\frac{\text{ft}}{s^2}$. The function $u(t)$ describing the displacement of the mass from the equilibrium position as a function of time t satisfies what initial value problem?

- **Solution:**
- Find m: $w = mg$ which implies
$$m = \frac{w}{g} = \frac{8 \text{ lb}}{32 \text{ ft/s\^{}2}} = \frac{1}{4}\frac{\text{lb}s^2}{ft}$$
- Find γ: $\gamma = 0$ since there is no damping.
- Find k: (Hooke's Law)
$$k = \frac{mg}{L} = \frac{8 \text{ lb}}{(1/2)\,\text{ft}} = \frac{16 \text{ lb}}{\text{ft}}.$$
- There is no external force so $F(t) = 0$.
- Thus
$$\frac{1}{4}u'' + 0u' + 16u = 0$$
hence
$$u'' + 64u = 0, \quad u(0) = 1, \;\; u'(0) = -2.$$

(2) A mass of 5 kg stretches spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass.

- **Solution:**
- Find m: We are given that
$$m = 5 \text{ kg}$$
- Find γ: Using $\gamma u' = 2\,\text{N}$ when $u' = 4$ cm/sec we have
$$\gamma = \frac{2 \text{ N}}{.04 \text{ m/sec}} = 50\frac{\text{N-sec}}{\text{m}}.$$
- Find k: (Hooke's Law)
$$k = \frac{mg}{L} = \frac{5 \cdot (9.8)}{.1\,\text{m}} = \frac{490 \text{ N}}{\text{m}}.$$
- The external force is given by so $F(t) = 10\sin(t/2)$.
- Thus
$$5u'' + 50u' + 490u = 10\sin(t/2)$$
hence
$$5u'' + 50u' + 490u = 10\sin(t/2), \quad u(0) = 0, \;\; u'(0) = .03\frac{\text{m}}{\text{s}}.$$

3.8. Problems

(1) A 64 lb mass stretches a spring 4 feet. The mass is displaced an additional 5 feet. and then released; and is in a medium with a damping coefficients $\gamma = 7\frac{\text{lb sec}}{\text{ft}}$. Suppose there is no external forcing. Formulate the IVP that governs the motion of this mass:

- **Solution:**
  - Find m: $w = mg$ which implies
$$m = \frac{w}{g} = \frac{64 \text{ lb}}{32 \text{ ft/s\^{}2}} = 2\frac{\text{lb}s^2}{ft}$$
  - Find γ: Given
$$\gamma = 7\frac{\text{lb sec}}{\text{ft}}.$$
  - Find k:
$$k = \frac{mg}{L} = \frac{64 \text{ lb}}{4ft} = 16\frac{\text{lb}}{ft}.$$
  - Thus the IVP is given by
$$2u'' + 7u' + 16u = 0, \quad u(0) = 5, \;\; u'(0) = 0$$

(2) A 32 lb mass stretches a spring 4 feet. The mass is displaced an additional 6 feet. and then released with an initial velocity of 3 $\frac{\text{ft}}{\text{sec}}$; and is in a medium with a damping coefficients $\gamma = 2\frac{\text{lb sec}}{\text{ft}}$. Suppose there is an external forcing due to wind given by $F(t) = 3\cos(3t)$. Formulate the IVP that governs the motion of this mass:

- **Solution:**
  - Find m: $w = mg$ which implies
$$m = \frac{w}{g} = \frac{32 \text{ lb}}{32 \text{ ft/s\^{}2}} = 1\frac{\text{lb}s^2}{ft}$$
  - Find γ: Given
$$\gamma = 2\frac{\text{lb sec}}{\text{ft}}.$$
  - Find k:
$$k = \frac{mg}{L} = \frac{32 \text{ lb}}{4ft} = 8\frac{\text{lb}}{ft}.$$
  - Thus the IVP is given by
$$u'' + 2u' + 8u = 3\cos(3t), \quad u(0) = 6, \;\; u'(0) = 3.$$

(3) Consider the following undamped harmonic oscillator with external forcing:
$$u'' + 5u = \sin(3t), \quad u(0) = 1, \;\; u'(0) = -1.$$
What is the natural frequency? What is the frequency for the external force? Will you get resonance? What is your guess for u_p? Solve the IVP.

- **Solution:**
- Recall that $r^2 + 5 = 0$ so $r = \pm\sqrt{5}i$, so that
$$u_h(t) = c_1 \cos\left(\sqrt{5}t\right) + c_2 \sin\left(\sqrt{5}t\right).$$
- Thus the **Natural Frequency** is $\omega_0 = \sqrt{5}$.
- The **External Frequency** from $\sin(3t)$ is $\omega = 3$. Since they don't match, then we will not get resonance.

- Thus our guess for u_p is
$$u_p = A\cos(3t) + B\sin(3t).$$
- Taking derivatives and plugging into the LHS we obtain
$$A = 0, B = -\frac{1}{4}$$
so that
$$u(t) = c_1\cos\left(\sqrt{5}t\right) + c_2\sin\left(\sqrt{5}t\right) - \frac{1}{4}\sin(3t).$$
using the initial conditions we get that
$$c_1 = 1, c_2 = -\frac{1}{4\sqrt{5}}$$
so that the solution to IVP is
$$u(t) = \cos\left(\sqrt{5}t\right) - \frac{1}{4\sqrt{5}}\sin\left(\sqrt{5}t\right) - \frac{1}{4}\sin(3t).$$

(4) Consider the following undamped harmonic oscillator with external forcing:
$$u'' + 16u = 7\cos(4t), \quad u(0) = 0, \ u'(0) = 0.$$
What is the natural frequency? What is the frequency for the external force? Will you get resonance? What is your guess for u_p? Solve the IVP.

- **Solution:**
- Recall that $r^2 + 16 = 0$ so $r = \pm 4i$, so that
$$u_h(t) = c_1\cos(4t) + c_2\sin(4t).$$
- Thus the **Natural Frequency** is $\omega_0 = 4$.
- The **External Frequency** from $7\cos(4t)$ is $\omega = 4$. Since they match, then we will get resonance!
- Thus our guess for u_p is
$$u_p = At\cos(4t) + Bt\sin(4t).$$
- Taking derivatives and plugging into the LHS we obtain
$$A = 0, B = \frac{7}{8}$$
so that
$$u(t) = c_1\cos(4t) + c_2\sin(4t) + \frac{7}{8}t\sin(4t).$$
using the initial conditions we get that $c_1 = c_2 = 0$ so that the solution to IVP is
$$u(t) = \frac{7}{8}t\sin(4t).$$

3.9. Problems

(1) Consider the following ODE

$$y'' + 16y = \frac{1}{\sin(4t)}.$$

(a) Find a particular solution to the ODE above using the method of variation of parameters.

- **Solution:**
- **Step1:** First find y_h is possible. In this case y_h will be given by solving the characteristic equation $r^2 + 16 = 0$ so that $r = \pm 4i$ hence

$$y_h(t) = c_1 \cos(4t) + c_2 \sin(4t).$$

Thus $y_1(t) = \cos(4t)$ and $y_2(t) = \sin(4t)$.

- **Step2:** Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{vmatrix} \\ &= 4\cos^2(4t) + 4\sin^2(4t) \\ &= 4\left[\cos^2(4t) + \sin^2(4t)\right] \\ &= 4 \cdot 1 = 4. \end{aligned}$$

- **Step3:** Use our formula with $g(t) = \frac{1}{\sin(3t)}$ and get

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt\right] \\ &= -\cos(4t)\left[\int \frac{1}{4}\frac{\sin(4t)}{\sin(4t)}dt\right] + \sin(4t)\left[\int \frac{\cos(4t)}{4}\frac{1}{\sin(4t)}dt\right] \\ &= -\cos(4t)\left[\int \frac{1}{4}dt\right] + \sin(4t)\left[\frac{1}{4}\int \frac{\cos(4t)}{\sin(4t)}dt\right] \\ &= -\cos(4t)\left[\frac{t}{4}\right] + \frac{1}{4}\sin(4t)\left[\int \frac{\cos(4t)}{\sin(4t)}dt\right] \end{aligned}$$

now since $\int \frac{\cos(4t)}{\sin(4t)}dt = \int \cot(4t)\text{dt}$ you can remember the antiderivative of $\int \cot(u)\,dt = \ln(\sin u) + C$

- Or use can use u-substitution with $u = \sin(4t)$ and get $du = 4\cos(4t)dt$ so that

$$\begin{aligned} \int \frac{\cos(4t)}{\sin(4t)}dt &= \int \frac{1}{u}\frac{du}{4} = \frac{1}{4}\ln|u| + C \\ &= \frac{1}{4}\ln|\sin(4t)| + C \end{aligned}$$

hence taking $C = 0$,

$$\begin{aligned} y_p(t) &= -\cos(4t)\left[\frac{t}{4}\right] + \frac{1}{4}\sin(4t)\left[\frac{1}{4}\ln|\sin(4t)|\right] \\ &= -\frac{t}{4}\cos(4t) + \frac{1}{16}\sin(4t)\ln|\sin(4t)| \end{aligned}$$

(b) What is the general solution to the ODE above.

- **Solution:**

- Then general solution is given by

$$\begin{aligned} y(t) &= y_h + y_p \\ &= c_1 \cos(4t) + c_2 \sin(4t) \\ &- \frac{t}{4}\cos(4t) + \frac{1}{16}\sin(4t)\ln|\sin(4t)|. \end{aligned}$$

(2) Find the general solution to

$$t^2 y'' - 4ty' + 6y = t^3, \quad t > 0$$

given that

$$y_1(t) = t^2, \;\; y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

- **Solution:**
- **Step1:** Since $y_1(t) = t^2, \;\; y_2(t) = t^3$ forms a fundamental set of solution, this means that the general solution for the homogeneous equation is

$$y_h = c_1 t^2 + c_2 t^3.$$

- **Step2:** Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} \\ &= 3t^4 - 2t^4 = t^4 \neq 0, \end{aligned}$$

- **Step3:** Rewrite the equation in the form $y'' + p(t)y' + q(t)y = g(t)$ and hence

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = t,$$

Use our formula with $g(t) = t$ and get

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt\right] \\ &= -t^2\left[\int \frac{t^3 \cdot t}{t^4}dt\right] + t^3\left[\int \frac{t^2 \cdot t}{t^4}dt\right] \\ &= -t^2\left[\int dt\right] + t^3\left[\int \frac{1}{t}dt\right] \\ &= -t^2[t] + t^3[\ln t] \\ &= -t^3 + t^3 \ln t \end{aligned}$$

hence the **general solution** is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= c_1 t + c_2 t^3 - t^3 + t^3 \ln t. \end{aligned}$$

(3) Find the general solution to

$$t^2 y'' - 3ty' + 3y = 8t^3, \quad t > 0$$

given that

$$y_1(t) = t, \;\; y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

- **Solution:**
- **Step1:** Since $y_1(t) = t, \;\; y_2(t) = t^3$ forms a fundamental set of solution, this means that the general solution for the homogeneous equation is

$$y_h = c_1 t + c_2 t^3.$$

- **Step2:** Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} \\ &= 3t^3 - t^3 = 2t^3 \neq 0, \end{aligned}$$

- **Step3:** Rewrite the equation in the form $y'' + p(t)y' + q(t)y = g(t)$ and hence

$$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = 8t, .$$

Use our formula with $g(t) = 8t$ and get

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt\right] \\ &= -t\left[\int \frac{t^3}{2t^3}8tdt\right] + t^3\left[\int \frac{t}{2t^3}8tdt\right] \\ &= -t\left[\int 4tdt\right] + t^3\left[\int \frac{4}{t}dt\right] \\ &= -t\left[2t^2\right] + t^3\left[4\ln t\right] \\ &= -2t^3 + 4t^3 \ln t \end{aligned}$$

hence the **general solution** is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= c_1 t + c_2 t^3 - 2t^3 + 4t^3 \ln t. \end{aligned}$$

(4) Find the general solution to

$$2t^2 y'' + ty' - 3y = 2t^{5/2}, \quad t > 0$$

given that

$$y_1(t) = t^{-1}, \;\; y_2(t) = t^{3/2}$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

- **Solution:**
- **Step1:** Since $y_1(t) = t, \;\; y_2(t) = t^{-2}$ forms a fundamental set of solution, this means that the general solution for the homogeneous equation is

$$y_h = c_1 t^{-1} + c_2 t^{3/2}.$$

- **<u>Step2:</u>** Find the Wronskian:

$$\begin{aligned}
W(y_1,y_2)(t) &= \begin{vmatrix} t^{-1} & t^{3/2} \\ -t^{-2} & \frac{3}{2}t^{1/2} \end{vmatrix} \\
&= t^{-1}\frac{3}{2}t^{1/2} - t^{3/2}\left(-t^{-2}\right) \\
&= \frac{3}{2}t^{-1/2} + t^{-1/2} \\
&= \frac{5}{2}t^{-1/2}
\end{aligned}$$

- **<u>Step3:</u>** Rewrite the equation in the form $y'' + p(t)y' + q(t)y = g(t)$ and hence

$$y'' + \frac{1}{2t}y' - \frac{3}{2t^2}y = \frac{2t^{5/2}}{2t^2} = t^{1/2}.$$

Use our formula with $g(t) = t^{1/2}$ and get

$$\begin{aligned}
y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W\left(y_1,y_2\right)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W\left(y_1,y_2\right)(t)}dt\right] \\
&= -t^{-1}\left[\int \frac{t^{3/2}}{\frac{5}{2}t^{-1/2}}t^{1/2}dt\right] + t^{3/2}\left[\int \frac{t^{-1}}{\frac{5}{2}t^{-1/2}}t^{1/2}dt\right] \\
&= -t^{-1}\left[\int \frac{2}{5}t^2t^{1/2}dt\right] + t^{3/2}\left[\int \frac{2}{5}t^{-1/2}t^{1/2}dt\right] \\
&= -t^{-1}\left[\int \frac{2}{5}t^{5/2}dt\right] + t^{3/2}\left[\int \frac{2}{5}dt\right] \\
&= -t^{-1}\left[\frac{2}{5}\frac{2}{7}t^{7/2}\right] + t^{3/2}\left[\frac{2}{5}t\right] \\
&= -\frac{4}{35}t^{5/2} + \frac{2}{5}t^{5/2} \\
&= \frac{10}{35}t^{5/2} \\
&= \frac{2}{7}t^{5/2}
\end{aligned}$$

hence the **general solution** is

$$\begin{aligned}
y(t) &= y_h + y_p \\
&= c_1t^{-1} + c_2t^{3/2} + \frac{2}{7}t^{5/2}.
\end{aligned}$$

CHAPTER 4

Higher Order Linear Equations

4.1. Problems

(1) What is the largest interval for which there exists a unique solution by the Existence and Uniqueness Theorem for the following IVP:

$$\begin{cases} (t-5)\, y^{(4)} - \frac{\ln(t+7)}{t} y'' + e^t y = \frac{t^2+1}{(t-1)} \\ y(2) = -1 \\ y'(2) = 1 \\ y''(2) = 2 \\ y'''(2) = 5. \end{cases}$$

- **Solution:**
- First rewrite

$$y^{(4)} - \frac{\ln(t+7)}{t\,(t-5)} y'' + \frac{e^t}{(t-5)} y = \frac{t^2+1}{(t-1)\,(t-5)}$$

- The function $p_1(t) = -\frac{\ln(t+7)}{t(t-5)}$ is continuous when $t \neq 0, 5$ and $t + 7 > 0$, or $t > -7$
- The function $p_2(t) = \frac{e^t}{(t-5)}$ is continuous when $t \neq 5$.
- The function $g(t) = \frac{t^2+1}{(t-1)(t-5)}$ is continuous when $t \neq 1, 5$
- All functions are simultaneously continuous (draw number lines to help you find out when p_1, p_2, g_2 are **all** continuous) when
  - $t \neq 0, 1, 5$ and $t > -7$

$$(-7, 0) \cup (0, 1) \cup (1, 5) \cup (5, \infty)$$

  since $t_0 = 2$ falls inside $(1, 5)$ then the solution to this IVP must have a domain as large as

$$I = (1, 5)\,,$$

  by the theorem.

(2) Find general solution of

$$y''' + 10y'' + 7y' - 18y = 0.$$

(Hint: $r^3 + 10r^2 + 7r - 18 = (r-1)\,(r+2)\,(r+9)$)

- **Solution:**
- The characteristic equation is given by

$$r^3 + 10r^2 + 7r - 18 = 0$$

  and by the hint

$$Z(r) = (r-1)\,(r+2)\,(r+9) = 0$$

which gives

$$r = 1, -2, -9$$

hence

$$y(t) = c_1 e^t + c_2 e^{-2t} + c_3 e^{-9t}$$

(3) Find general solution of

$$y^{(4)} - 10y''' + 36y'' - 54y' + 27y = 0.$$

(Hint: $r^4 - 10r^3 + 36r^2 - 54r + 27 = (r-1)(r-3)^3$)

- **Solution:**
- The characteristic equation is given by

$$r^4 - 10r^3 + 36r^2 - 54r + 27 = 0$$

and by the hint

$$Z(r) = (r-1)(r-3)^3 = 0$$

which gives

$$r = 1, 3, 3, 3$$

hence

$$y(t) = c_1 e^t + c_2 e^{3t} + c_3 t e^{3t} + c_4 t^2 e^{3t}$$

(4) Find general solution of

$$y^{(5)} - 4y^{(4)} + 13y''' - 36y'' + 36y' = 0.$$

(Hint: $r^5 - 4r^4 + 13r^3 - 36r^2 + 36r = r(r-2)^2(r^2+9)$)

- **Solution:**
- The characteristic equation is given by

$$r^5 - 4r^4 + 13r^3 - 36r^2 + 36r = 0$$

and by the hint

$$Z(r) = r(r-2)^2(r^2+9) = 0$$

which gives

$$r = 0, 2, 2, \pm 3i$$

hence

$$y(t) = c_1 + c_2 e^{2t} + c_3 t e^{2t} + c_4 \cos(3t) + c_5 \sin(3t).$$

(5) Find general solution of

$$y^{(4)} + 11y'' + 18y = 0.$$

(Hint: $r^4 + 11r^2 + 18 = (r^2+2)(r^2+9)$)

- **Solution:**
- The characteristic equation is given by

$$r^4 + 11r^2 + 18r = 0$$

and by the hint

$$Z(r) = (r^2+2)(r^2+9) = 0$$

which gives

$$r = \pm\sqrt{2}i, \pm 3i$$

hence

$$y(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + c_4 \cos(3t) + c_4 \sin(3t).$$

(6) Find general solution of

$$y^{(6)} + 32y^{(4)} + 256y'' = 0.$$

(Hint: $r^6 + 32r^4 + 256r^2 = r^2\left(r^2+16\right)^2$)

- **Solution:**
- The characteristic equation is given by

$$r^6 + 32r^4 + 256r^2 = 0$$

and by the hint

$$Z(r) = r^2\left(r^2+16\right)^2 = 0$$

which gives

$$r = 0, 0, \pm 4i, \pm 4i$$

hence

$$\begin{aligned} y(t) &= c_1 + c_2 t \\ &+ c_3 \cos(4t) + c_4 \sin(4t) \\ &+ c_5 t \cos(4t) + c_6 t \sin(4t). \end{aligned}$$

4.2. Problems

(1) Consider

$$y''' - 4y'' - 11y' + 30y = 4e^{-3t} + \cos t.$$

Find the general form of y_p. (Hint: $r^3 - 4r^2 - 11r + 30 = (r+3)(r-2)(r-5)$)

- **Solution:**
- **Step1:** We find y_h: Solve the characteristic equation

$$r^3 - 4r^2 - 11r + 30 = 0$$
$$(r+3)(r-2)(r-5) = 0$$

so that $r = -3, 2, 5$ hence

$$y_h = c_1 e^{-3t} + c_2 e^{2t} + c_3 e^{5t}$$

- **Step2:** Find y_p:
  - **First Guess:** $y_p = Ae^{-3t} + B\cos t + C\sin t$.
  - **Second Guess:** $y_p = Ate^{-3t} + B\cos t + C\sin t$. And this is the final correct guess.

(2) Consider

$$y^{(4)} + 8y''' + 16y'' = t + e^t.$$

Find the general form of y_p(Hint: $r^4 + 8r^3 + 16r^2 = r^2(r+4)^2$)

- **Solution:**
- **Step1:** We find y_h: Solve the characteristic equation

$$r^4 + 8r^3 + 16r^2 = 0$$
$$r^2(r+4)^2 = 0$$

so that $r = 0, 0, -4, -4$ hence

$$y_h = c_1 + c_2 t + c_3 e^{-4t} + c_4 t e^{-4t}$$

- **Step2:** Find y_p:
  - **First Guess:** $y_p = (At + B) + Ce^t$.
  - **Second Guess:** $y_p = \left(At^2 + Bt\right) + Ce^t$.
  - **Third Guess:** $y_p = \left(At^3 + Bt^2\right) + Ce^t$. And this is the final correct guess.

(3) Consider

$$y^{(4)} - 10y''' + 36y'' - 54y' + 27y = 2te^t + \cos(3t),$$

and suppose you know that $y_h = c_1 e^t + c_2 e^{-3t} + c_3 t e^{-3t} + c_4 t^2 e^{-3t}$. Find the general form of y_p

- **Solution:**
- **Step1:** We find y_h: This is given to us

$$y_h = c_1 e^t + c_2 e^{-3t} + c_3 t e^{-3t} + c_4 t^2 e^{-3t}$$

- **Step2:** Find y_p:
  - **First Guess:** $y_p = (At + B)e^t + C\cos(3t) + D\sin(3t)$.
  - **Second Guess:** $y_p = \left(At^2 + Bt\right)e^t + C\cos(3t) + D\sin(3t)$. And this is the final correct guess.

(4) Consider

$$y^{(4)} - 2y''' = 2t + 1.$$

Find the general form of y_p. (Hint: $r^4 - 2r^3 = r^3(r-2)$)

- **Solution:**

- **Step1:** We find y_h: Solve the characteristic equation

$$r^4 - 2r^3 = 0$$
$$r^3 (r-2) = 0$$

so that $r = 0, 0, 0, 2$ hence

$$y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{2t}$$

- **Step2:** Find y_p:
  - **First Guess:** $y_p = At + B$.
  - **Second Guess:** $y_p = At^2 + Bt$.
  - **Third Guess:** $y_p = At^3 + Bt^2$.
  - **Fourth Guess:** $y_p = At^4 + Bt^3$. And this is the final correct guess since there is no repeats with y_h.

CHAPTER 5

Systems of First Order Linear Equations

5.1. Problems

(1) Show that the functions

$$x_1(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$$
$$x_2(t) = \frac{1}{3}e^t - \frac{4}{3}e^{-2t}$$

solve the following system of first order differential equations IVP

$$\begin{cases} x_1' = x_2 & x_1(0) = 1 \\ x_2' = 2x_1 - x_2 & x_2(0) = -1 \end{cases}.$$

- **Solution:**
- First let's check the initial conditions hold,

$$x_1(0) = \frac{1}{3}e^0 + \frac{2}{3}e^0 = 1$$
$$x_2(0) = \frac{1}{3}e^0 - \frac{4}{3}e^0 = -1$$

- Now we plug x_1 and x_2 into the LHS,

$$\begin{aligned} \text{LHS} &= \begin{cases} x_1' \\ x_2' \end{cases} \\ &= \begin{cases} \frac{d}{dt}\left(\frac{1}{3}e^t + \frac{2}{3}e^{-2t}\right) \\ \frac{d}{dt}\left(\frac{1}{3}e^t - \frac{4}{3}e^{-2t}\right) \end{cases} \\ &= \begin{cases} \frac{1}{3}e^t - \frac{4}{3}e^{-2t} \\ \frac{1}{3}e^t + \frac{8}{3}e^{-2t} \end{cases} \end{aligned}$$

and now we plug into the RHS,

$$\text{RHS} = \begin{cases} x_2 \\ 2x_1 - x_2 \end{cases}$$
$$= \begin{cases} \frac{1}{3}e^t - \frac{4}{3}e^{-2t} \\ 2\left(\frac{1}{3}e^t + \frac{2}{3}e^{-2t}\right) - \left(\frac{1}{3}e^t - \frac{4}{3}e^{-2t}\right) \end{cases}$$
$$= \begin{cases} \frac{1}{3}e^t - \frac{4}{3}e^{-2t} \\ \frac{2}{3}e^t + \frac{4}{3}e^{-2t} - \frac{1}{3}e^t + \frac{4}{3}e^{-2t} \end{cases}$$
$$= \begin{cases} \frac{1}{3}e^t - \frac{4}{3}e^{-2t} \\ \frac{1}{3}e^t + \frac{8}{3}e^{-2t} \end{cases}$$

since

$$\text{LHS} = \begin{cases} \frac{1}{3}e^t - \frac{4}{3}e^{-2t} \\ \frac{1}{3}e^t + \frac{8}{3}e^{-2t} \end{cases} = \text{RHS}$$

then these functions solve the IVP.

(2) Show that the functions

$$x_1(t) = \sin(2t)$$
$$x_2(t) = \cos(2t)$$

solve the following system of first order differential equations IVP

$$\begin{cases} x_1' = 2x_2 & x_1(0) = 0 \\ x_2' = -2x_1 & x_2(0) = 1 \end{cases}.$$

- **Solution:**
- First let's check the initial conditions hold,

$$x_1(0) = \sin 0 = 0$$
$$x_2(0) = \cos 0 = 1$$

- Now we plug x_1 and x_2 into the LHS,

$$\text{LHS} = \begin{cases} x_1' \\ x_2' \end{cases}$$
$$= \begin{cases} \frac{d}{dt}(\sin(2t)) \\ \frac{d}{dt}(\cos(2t)) \end{cases}$$
$$= \begin{cases} 2\cos(2t) \\ -2\sin(2t) \end{cases}$$

and now we plug into the RHS,

$$\begin{aligned} \text{RHS} &= \begin{cases} 2x_2 \\ -2x_1 \end{cases} \\ &= \begin{cases} 2\left(\cos\left(2t\right)\right) \\ -2\left(\cos\left(2t\right)\right) \end{cases} \\ &= \begin{cases} 2\cos\left(2t\right) \\ -2\cos\left(2t\right) \end{cases} \end{aligned}$$

since

$$\text{LHS} = \begin{cases} 2\cos\left(2t\right) \\ -2\sin\left(2t\right) \end{cases} = \text{RHS}$$

then these functions solve the IVP.

(3) Turn the following second order ODE

$$y'' + y' + 2y = t^2$$

into a system of first differential equations.

- **Solution:**
- Goal: We let $x_1 = y$, $x_2 = y'$ and set up the following system:

$$\begin{aligned} x_1' &= ? \\ x_2' &= ? \end{aligned}$$

- To do so, we start with what we defined and take derivatives:

$$\begin{aligned} x_1 = y &\quad \Longrightarrow \quad x_1' = y' = x_2 \\ x_2 = y' &\quad \Longrightarrow \quad x_2' = -y' - 2y + t^2 \end{aligned}$$

hence

$$\begin{aligned} x_2' &= -y' - 2y + t^2 \\ &= -x_2 - 2x_1 + t^2, \text{ by definition} \end{aligned}$$

thus

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_2 - 2x_1 + t^2 \end{cases}$$

(4) Turn the following second order ODE

$$y'' - 2y' + 10y = 0.$$

into a system of first differential equations.

- **Solution:**
- Goal: We let $x_1 = y$, $x_2 = y'$ and set up the following system:

$$\begin{aligned} x_1' &= ? \\ x_2' &= ? \end{aligned}$$

- To do so, we start with what we defined and take derivatives:

$$\begin{aligned} x_1 = y \quad &\Longrightarrow \quad x_1' = y' = x_2 \\ x_2 = y' \quad &\Longrightarrow \quad x_2' = -10y + 2y' \end{aligned}$$

  hence

$$\begin{aligned} x_2' &= -10y + 2y' \\ &= -10x_1 + 2x_2, \text{ by definition} \end{aligned}$$

  thus

$$\begin{cases} x_1' = x_2 \\ x_2' = -10x_1 + 2x_2 \end{cases}$$

5.2. Problems

(1) Let $A = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, find $A\boldsymbol{x}$, $4\boldsymbol{x}$, and $\boldsymbol{x}+\boldsymbol{y}$.

- **Solution:**
- We have

$$\begin{aligned} A\boldsymbol{x} &= \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2+2 \\ -10+3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -7 \end{pmatrix}. \end{aligned}$$

and

$$4\boldsymbol{x} = 4\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$$

and finally,

$$\begin{aligned} \boldsymbol{x}+\boldsymbol{y} &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 6 \end{pmatrix}. \end{aligned}$$

(2) Turn the following system of first order equations into matrix-product form:

(a) Given by:

$$\begin{cases} x_1' = 3x_2 \\ x_2' = 9x_1 - 3x_2 \end{cases}$$

- **Solution:**
- Letting

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 3 \\ 9 & -3 \end{pmatrix}\boldsymbol{x}$$

(b) Given by:

$$\begin{cases} x_1' = -x_1 + 2x_2 \\ x_2' = 7x_1 + 5x_2 \end{cases}$$

- **Solution:**
- Letting

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then

$$\boldsymbol{x}' = \begin{pmatrix} -1 & 2 \\ 7 & 5 \end{pmatrix}\boldsymbol{x}$$

(3) Turn the following vector valued ODE into a system of first order equations.

(a) Given by:

$$\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 3 & 17 \end{pmatrix} \boldsymbol{x}$$

where $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

- **Solution:**
- Since $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ then $\boldsymbol{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$.
- By multiplying

$$\begin{aligned} \begin{pmatrix} 3 & -2 \\ 3 & 17 \end{pmatrix} \boldsymbol{x} &= \begin{pmatrix} 3 & -2 \\ 3 & 17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 3x_1 - 2x_2 \\ 3x_1 + 17x_2 \end{pmatrix} \end{aligned}$$

thus

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 3 & -2 \\ 3 & 17 \end{pmatrix} \boldsymbol{x} \\ \iff \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} &= \begin{pmatrix} 3x_1 - 2x_2 \\ 3x_1 + 17x_2 \end{pmatrix} \end{aligned}$$

which means that

$$\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 3x_1 + 17x_2 \end{cases}$$

(b) Given by:

$$\boldsymbol{Y}' = \begin{pmatrix} 0 & 4 \\ 7 & 3 \end{pmatrix} \boldsymbol{Y}$$

where $\boldsymbol{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

- **Solution:**
- Since $\boldsymbol{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ then $\boldsymbol{Y}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$.
- By multiplying

$$\begin{aligned} \begin{pmatrix} 0 & 4 \\ 7 & 3 \end{pmatrix} \boldsymbol{Y} &= \begin{pmatrix} 0 & 4 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 4y \\ 7x + 3y \end{pmatrix} \end{aligned}$$

thus

$$\begin{aligned} \boldsymbol{Y}' &= \begin{pmatrix} 0 & 4 \\ 7 & 3 \end{pmatrix} \boldsymbol{Y} \\ \iff \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} &= \begin{pmatrix} 4y \\ 7x + 3y \end{pmatrix} \end{aligned}$$

which means that

$$\begin{cases} x' = 4y \\ y' = 7x + 3y \end{cases}$$

(4) Find the equilibrium solutions of the following system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

- **Solution:**
- The equilibrium solutions is the solution of the following equation:

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

hence involves solving the following system of algebraic equations:

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_1 + 3x_2 = 0 \end{cases}$$

and it turn out that the only solution is

$$\mathbf{x}(t) = (0, 0).$$

(5) Find the equilibrium solutions of the following system:

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x}.$$

- **Solution:**
- The equilibrium solutions are the solution of the following equation:

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

hence involves solving the following system of algebraic equations:

$$\begin{cases} 3x_1 - x_2 = 0 \\ 9x_1 - 3x_2 = 0 \end{cases}$$

since these equations are multiple of each other than there is a whole line of solutions:

$$x_2 = 3x_1$$

and it turn out that the solution are of then form:

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

- Using a generic constant c, we have that there are an infinite number of equilibrium solutions:

$$\mathbf{x}(t) = c \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(6) Check if the following vector functions satisfy the following differential equations.

(a) Where

$$\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(t) = \begin{pmatrix} 4e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

- **Solution:**

- First we plug $\boldsymbol{x}$ into the LHS and obtain,

$$\begin{aligned}\text{LHS} &= \boldsymbol{x}' \\ &= \frac{d}{dt}\begin{pmatrix} 4e^{2t} \\ 2e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 8e^{2t} \\ 4e^{2t} \end{pmatrix}\end{aligned}$$

and now we plug $\boldsymbol{x}$ into the RHS and obtain

$$\begin{aligned}\text{RHS} &= \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\boldsymbol{x} \\ &= \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 4e^{2t} \\ 2e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 12e^{2t} - 4e^{2t} \\ 8e^{2t} - 4e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 8e^{2t} \\ 4e^{2t} \end{pmatrix}\end{aligned}$$

since the LHS equals the RHS, then this vector function satisfies the ODE.

(b) Where

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\boldsymbol{x}, \quad \boldsymbol{x}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t}.$$

- **Solution:**
- First we plug $\boldsymbol{x}$ into the LHS and obtain,

$$\begin{aligned}\text{LHS} &= \boldsymbol{x}' \\ &= \frac{d}{dt}\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}3e^{3t} \\ &= \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix}\end{aligned}$$

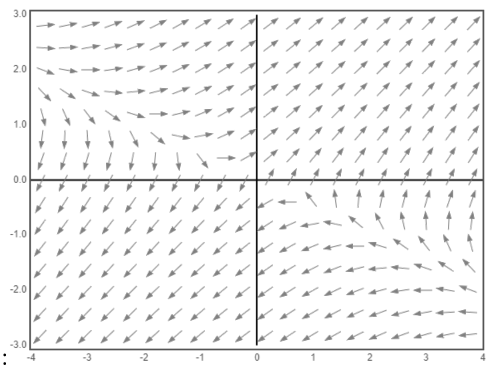
and now we plug $\boldsymbol{x}$ into the RHS and obtain

$$\begin{aligned}\text{RHS} &= \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} \\ &= \begin{pmatrix} 1+6 \\ 2+4 \end{pmatrix}e^{3t} \\ &= \begin{pmatrix} 7 \\ 6 \end{pmatrix}e^{3t} \\ &= \begin{pmatrix} 7e^{3t} \\ 6e^{3t} \end{pmatrix}\end{aligned}$$

since the LHS DOES NOT equal to the RHS, then this vector function satisfies is NOT a solution to the given ODE.

(7) Consider the following differential equation

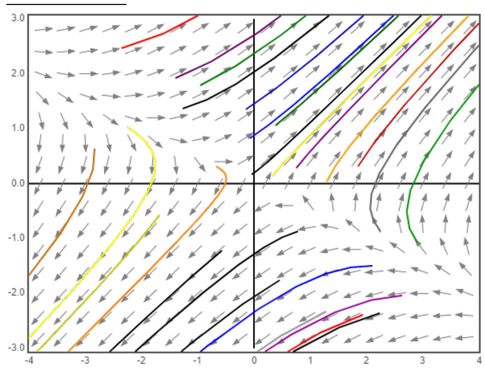
$$\boldsymbol{x}' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \boldsymbol{x},$$



with corresponding direction field:

(a) Using the direction field draw a Phase portrait. (A Phase portrait is graph a several possible different solutions of the ODE)

- **Solution:**



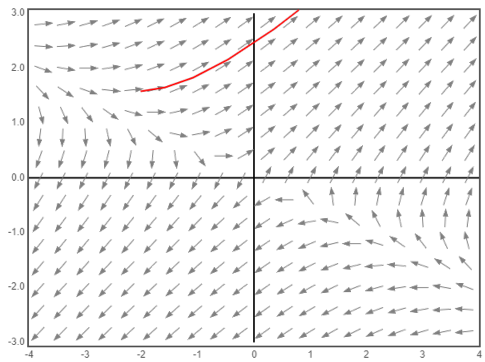
-

(b) Consider the following IVP:

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} -2 \\ 1.5 \end{pmatrix}$$

draw the unique soluton to this IVP and use it to **predict** long term behavior of $\lim_{t\to\infty} x_1(t)$ and $\lim_{t\to\infty} x_2(t)$.

- **Solution:**

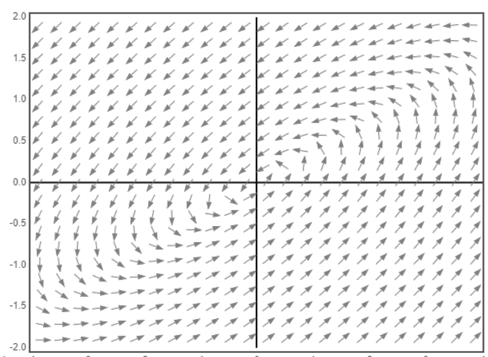
- 

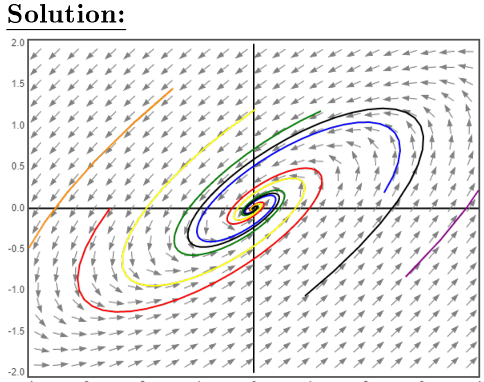
- We have

$$\lim_{t\to\infty} x_1(t) = +\infty,$$
$$\lim_{t\to\infty} x_2(t) = +\infty,$$

(8) Consider the following differential equation

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -2 \end{pmatrix} \boldsymbol{x},$$

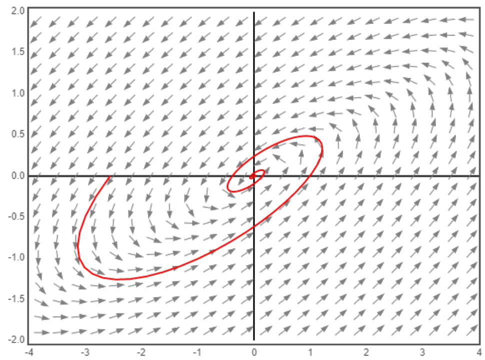
with corresponding direction field: 

(a) Using the direction field draw a Phase portrait. (A Phase portrait is graph a several possible different solutions of the ODE)
- **Solution:**
- 

(b) Consider the following IVP:

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} -2.5 \\ 0 \end{pmatrix}$$

draw the unique soluton to this IVP and use it to **predict** long term behavior of $\lim_{t\to\infty} x_1(t)$ and $\lim_{t\to\infty} x_2(t)$.

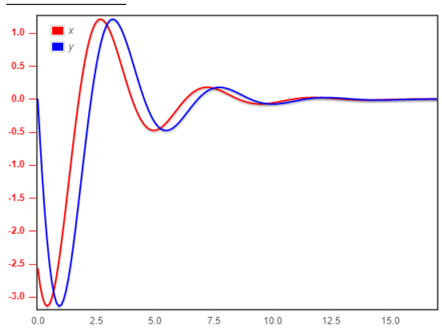
- **Solution:**
- 
- Hence we have

$$\lim_{t\to\infty} \boldsymbol{x}(t) = (0,0)\,.$$

so that

$$\lim_{t\to\infty} x_1(t) = 0,$$
$$\lim_{t\to\infty} x_2(t) = 0,$$

(c) Try to draw what the graphs of $x_1(t)$ and $x_2(t)$ are individually as functions of t, for the same IVP above.

- **Solution:**
- 

5.3. Problems

(1) Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

(a) Show that the two functions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

are solutions to the system above.

- **Solution:**
- We plug these functions into the LHS and RHS and check that their the same.
- Now we check $\mathbf{x}^{(1)}(t)$ and

$$\begin{aligned} \text{LHS} &= \frac{d\mathbf{x}^{(1)}}{dt} \\ &= \frac{d}{dt} \begin{pmatrix} 0 \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ e^t \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}^{(1)} \\ &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} 0+0 \\ 0+e^t \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ e^t \end{pmatrix} \end{aligned}$$

since LHS=RHS then $\mathbf{x}^{(1)}$ is a solution.

- First we check $\mathbf{x}^{(2)}(t)$ and

$$\begin{aligned} \text{LHS} &= \frac{d\mathbf{x}^{(2)}}{dt} \\ &= \frac{d}{dt} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}\text{RHS} &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}^{(2)} \\ &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} + 0 \\ e^{2t} + e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}\end{aligned}$$

since LHS=RHS that $\mathbf{x}^{(2)}(t)$ is a solution.

(b) Solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

- **Solution:**
- Using the general solution thereom we must check the solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent. That is, we check that

$$W\left[\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\right] = \begin{vmatrix} 0 & e^{2t} \\ e^{t} & e^{2t} \end{vmatrix} = -e^{3t} \neq 0$$

and since its not equal to zero then these are linearly independent solutions so that the general solution is given by

$$\begin{aligned}\mathbf{x}(t) &= c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} 0 \\ e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}\end{aligned}$$

using the initial conditions we have

$$\begin{aligned}\begin{pmatrix} -2 \\ -1 \end{pmatrix} &= \mathbf{x}(0) \\ &= c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_2 \\ c_1 + c_2 \end{pmatrix}\end{aligned}$$

hence we have to solve the system of equations

$$\begin{cases} c_2 = -2 \\ c_1 + c_2 = -1 \end{cases}$$

and get $c_1 = 1$ and $c_2 = -2$ so that

$$\mathbf{x}(t) = \begin{pmatrix} 0 \\ e^{t} \end{pmatrix} + 2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}.$$

(2) Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}$$

(a) Check that the two functions are solutions to the system. If they are not solutions, then stop and do not do parts (b) and (c).

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} 4e^{-3t} \\ 4e^{-3t} \end{pmatrix}$$

- **Solution:**
- We plug these functions into the LHS and RHS and check that their the same.
- Now we check $\mathbf{x}^{(1)}(t)$ and

$$\begin{aligned} \text{LHS} &= \frac{d\mathbf{x}^{(1)}}{dt} \\ &= \frac{d}{dt}\begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}^{(1)} \\ &= \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-3t} - e^{-3t} \\ 2e^{-3t} - 5e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix} \end{aligned}$$

since LHS=RHS then $\mathbf{x}^{(1)}$ is a solution.

- Similarly you'll see that $\mathbf{x}^{(2)}$ is also a solution.

(b) Are $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ linearly independent? If they are not, then stop and do not move on to part (c).

- **Solution:**
- By a theorem from the book, it is enough to check that $\mathbf{x}^{(1)}(0)$ and $\mathbf{x}^{(1)}(0)$ are linearly independent. Since

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{x}^{(2)}(0) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

- There is two ways of checking if these are linearly independent vectors. For one, we can compute the Wronskian (determinant) and note that

$$W[\mathbf{x}^{(1)}(0), \mathbf{x}^{(1)}(0)] = \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} = 0$$

hence it's not linearly independent.

- Or, we can see that these vectors DO lie in the same line, hence they are NOT linearly independent.
- We do not proceeed into the next question.

(c) Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}.$$

(3) Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}$$

(a) Check that the two functions are solutions to the system. If they are not solutions, then stop and do not do parts (b) and (c).

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} 2e^{-3t} + e^{-4t} \\ 2e^{-3t} + 2e^{-4t} \end{pmatrix}$$

- **Solution:**
- By plugging both functions into the LHS and RHS, you'll see that they are both solutions.

(b) Are $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ linearly independent? If they are not, then stop and do not move on to part (c).

- **Solution:**
- By a theorem from the book, it is enough to check that $\mathbf{x}^{(1)}(0)$ and $\mathbf{x}^{(1)}(0)$ are linearly independent. Since

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

and

$$\mathbf{x}^{(2)}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

- There is two ways of checking if these are linearly independent vectors. For one, we can compute the Wronskian (determinant) and check its not zero. Or, we can see that these vectors do not lie in the same line, hence they are linearly independent.

(c) Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}.$$

- **Solution:**
- By general solution theorem, since these we found two linearly independent solutions then the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-3t} + e^{-4t} \\ 2e^{-3t} + 2e^{-4t} \end{pmatrix}. \end{aligned}$$

5.4. Problems

(1) Find the general solution of the given system of differential equations:

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

- **Solution:**
- Step 1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \det \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} = 0 \\ &\iff (1-\lambda)(-4-\lambda) - (-2)\cdot 3 = 0. \\ &\iff \lambda^2 + 3\lambda - 4 + 6 = 0 \\ &\iff \lambda^2 + 3\lambda + 2 = 0 \\ &\iff (\lambda+1)(\lambda+2) = 0 \\ &\iff \lambda = -1, -2. \end{aligned}$$

- Step2: Find the eigenvectors.
- For $\lambda_1 = -1$ we have

$$\begin{aligned} A\mathbf{v} = -\mathbf{v} &\iff \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\iff \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 - 4x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\iff \begin{cases} x_1 - 2x_2 = -x_1 & \implies x_2 = x_1 \\ 3x_1 - 4x_2 = -x_2 \end{cases} \end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for any x_1. So choose $x_1 = 1$ and we pick the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(note that there are infinitely many possibilities for eigenvectors) for example we could have chosen the eigenvector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ since any multiple is also an eigenvector.

- For $\lambda_2 = -2$we have

$$\begin{aligned} A\mathbf{v} = -2\mathbf{v} &\iff \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\iff \begin{cases} x_1 - 2x_2 = -2x_1 \\ 3x_1 - 4x_2 = -2x_2 & \implies x_1 = \frac{2}{3}x_2 \end{cases} \end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x_2 \\ x_2 \end{pmatrix}$ for any x_2. So choose $x_1 = 3$ and we pick the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- Hence we have the following eigenvalues with their corresponding eigenvectors:

$$\lambda_1 = -1, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -2, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

- Hence the general solution is given by $\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \mathbf{v}_2$ so that the **general solution** is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}.$$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

- **Solution:**
- Step 1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 \quad &\Longleftrightarrow \quad \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} = 0 \\ &\Longleftrightarrow \quad (1-\lambda)(-2-\lambda) - 1 \cdot 4 = 0. \\ &\Longleftrightarrow \quad \lambda^2 + \lambda - 2 - 4 = 0 \\ &\Longleftrightarrow \quad \lambda^2 + \lambda - 6 = 0 \\ &\Longleftrightarrow \quad (\lambda - 2)(\lambda + 3) = 0 \\ &\Longleftrightarrow \quad \lambda = 2, -3. \end{aligned}$$

- Step2: Find the eigenvectors.
- For $\lambda_1 = 2$ we have

$$\begin{aligned} A\mathbf{v} = -\mathbf{v} \quad &\Longleftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\Longleftrightarrow \quad \begin{cases} x_1 + x_2 = 2x_1 \\ 4x_1 - 2x_2 = 2x_2 \end{cases} \Longrightarrow x_2 = x_1 \end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for any x_1. So choose $x_1 = 1$ and we pick the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- For $\lambda_2 = -3$we have

$$\begin{aligned} A\mathbf{v} = -\mathbf{v} \quad &\Longleftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\Longleftrightarrow \quad \begin{cases} x_1 + x_2 = -3x_1 \\ 4x_1 - 2x_2 = -3x_2 \end{cases} \Longrightarrow x_2 = -4x_1 \end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -4x_1 \end{pmatrix}$ for any x_1. So choose $x_1 = 1$ and we pick the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

- Hence we have the following eigenvalues with their corresponding eigenvectors:

$$\lambda_1 = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -3, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

- Hence the general solution is given by $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\mathbf{v}_2$ so that the **general solution** is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

(c) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \mathbf{x}$

- **Solution:**
- Step 1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 \quad &\Longleftrightarrow \quad \det \begin{pmatrix} 3-\lambda & 2 \\ 0 & -2-\lambda \end{pmatrix} = 0 \\ &\Longleftrightarrow \quad (3-\lambda)(-2-\lambda) - 2\cdot 0 = 0. \\ &\Longleftrightarrow \quad \lambda^2 - \lambda - 6 = 0 \\ &\Longleftrightarrow \quad (\lambda+2)(\lambda-3) = 0 \\ &\Longleftrightarrow \quad \lambda = -2, 3. \end{aligned}$$

- Step2: Find the eigenvectors.
- For $\lambda_1 = -2$ we have

$$\begin{aligned} A\mathbf{v} = -\mathbf{v} \quad &\Longleftrightarrow \quad \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\Longleftrightarrow \quad \begin{cases} 3x_1 + 2x_2 = -2x_1 & \Longrightarrow x_2 = \frac{-5}{2}x_1 \\ -2x_2 = -2x_2 \end{cases} \end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{-5}{2}x_1 \end{pmatrix}$ for any x_1. So choose $x_1 = 2$ and we pick the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

(note that there are infinitely many possibilities for eigenvectors)

- For $\lambda_2 = 3$we have

$$
\begin{aligned}
A\mathbf{v} = -\mathbf{v} \quad &\iff \quad \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\iff \quad \begin{cases} 3x_1 + 2x_2 = 3x_1 & \Longrightarrow x_2 = 0 \\ -2x_2 = 3x_2 & \Longrightarrow x_2 = 0 \end{cases}
\end{aligned}
$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ for any x_1. So choose $x_1 = 1$ and we pick the eigenvector

$$
\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

- Hence we have the following eigenvalues with their corresponding eigenvectors:

$$
\begin{aligned}
\lambda_1 = -2, \qquad & \mathbf{v}_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix} \\
\lambda_2 = 3, \qquad & \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}
\end{aligned}
$$

- Hence the general solution is given by $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\mathbf{v}_2$ so that the **general solution** is

$$
\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}.
$$

(d) The system

$$
\begin{aligned}
x_1' &= 3x_1 + 4x_2 \\
x_2' &= x_1
\end{aligned}
$$

- **Solution:**
- Converting this system into matrix-vector notation we have

$$
\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} \mathbf{x}
$$

- Step 1: Find the eigenvalues:

$$
\begin{aligned}
\det(A - \lambda I) = 0 \quad &\iff \quad \det \begin{pmatrix} 3-\lambda & 4 \\ 1 & 0-\lambda \end{pmatrix} = 0 \\
&\iff \quad (3-\lambda)(-\lambda) - 4 \cdot 1 = 0. \\
&\iff \quad \lambda^2 - 3\lambda - 4 = 0 \\
&\iff \quad (\lambda - 4)(\lambda + 1) = 0 \\
&\iff \quad \lambda = 4, -1.
\end{aligned}
$$

- Step2: Find the eigenvectors.

- For $\lambda_1 = 4$ we have

$$A\mathbf{v} = -\mathbf{v} \quad \Longleftrightarrow \quad \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\Longleftrightarrow \quad \begin{cases} 3x_1 + 4x_2 = 4x_1 \\ x_1 = 4x_2 \end{cases} \quad \Longrightarrow \; x_1 = 4x_2$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_2 \\ x_2 \end{pmatrix}$ for any x_2. So choose $x_2 = 1$ and we pick the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

(note that there are infinitely many possibilities for eigenvectors)

- For $\lambda_2 = -1$ we have

$$A\mathbf{v} = -\mathbf{v} \quad \Longleftrightarrow \quad \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\Longleftrightarrow \quad \begin{cases} 3x_1 + 4x_2 = -x_1 \\ x_1 = -x_2 \end{cases} \quad \Longrightarrow \; x_1 = -x_2$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix}$ for any x_2. So choose $x_2 = 1$ and we pick the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- Hence we have the following eigenvalues with their corresponding eigenvectors:

$$\lambda_1 = 4, \qquad \mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -1, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

- Hence the general solution is given by $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\mathbf{v}_2$ so that the **general solution** is

$$\mathbf{x} = c_1 e^{4t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- Simplifying we have

$$\begin{aligned} \mathbf{x} &= c_1 e^{4t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 4c_1 e^{4t} \\ c_1 e^{4t} \end{pmatrix} + \begin{pmatrix} -c_2 e^{-t} \\ c_2 e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} 4c_1 e^{4t} - c_2 e^{-t} \\ c_1 e^{4t} + c_2 e^{-t} \end{pmatrix} \end{aligned}$$

hence

$$x_1(t) = 4c_1e^{4t} - c_2e^{-t},$$
$$x_2(t) = c_1e^{4t} + c_2e^{-t}.$$

(2) Solve the following initial value problems and find $x_1(t)$ and $x_2(t)$.

(a) Where the IVP is given by

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\mathbf{x} \quad , \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- **Solution:**
- Step 1: Find the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) = 0 \quad &\iff \quad \det\begin{pmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{pmatrix} = 0 \\
&\iff \quad (-4-\lambda)(-3-\lambda) - 1 \cdot 2 = 0. \\
&\iff \quad \lambda^2 + 7\lambda + 12 - 2 = 0 \\
&\iff \quad \lambda^2 + 7\lambda + 10 = 0 \\
&\iff \quad (\lambda+2)(\lambda+5) = 0 \\
&\iff \quad \lambda = -2, -5.
\end{aligned}$$

- Step2: Find the eigenvectors.
- For $\lambda_1 = -2$ we have

$$\begin{aligned}
A\mathbf{v} = -\mathbf{v} \quad &\iff \quad \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\iff \quad \begin{cases} -4x_1 + x_2 = -2x_1 & \implies x_2 = 2x_1 \\ 2x_1 - 3x_2 = -2x_2 \end{cases}
\end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix}$ for any x_1. So choose $x_2 = 1$ and we pick the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- For $\lambda_2 = -5$ we have

$$\begin{aligned}
A\mathbf{v} = -\mathbf{v} \quad &\iff \quad \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\iff \quad \begin{cases} -4x_1 + x_2 = -5x_1 & \implies x_2 = -x_1 \\ 2x_1 - 3x_2 = -5x_2 \end{cases}
\end{aligned}$$

hence the eigenvectors are of them form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$ for any x_2. So choose $x_2 = 1$ and we pick the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Hence we have the following eigenvalues with their corresponding eigenvectors:

$$\lambda_1 = -2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\lambda_2 = -5, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Hence the general solution is given by $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\mathbf{v}_2$ so that the **general solution** is

$$\mathbf{x} = c_1 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Step 3: Find the c_1, c_2 using the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so that

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{x}(0) &= c_1 e^0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 + c_2 \\ 2c_1 - c_2 \end{pmatrix} \end{aligned}$$

this means we must solve

$$\begin{cases} c_1 + c_2 = 1 \\ 2c_1 - c_2 = 0 \end{cases}$$

which means $c_1 = \frac{1}{3}$ and $c_2 = \frac{2}{3}$ so that

$$\mathbf{x}(t) = \frac{1}{3}e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{3}e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Now simplifying so that we can extract x_1 and x_2 separately we have

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{3}e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{3}e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}e^{-2t} \\ \frac{2}{3}e^{-2t} \end{pmatrix} + \begin{pmatrix} \frac{2}{3}e^{-5t} \\ -\frac{2}{3}e^{-5t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}e^{-2t} + \frac{2}{3}e^{-5t} \\ \frac{2}{3}e^{-2t} - \frac{2}{3}e^{-5t} \end{pmatrix} \end{aligned}$$

so that

$$x_1(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{-5t}$$

and

$$x_2(t) = \frac{2}{3}e^{-2t} - \frac{2}{3}e^{-5t}.$$

(b) Where the IVP is given by

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\mathbf{x} \quad , \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- **Solution:**

- Recall from the previous example, that the **general solution** is

$$\mathbf{x} = c_1 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Find the c_1, c_2 using the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so that

$$\begin{aligned}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{x}(0) &= c_1 e^0\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^0\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 + c_2 \\ 2c_1 - c_2 \end{pmatrix}\end{aligned}$$

  this means we must solve

$$\begin{cases} c_1 + c_2 = 1 \\ 2c_1 - c_2 = 2 \end{cases}$$

  which means $c_1 = 1$ and $c_2 = 0$ so that

$$\mathbf{x}(t) = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \cdot e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Now simplifying so that we can extract x_1 and x_2 separately we have

$$\begin{aligned}\mathbf{x}(t) &= e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}\end{aligned}$$

  so that

$$x_1(t) = e^{-2t}$$

  and

$$x_2(t) = 2e^{-2t}.$$

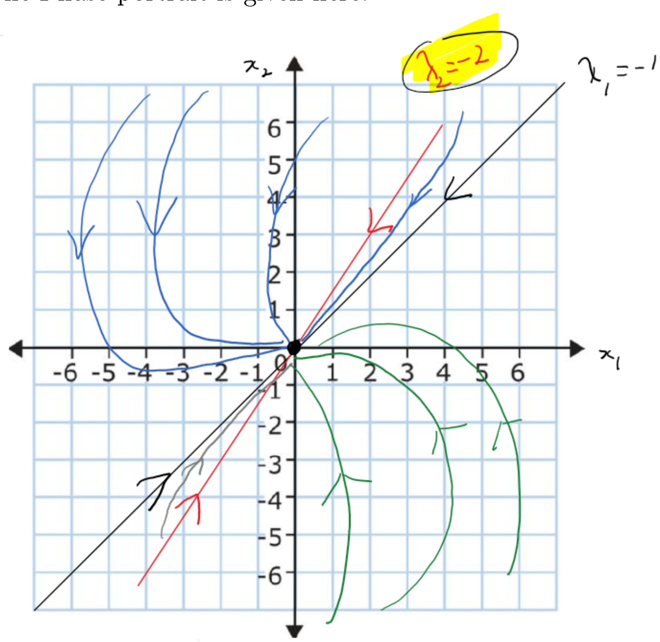
5.5. Problems

(1) Sketch the Phase portrait for the following systems and classify the equilibrium solution for the following systems.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$ and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = -1, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -2, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

- **Solution:**
- The equilibrium solution (the dot in the origin is the equilibrium solution) is classified as a **assymtotically stable** (**sink**) since both eigenvalues are negative.
- The Phase portrait is given here:



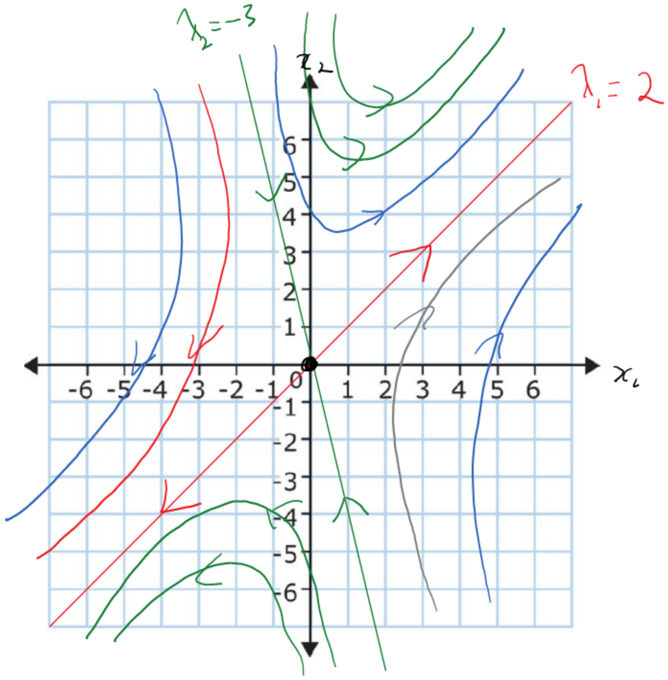
-
- Note that since λ_2 has the biggest magnitude, then all other solution come from the general direction related to the straight line solution related to λ_2

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -3, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

- **Solution:**
- The equilibrium solution (the dot in the origin is the only equilibrium solution) is classified as a **saddle** since the eigenvalues have different signs.

- The Phase portrait is given here:



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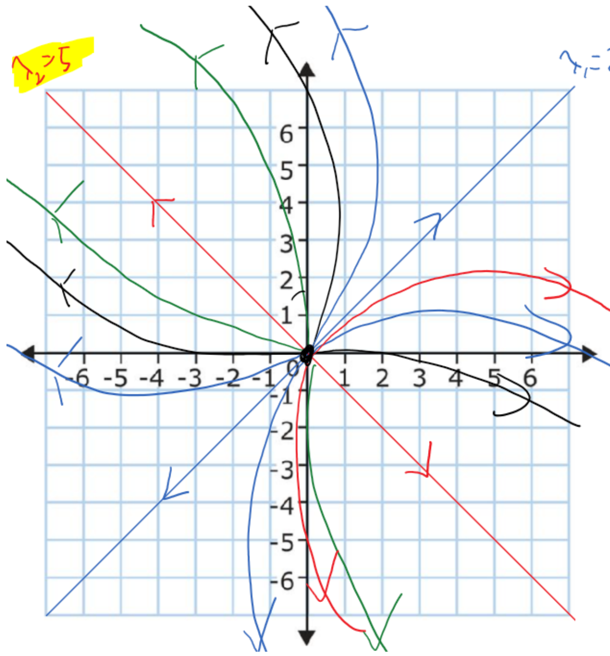
(c) $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = 5, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- **Solution:**
- The equilibrium solution (the dot in the origin is the only equilibrium solution) is classified as a **assymtotically unstable** (**source**) since both eigenvalues are positive.
- The Phase portrait is given here:



-
- Note that since λ_2 has the biggest magnitude, then all other solutions are goiung in the general direction related to the straight line solution related to λ_2

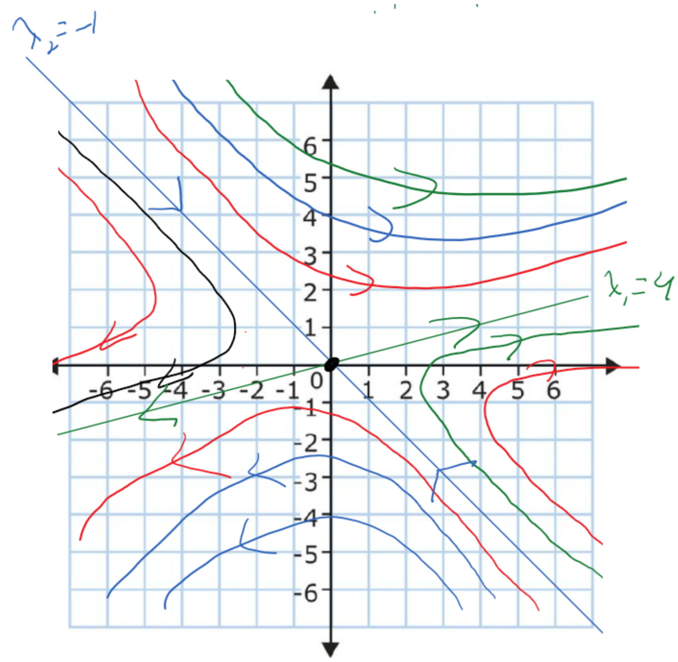
(d) The system

$$x_1' = 3x_1 + 4x_2$$
$$x_2' = x_1$$

and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = 4, \qquad \mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -1, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

- **Solution:**
- The equilibrium solution (the dot in the origin is the equilibrium solution) is classified as a **saddle** since the eigenvalues have different signs.
- The Phase portrait is given here:



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(2) Suppose A is a matrix and consider the following system

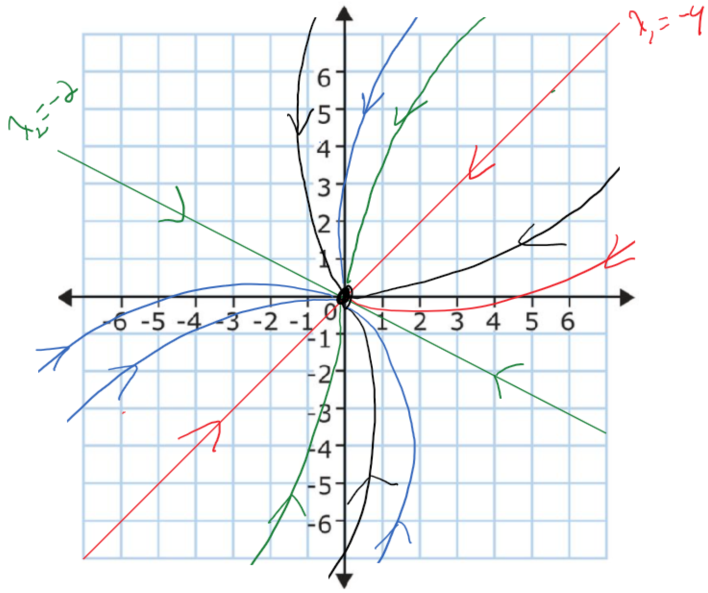
$$\mathbf{x}' = A\mathbf{x}.$$

(a) Suppose the matrix A has the following associated eigenvalues with corresponding eigenvectors

$$\begin{aligned} \lambda_1 &= -4, & \mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 &= -2, & \mathbf{v}_2 &= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{aligned}.$$

Draw the Phase portrait for $\mathbf{x}' = A\mathbf{x}$.

- **Solution:**
- The Phase portrait is given here:

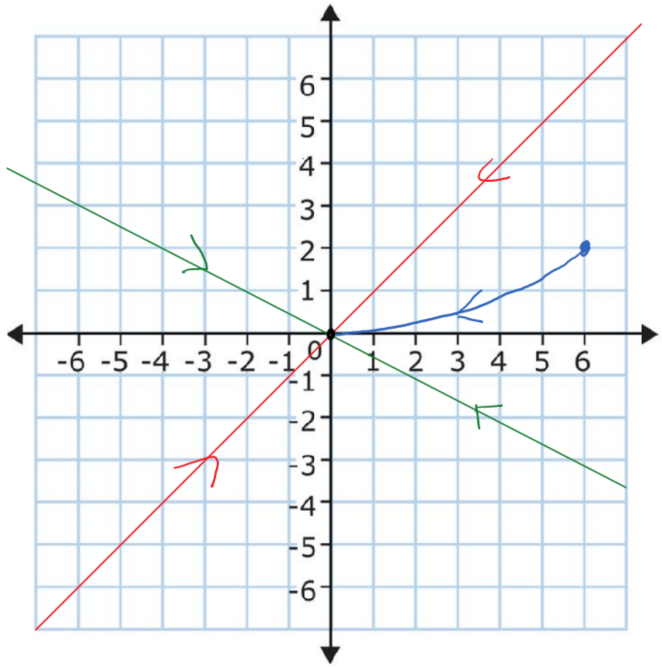


-
- Note that since λ_1 has the biggest magnitude, then all other solution come from the general direction related to the straight line solution related to λ_1

(b) Consider the same matrix as in Part (a). Draw the trajectory curve for $t \geq 0$ of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

- **Solution:**
- The solution to the IVP $\mathbf{x}$ is drawn in **blue** the following graph. The straight-line solutions are there just for reference. The **blue** curve is the actual solution of the IVP.



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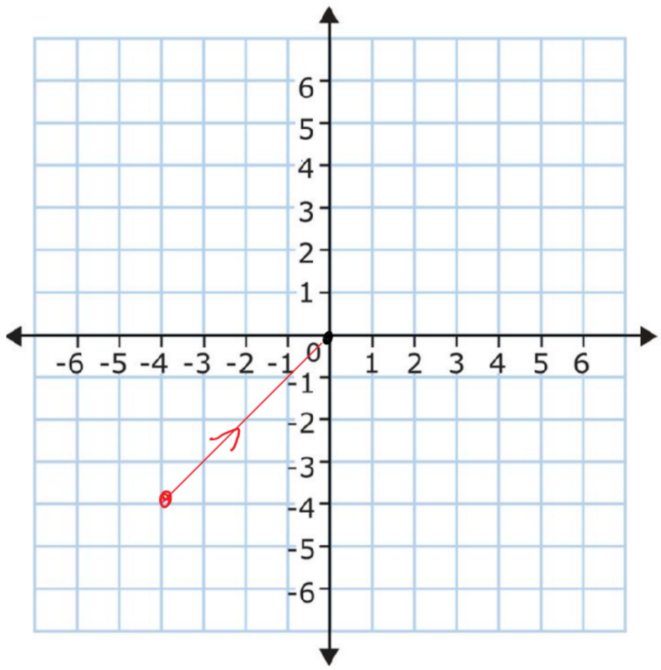
(c) Consider the same matrix as in Part (a). Draw the trajectory curve of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

- **Solution:**
- The solution to the IVP $\mathbf{x}$ is drawn in **RED** the following graph. Note that since the initial point $(-4, -4)$ is located where the straight-line solution is locarted, then its simply the straight-line solution.



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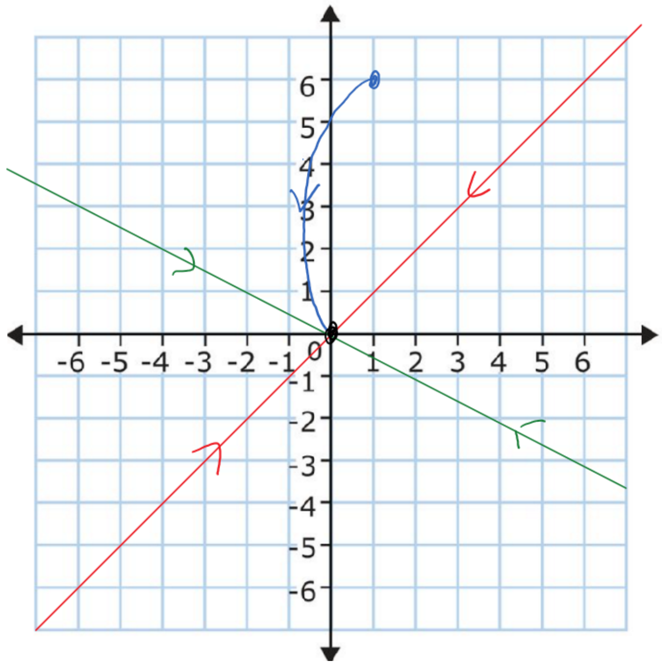
(d) Consider the same matrix as in Part (a). Draw the trajectory curve of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

- **Solution:**
- The solution to the IVP $\mathbf{x}$ is drawn in **blue** the following graph. The straight-line solutions are there just for reference. The **blue** curve is the actual solution of the IVP.



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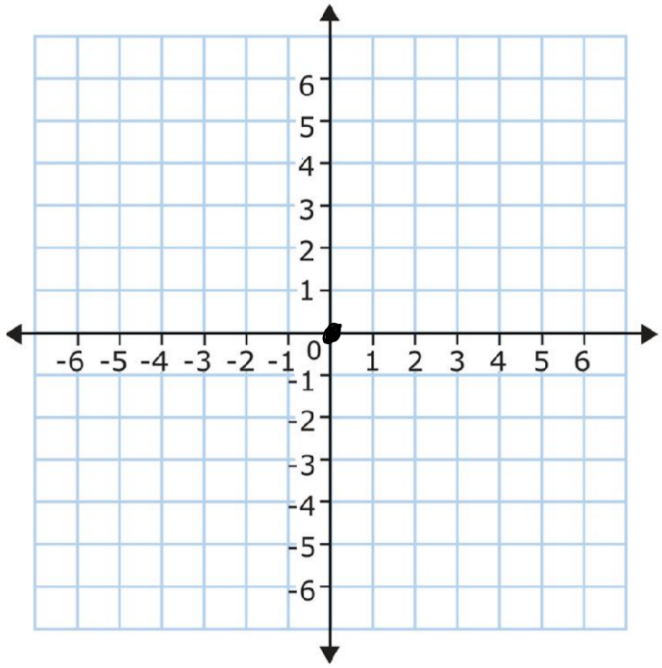
(e) Consider the same matrix as in Part (a). Draw the trajectory curve of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- **Solution:**
- The solution to the IVP $\mathbf{x}$ IS the equilibrium solution. Since it's the equilibrium solution, the point is graphed as a dot, because it stays constant over time..


6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-6 -5 -4 -3 -2 -1
1 2 3 4 5 6

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5.6. Problems

(1) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 0 & 9 \\ -9 & 0 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution

- **Solution:**
- Step1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \begin{vmatrix} -\lambda & 9 \\ -9 & -\lambda \end{vmatrix} = 0 \\ &\iff \lambda^2 + 81 = 0 \\ &\iff \lambda = \pm 9i. \end{aligned}$$

- Step2: Pick the eigenvalue $\lambda = 9i$ and find a corresponding eigenvector:

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\iff \begin{pmatrix} 0 & 9 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 9i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\iff \begin{cases} 9x_2 = 9ix_1 & \Longrightarrow x_2 = ix_1 \\ -9x_1 = 9ix_2 \end{cases} \end{aligned}$$

hence the eigenvectors are of the form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ ix_1 \end{pmatrix}$ for any x_1. By choosing $x_1 = 1$ we can pick the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

- Step3: Find the **complex solution** $\mathbf{x}_c = \mathbf{x}_{re} + i\mathbf{x}_{im}$ for this eigenvalue and eigenvector

$$\begin{aligned} \mathbf{x}_c &= e^{\lambda t}\mathbf{v} \\ &= e^{(9t)i} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= (\cos(9t) + i\sin(9t)) \begin{pmatrix} 1 \\ i \end{pmatrix}, \text{ by Euler's Formula} \\ &= \begin{pmatrix} \cos(9t) + i\sin(9t) \\ i\cos(9t) + i^2\sin(9t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(9t) + i\sin(9t) \\ i\cos(9t) - \sin(9t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(9t) + i\sin(9t) \\ -\sin(9t) + i\cos(9t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(9t) \\ -\sin(9t) \end{pmatrix} + i \begin{pmatrix} \sin(9t) \\ \cos(9t) \end{pmatrix} \\ &= \mathbf{x}_{re} + i\mathbf{x}_{im} \end{aligned}$$

- Taking the real and imaginary parts, we obtain the general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(9t) \\ -\sin(9t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(9t) \\ \cos(9t) \end{pmatrix}.$$

(b) Solve the IVP:

$$\mathbf{x}' = \begin{pmatrix} 0 & 9 \\ -9 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and find $x_1(t)$ and $x_2(t)$.

- **Solution:**
- We use the initial condition and get

$$\begin{aligned} \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= \mathbf{x}(0) \\ &= c_1 \begin{pmatrix} \cos(0) \\ -\sin(0) \end{pmatrix} + c_2 \begin{pmatrix} \sin(0) \\ \cos(0) \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

  hence $c_1 = 1$ and $c_2 = 2$.
- Thus the particular solution is

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} \cos(9t) \\ -\sin(9t) \end{pmatrix} + 2 \begin{pmatrix} \sin(9t) \\ \cos(9t) \end{pmatrix} \\ &= \begin{pmatrix} \cos(9t) + 2\sin(9t) \\ -\sin(9t) + 2\cos(9t) \end{pmatrix} \end{aligned}$$

  hence

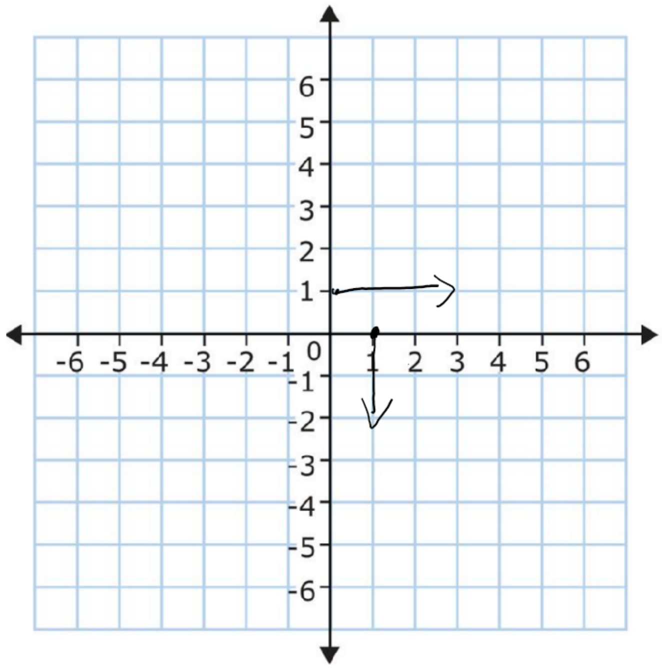
$$\begin{aligned} x_1(t) &= \cos(9t) + 2\sin(9t), \\ x_2(t) &= -\sin(9t) + 2\cos(9t). \end{aligned}$$

(c) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)

- **Solution:**
- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$\begin{aligned} A \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ -9 \end{pmatrix}, \\ A \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 9 \\ 0 \end{pmatrix} \end{aligned}$$

  hence the spirals **clockwise**.
- See here:



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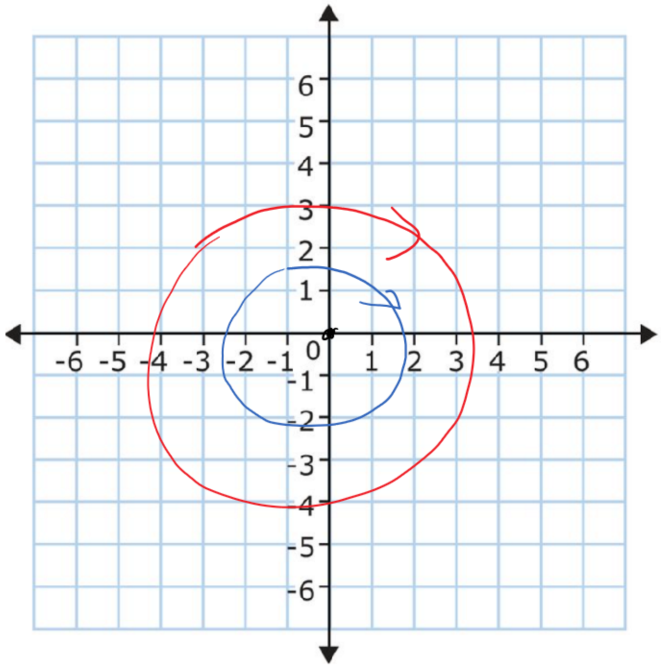
(d) Classify the Equilibrium solution
- **Solution:**
- The eigenvalues are

$$\lambda = \pm 9i.$$

since $\alpha = 0$, then this is called an **center.**

(e) Draw the Phase Portrait
- **Solution:**
- Plotting the Phase portrait, we draw elliptical shapes going counterclockwise:



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(2) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution

- **Solution:**
- Step1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0 \\ &\iff (-1-\lambda)^2 + 4 = 0 \\ &\iff (-1-\lambda)^2 = -4 \\ &\iff -1-\lambda = \pm 2i \\ &\iff \lambda = -1 \pm 2i. \end{aligned}$$

- Step2: Pick the eigenvalue $\lambda = -1 + 2i$ and find a corresponding eigenvector:

$$\begin{aligned} & A\mathbf{v} = \lambda \mathbf{v} \\ \iff & \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-1+2i) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \iff & \begin{cases} 9x_1 - 4x_2 = (-1+2i)\, x_1 \\ x_1 - x_2 = (-1+2i)\, x_2 \end{cases} \implies x_1 = (2i)\, x_2 \end{aligned}$$

hence the eigenvectors are of the form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (2i)\,x_2 \\ x_2 \end{pmatrix}$ for any x_2. By choosing $x_2 = 1$ we can pick the eigenvector

$$\mathbf{v} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

- Step3: Find the **complex solution** $\mathbf{x}_c = \mathbf{x}_{re} + i\mathbf{x}_{im}$ for this eigenvalue and eigenvector

$$\begin{aligned}
\mathbf{x}_c &= e^{\lambda t}\mathbf{v} \\
&= e^{(-1+2i)t}\begin{pmatrix} 2i \\ 1 \end{pmatrix} \\
&= e^{-t}e^{(2t)i}\begin{pmatrix} 2i \\ 1 \end{pmatrix} \\
&= e^{-t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 2i \\ 1 \end{pmatrix}, \text{ by Euler's Formula} \\
&= e^{-t}\begin{pmatrix} 2i\cos(2t) + 2i^2\sin(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} 2i\cos(2t) - 2\sin(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} -2\sin(2t) + 2i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{pmatrix} \\
&= e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + ie^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} \\
&= \mathbf{x}_{re} + i\mathbf{x}_{im}
\end{aligned}$$

- Taking the real and imaginary parts, we obtain the general solution

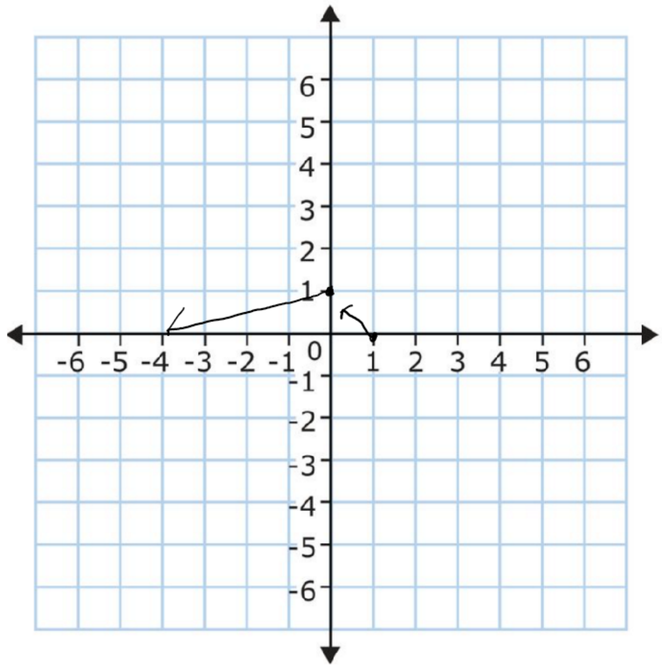
$$\mathbf{x}(t) = c_1e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}.$$

(b) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)

- **Solution:**
- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

hence the spirals go **counter-clockwise**.



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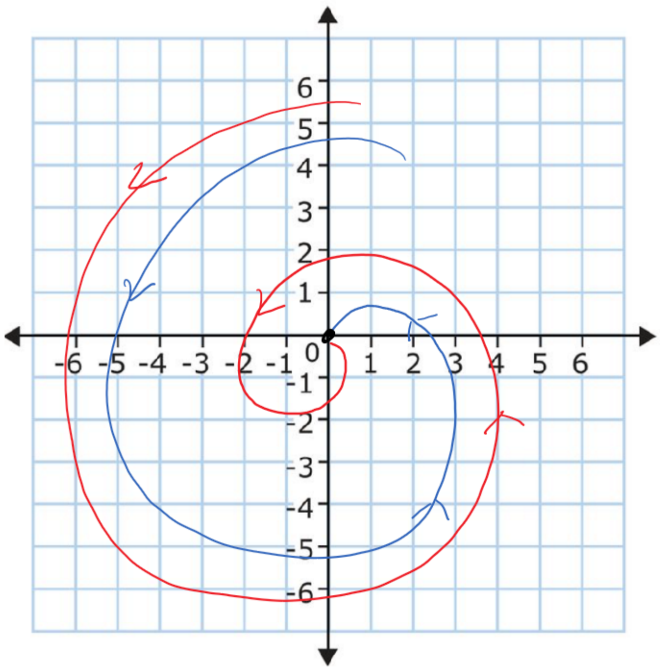
(c) Classify the Equilibrium solution
- **Solution:**
- Recall that the eigenvalues are

$$\lambda = -1 \pm 2i.$$

since $\alpha < 0$, then this is called an **asymptotically stable spiral.**

(d) Draw the Phase Portrait
- **Solution:**
- Plotting the Phase portrait, we draw spirals going counter-clockwise going towards the origin:



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(3) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution

- **Solution:**
- Step1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0 \\ &\iff (2-\lambda)(-2-\lambda) + 5 = 0 \\ &\iff \lambda^2 - 4 + 5 = 0 \\ &\iff \lambda^2 = -1 \\ &\iff \lambda = \pm i \end{aligned}$$

- Step2: Pick the eigenvalue $\lambda = i$ and find a corresponding eigenvector:

$$\begin{aligned} & A\mathbf{v} = \lambda \mathbf{v} \\ \iff & \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \iff & \begin{cases} 2x_1 - 5x_2 = ix_1 \\ x_1 - 2x_2 = ix_2 \end{cases} \implies x_1 = (2+i)\, x_2 \end{aligned}$$

hence the eigenvectors are of the form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (2+i)\,x_2 \\ x_2 \end{pmatrix}$ for any x_2. By choosing $x_2 = 1$ we can pick the eigenvector

$$\mathbf{v} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$$

- Step3: Find the **complex solution** $\mathbf{x}_c = \mathbf{x}_{re} + i\mathbf{x}_{im}$ for this eigenvalue and eigenvector

$$\begin{aligned}
\mathbf{x}_c &= e^{\lambda t}\mathbf{v} \\
&= e^{it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= (\cos(t) + i\sin(t))\begin{pmatrix} 2+i \\ 1 \end{pmatrix}, \text{ by Euler's Formula} \\
&= \begin{pmatrix} (\cos(t) + i\sin(t))(2+i) \\ \cos(t) + i\sin(t) \end{pmatrix} \\
&= \begin{pmatrix} 2\cos(t) + 2i\sin(t) + i\cos(t) + i^2\sin(t) \\ \cos(t) + i\sin(t) \end{pmatrix} \\
&= \begin{pmatrix} 2\cos(t) + 2i\sin(t) + i\cos(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{pmatrix} \\
&= \begin{pmatrix} 2\cos(t) - \sin(t) + i(2\sin(t) + \cos(t)) \\ \cos(t) + i\sin(t) \end{pmatrix} \\
&= \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} \\
&= \mathbf{x}_{re} + i\mathbf{x}_{im}
\end{aligned}$$

- Taking the real and imaginary parts, we obtain the general solution

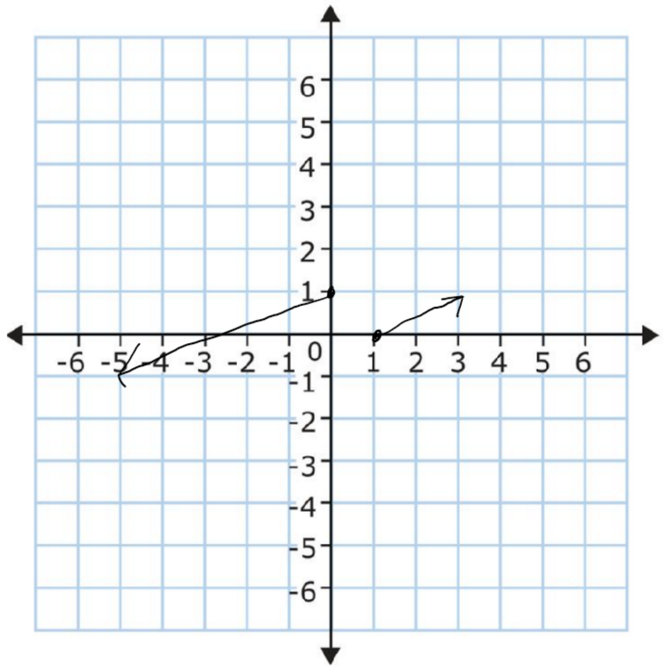
$$\mathbf{x}(t) = c_1\begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + c_2\begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix}.$$

(b) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)

- **Solution:**
- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

hence the oscillations are going **counter-clockwise**.



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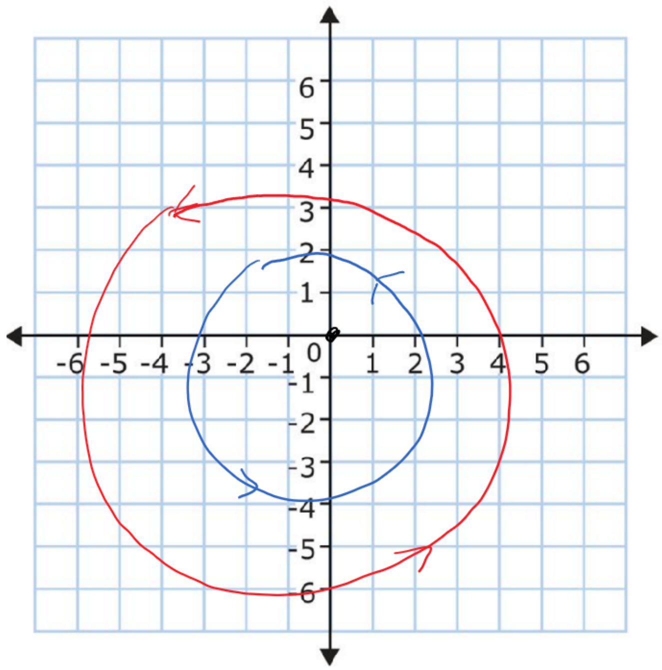
(c) Classify the Equilibrium solution
- **Solution:**
- Recall that the eigenvalues are

$$\lambda = \pm i.$$

since $\alpha = 0$, then this is called an **center.**

(d) Draw the Phase Portrait
- **Solution:**
- Plotting the Phase portrait, we draw elliptical shapes going counter-clockwise:



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(4) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution

- **Solution:**
- Step1: Find the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = 0 \\ &\iff (1-\lambda)(-3-\lambda) + 5 = 0 \\ &\iff \lambda^2 + 2\lambda - 3 + 5 = 0 \\ &\iff \lambda^2 + 2\lambda + 2 = 0 \\ &\iff \lambda^2 + 2\lambda + 1 + 1 = 0 \\ &\iff (\lambda+1)^2 = -1 \\ &\iff \lambda = -1 \pm i \end{aligned}$$

- Step2: Pick the eigenvalue $\lambda = -1 + i$ and find a corresponding eigenvector:

$$A\mathbf{v} = \lambda \mathbf{v}$$

$$\iff \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-1+i) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\iff \begin{cases} x_1 - x_2 = (-1+i)\,x_1 & \text{pick this equation} \\ 5x_1 - 3x_2 = (-1+i)\,x_2 \end{cases}$$

then

$$\begin{aligned} x_1 - x_2 = (-1+i)\,x_1 &\iff x_2 = x_1 - (-1+i)\,x_1 \\ &\iff x_2 = x_1 + (1-i)\,x_1 \\ &\iff x_2 = (2-i)\,x_1 \end{aligned}$$

hence the eigenvectors are of the form $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ (2-i)\,x_1 \end{pmatrix}$ for any x_1. By choosing $x_2 = 1$ we can pick the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}.$$

- Step3: Find the **complex solution** $\mathbf{x}_c = \mathbf{x}_{re} + i\mathbf{x}_{im}$ for this eigenvalue and eigenvector

$$\begin{aligned} \mathbf{x}_c &= e^{\lambda t}\mathbf{v} \\ &= e^{(-1+i)t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} \\ &= e^{-t}e^{(t)i}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} \\ &= e^{-t}\left(\cos(t) + i\sin(t)\right)\begin{pmatrix} 1 \\ 2-i \end{pmatrix} \text{ by Euler's Formula} \\ &= e^{-t}\begin{pmatrix} \cos(t) + i\sin(t) \\ (\cos(t) + i\sin(t))(2-i) \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos(t) + i\sin(t) \\ 2\cos(t) + i2\sin(t) - i\cos(t) - i^2\sin(t) \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos(t) + i\sin(t) \\ 2\cos(t) + i2\sin(t) - i\cos(t) + \sin(t) \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos(t) + i\sin(t) \\ 2\cos(t) + \sin(t) + i\left(2\sin(t) - \cos(t)\right) \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix} \\ &= \mathbf{x}_{re} + i\mathbf{x}_{im} \end{aligned}$$

- Taking the real and imaginary parts, we obtain the general solution

$$\mathbf{x}(t) = c_1e^{-t}\begin{pmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + c_2e^{-t}\begin{pmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}.$$

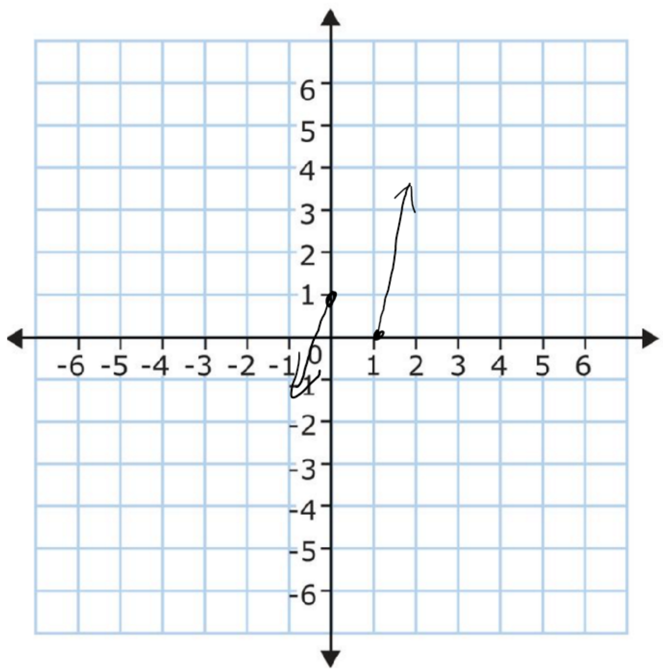
(b) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)

- **Solution:**

- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

hence the oscillations are going **counter-clockwise**.



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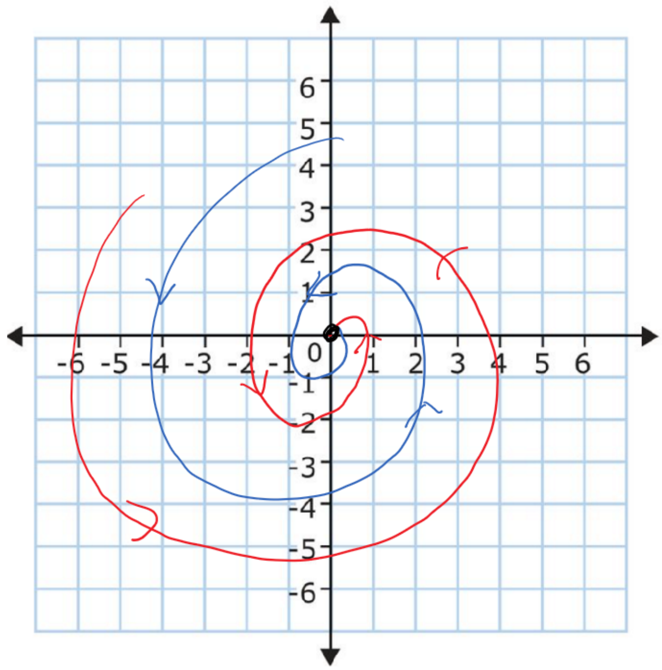
(c) Classify the Equilibrium solution
- **Solution:**
- Recall that the eigenvalues are

$$\lambda = -1 \pm i.$$

since $\alpha < 0$, then this is called an **asymptotically stable spiral.**

(d) Draw the Phase Portrait
- **Solution:**
- Plotting the Phase portrait, we draw spirals going counter-clockwise going towards the origin:



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(5) Draw the Phase portrait of the following system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x},$$

and classify the equilibrium solution.

- **Solution:**
- First we find the eigenvalues and get

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

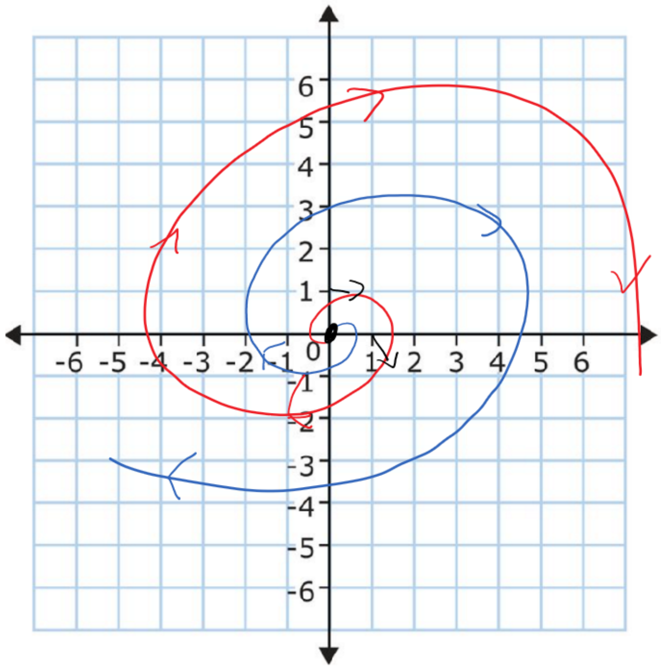
since $\alpha > 0$, then this is called an **asymptotically unstable spiral.**

- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

hence the spirals **clockwise**.

- Plotting the Phase portrait, we draw spirals going clockwise coming from the origin:



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(6) Draw the Phase portrait of the following system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{x}.$$

- **Solution:**
- First we find the eigenvalues and get

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i.$$

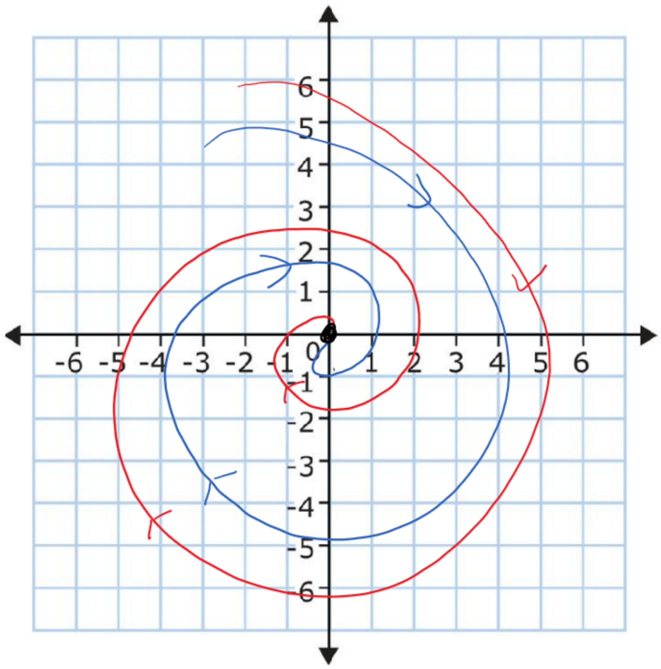
since $\alpha < 0$, then this is called an **asymptotically stable spiral.**

- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$
$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

hence the spirals **clockwise**.

- Plotting the Phase portrait, we draw spirals going clockwise going towards the origin:


6
5
4
3
2
1
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6
-1
-2
-3
-4
-5
-6

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5.7. Problems

(1) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution
- **Solution:**
- The characterisctic equation is

$$(-2-\lambda)^2 = 0$$

hence $\lambda = -2, -2$
- Then

$$\begin{aligned} \mathbf{v}_0 &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \text{ initial condition} \\ \mathbf{v}_1 &= (A - \lambda I)\,\mathbf{v}_0 \\ &= (A + 2I)\,\mathbf{v}_0 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ x_0 \end{pmatrix} \end{aligned}$$

- The general solution is given by

$$\begin{aligned} \mathbf{x}(t) &= e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1 \\ &= e^{-2t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t}\begin{pmatrix} 0 \\ x_0 \end{pmatrix} \end{aligned}$$

(b) Solve the IVP

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- **Solution:**
- Plugging in the intial condition we have

$$\begin{aligned} \mathbf{x}(t) &= e^{-2t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t}\begin{pmatrix} 0 \\ x_0 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + te^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

(c) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
- **Solution:**

- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$
$$A\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

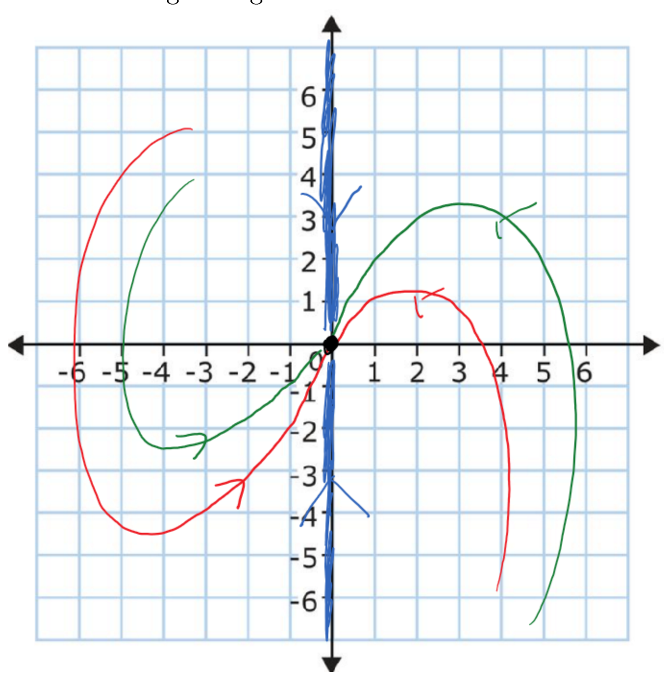
hence the oscilattions will be **counter-clockwise**.

(d) Classify the Equilibrium solution
- **Solution:**
  - Since this is a real repeates case with eigenvalues

$$\lambda = -2, -2.$$

then this is called an **asymptotically stable improper node**, or **an almost spiral.**

(e) Draw the Phase Portrait
- **Solution:**
- We know $\mathbf{v}_1 = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$ is an eigenvector, say we choose $x_0 = 1$ then $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then drawing this straight-line solution, and drawing counter-clockwise almost spirals going towards the origin we get:



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(2) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution
- **Solution:**
- The characterisctic equation is

$$(-2-\lambda)(-4-\lambda)+1 = (\lambda+3)^2 = 0$$

hence $\lambda = -3, -3$. This gives real repeated eigenvalues
- Then

$$\begin{aligned}
\mathbf{v}_0 &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \text{ initial condition} \\
\mathbf{v}_1 &= (A - \lambda I)\,\mathbf{v}_0 \\
&= (A + 3I)\,\mathbf{v}_0 \\
&= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\
&= \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}
\end{aligned}$$

- The general solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1 \\
&= e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t}\begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}
\end{aligned}$$

(b) Solve the IVP

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

- **Solution:**
- Plugging in the intial condition we have

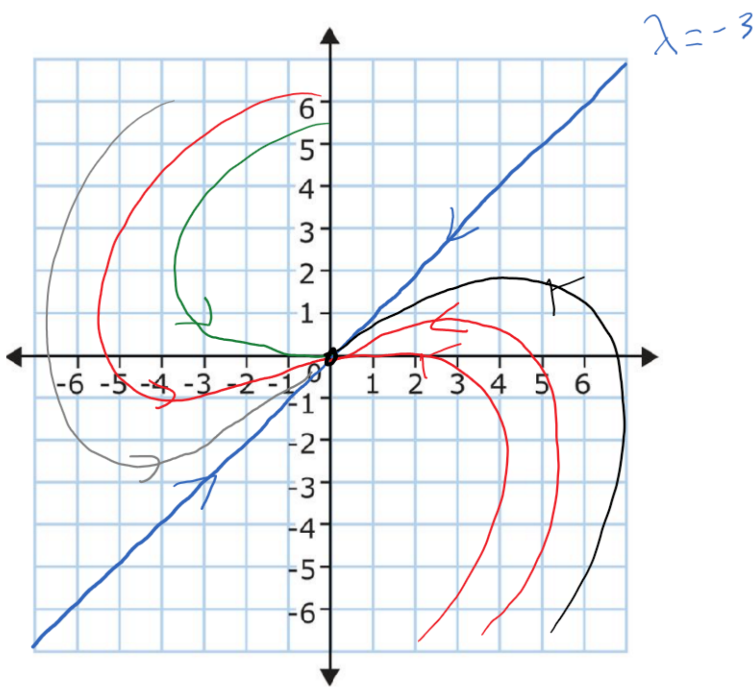
$$\begin{aligned}
\mathbf{x}(t) &= e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t}\begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix} \\
&= e^{-3t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + te^{-3t}\begin{pmatrix} -1-3 \\ -1-3 \end{pmatrix} \\
&= e^{-3t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + te^{-3t}\begin{pmatrix} -4 \\ -4 \end{pmatrix}
\end{aligned}$$

(c) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
- **Solution:**

- By testing at $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that the vector field at those points would have the following directions:
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix},$$
hence the oscilattions will be **counter-clockwise**.

(d) Classify the Equilibrium solution
- **Solution:**
- Since this is a real repeates case with eigenvalues
$$\lambda = -3, -3.$$
then this is called an **asymptotically stable improper node**, or **an almost spiral.**

(e) Draw the Phase Portrait
- **Solution:**
- We know $\mathbf{v}_1 = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$ is an eigenvector. Then drawing this straight-line solution, and also drawing counter-clockwise almost spirals going towards the origin we get:



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(3) Consider the following system
$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}\mathbf{x}$$

(a) Find the general solution
- **Solution:**

- The characteristic equation is

$$\lambda^2 + \lambda = 0$$

hence $\lambda = 0, 1$.

- We find the the corresponding eigenvectors are

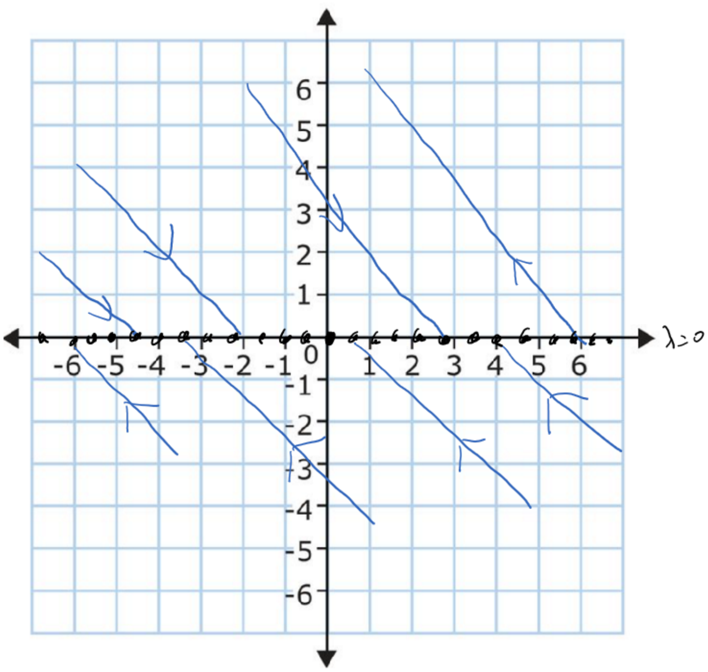
$$\lambda_1 = 0 \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = -1 \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

hence

$$\mathbf{x}(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(b) Draw the Phase Portrait

- **Solution:**
- Note that since $\det A = 0$ (which always happens when one of the eigenvalues are zero) then the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to a whole line of equilibrium solutions. The rest are straight-line solutions going into the equilibrium solutions in the direction of $\mathbf{v}_2$:



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(4) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution

- **Solution:**
- The characteristic equation is

$$\lambda^2 - 5\lambda = 0$$

hence $\lambda = 0, 5$.

- We find the the corresponding eigenvectors are

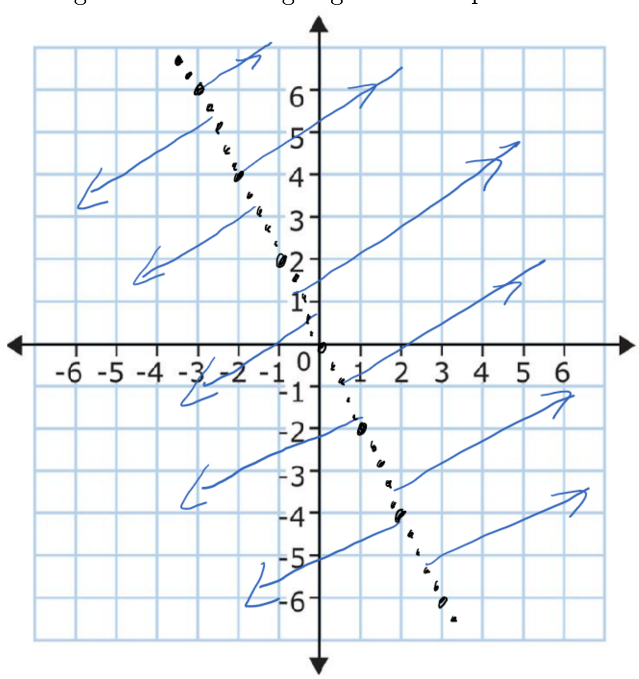
$$\lambda_1 = 0 \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$\lambda_2 = 5 \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

hence

$$\mathbf{x}(t) = c_1 e^{0t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) Draw the Phase Portrait

- **Solution:**
- Note that since $\det A = 0$ (which always happens when one of the eigenvalues are zero) then the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ corresponds to a whole line of equilibrium solutions. The rest are straight-line solutions going into the equilibrium solutions in the direction of $\mathbf{v}_2$:



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CHAPTER 6

The Laplace Transform

6.1. Problems

(1) Use the definition of Laplace transform to find the Laplace transform of $f(t) = 1$. That is, find $\mathcal{L}\{1\}$.

- **Solution:**
- We compute $\mathcal{L}\{f(t)\}$ with $f(t) = 1$

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^\infty e^{-st}dt \\ &= \left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=\infty} \\ &= \lim_{t\to\infty}\left(-\frac{e^{-st}}{s}\right) - \left(-\frac{e^{-0}}{s}\right) \\ &= 0 - \left(-\frac{1}{s}\right) \\ &= \frac{1}{s}.\end{aligned}$$

as long as $s > 0$.

(2) Use the definition of Laplace transform to find the Laplace transform of $f(t) = t$. That is, find $\mathcal{L}\{t\}$.

- **Solution:**
- We compute $\mathcal{L}\{f(t)\}$ with $f(t) = t$

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^\infty te^{-st}dt,\end{aligned}$$

thus we use integration by parts on

$$\begin{aligned}u = t, dv &= e^{-st}dt \\ du = dt, v &= -\frac{1}{s}e^{-st}\end{aligned}$$

and get that

$$\begin{aligned}\int te^{-st}dt &= uv - \int vdu \\ &= -\frac{t}{s}e^{-st} - \int \left(-\frac{1}{s}e^{-st}\right) dt \\ &= -\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt \\ &= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st},\end{aligned}$$

hence

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty te^{-st}dt \\ &= \lim_{t\to\infty}\left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right) - \left(-\frac{0}{s}e^{0} - \frac{1}{s^2}e^{0}\right) \\ &= (0-0) - \left(0 - \frac{1}{s^2}e^{0}\right) \\ &= \frac{1}{s^2},\end{aligned}$$

as long as $s > 0$.

(3) Use the definition of Laplace transform to find the Laplace transform of $f(t) = t^2$. That is, find $\mathcal{L}\{t^2\}$.

- **Solution:**
- We compute $\mathcal{L}\{f(t)\}$ with $f(t) = t^2$

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^\infty t^2e^{-st}dt,\end{aligned}$$

thus we use integration by parts, or tabular integration,

$$\begin{array}{ccc} u & & dv \\ t^2 & \overset{(+)}{\searrow} & e^{-st} \\ 2t & \overset{(-)}{\searrow} & -\frac{e^{-st}}{s} \\ 2 & \overset{(+)}{\searrow} & \frac{e^{-st}}{s^2} \\ 0 & & -\frac{e^{-st}}{s^3} \end{array}$$

and get that

$$\int te^{-st}dt = -t^2\frac{e^{-st}}{s} - 2t\frac{e^{-st}}{s^2} - 2\frac{e^{-st}}{s^3}$$

hence

$$
\begin{aligned}
\mathcal{L}\left\{t^2\right\} &= \int_0^\infty te^{-st}dt \\
&= \lim_{t\to\infty}\left(-t^2\frac{e^{-st}}{s} - 2t\frac{e^{-st}}{s^2} - 2\frac{e^{-st}}{s^3}\right) - \left(-0^2\frac{e^0}{s} - 2\cdot 0\frac{e^0}{s^2} - 2\frac{e^0}{s^3}\right) \\
&= \lim_{t\to\infty}\left(-t^2\frac{e^{-st}}{s} - 2t\frac{e^{-st}}{s^2} - 2\frac{e^{-st}}{s^3}\right) - \left(-2\frac{1}{s^3}\right) \\
&= (0+0+0) - \left(-2\frac{1}{s^3}\right) \\
&= \frac{2}{s^3}.
\end{aligned}
$$

as long as $s > 0$.

(4) Use the properties of Laplace transform and the following facts

$$
\begin{aligned}
\mathcal{L}\left\{1\right\} &= \frac{1}{s}, s > 0 \\
\mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a}, s > a, \\
\mathcal{L}\left\{t\right\} &= \frac{1}{s^2}, s > 0, \\
\mathcal{L}\left\{t^2\right\} &= \frac{2}{s^3}, s > 0, \\
\mathcal{L}\left\{\sin(at)\right\} &= \frac{a}{s^2+a^2}, s > 0, \\
\mathcal{L}\left\{\cos(at)\right\} &= \frac{s}{s^2+a^2}, s > 0,
\end{aligned}
$$

to compute the Laplace transforms of the following functions.

(a) $\mathcal{L}\left\{2e^{5t} + 7\cos(3t) + 2t\right\} =$
- **Solution:**
- Using Linearity and the formulas we above

$$
\begin{aligned}
&= \mathcal{L}\left\{2e^{5t} + 7\cos(3t) + 2t\right\} \\
&= 2\mathcal{L}\left\{e^{5t}\right\} + 7\mathcal{L}\left\{\cos(3t)\right\} + 2\mathcal{L}\left\{t\right\} \\
&= \frac{2}{s-5} + 7\frac{s}{s^2+3^2} + \frac{2}{s^2} \\
&= \frac{2}{s-5} + \frac{7s}{s^2+9} + \frac{2}{s^2}
\end{aligned}
$$

(b) $\mathcal{L}\left\{-7e^{-9t} - 5t^2 - 5\sin(3t)\right\} =$
- **Solution:**

- Using Linearity and the formulas we above

$$= \mathcal{L}\left\{-7e^{-9t} - 5t^2 - 5\sin(3t)\right\}$$
$$= -7\mathcal{L}\left\{e^{-9t}\right\} - 5\mathcal{L}\left\{t^2\right\} - 5\mathcal{L}\left\{\sin(3t)\right\}$$
$$= -\frac{7}{s-(-9)} - 5\frac{2}{s^3} - 5\frac{3}{s^2+3^2}$$
$$= -\frac{7}{s+9} - \frac{10}{s^3} - \frac{15}{s^2+9}$$

(c) $\mathcal{L}\left\{-5\sin(\sqrt{7}t) + 2 + 5t\right\} =$

- **Solution:**
- Using Linearity and the formulas we above

$$= \mathcal{L}\left\{-5\sin(\sqrt{7}t) + 2 + 5t\right\}$$
$$= -5\mathcal{L}\left\{\sin(\sqrt{7}t)\right\} + 2\mathcal{L}\left\{1\right\} + 5\mathcal{L}\left\{t\right\}$$
$$= -5\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2} + \frac{2}{s} + 5\frac{1}{s^2}$$
$$= \frac{-5\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2} + \frac{2}{s} + \frac{5}{s^2}$$

(d) $\mathcal{L}\left\{4e^{-t} - 6e^{3t} + \cos(3t)\right\} =$

- Using Linearity and the formulas we above

$$= \mathcal{L}\left\{4e^{-t} - 6e^{3t} + \cos(3t)\right\}$$
$$= 4\mathcal{L}\left\{e^{-t}\right\} - 6\mathcal{L}\left\{e^{3t}\right\} + \mathcal{L}\left\{\cos(3t)\right\}$$
$$= 4\frac{1}{s-(-1)} - 6\frac{1}{s-3} + \frac{s}{s^2+3^2}$$
$$= \frac{4}{s+1} - 6\frac{1}{s-3} + \frac{s}{s^2+9}$$

6.2. Problems

(1) Use the table of Laplace Transforms to help you compute the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\frac{5}{s-6}\right\}$

- **Solution:**
- We use $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5}{s-6}\right\} &= 5\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}\\ &= 5e^{6t}.\end{aligned}$$

(b) $\mathcal{L}^{-1}\left\{\frac{5}{7-s}+\frac{1}{s+3}\right\}$

- **Solution:**
- We use $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5}{7-s}+\frac{1}{s+3}\right\} &= -5\mathcal{L}^{-1}\left\{\frac{1}{s-7}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-(-3)}\right\}\\ &= -5e^{7t}+e^{-3t}\end{aligned}$$

(c) $\mathcal{L}^{-1}\left\{\frac{3}{s+9}-\frac{10}{s^2}\right\}$

- **Solution:**
- We use $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ and $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ with $n=1$ so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{s+9}-\frac{10}{s^2}\right\} &= 3\mathcal{L}^{-1}\left\{\frac{1}{s-(-9)}\right\} - 10\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\\ &= 3e^{-9t}-10t.\end{aligned}$$

(d) $\mathcal{L}^{-1}\left\{\frac{3}{s^2+7}+\frac{2}{(s-5)^3}\right\}$

- **Solution:**
- We use $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$ with $b=\sqrt{7}$ and $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$ with $n=2, a=5$ so that $\mathcal{L}\{t^2e^{5t}\} = \frac{2}{(s-5)^3}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{s^2+7}+\frac{2}{(s-5)^3}\right\} &= \frac{3}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-5)^3}\right\}\\ &= \frac{3}{\sqrt{7}}\sin(\sqrt{7}t)+t^2e^{5t}.\end{aligned}$$

(e) $\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+36}\right\}$

- **Solution:**
- We use $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$ with $a=3, b=6$ so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+36}\right\} &= \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+6^2}\right\}\\ &= e^{3t}\cos(6t).\end{aligned}$$

(f) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}+\frac{2}{s}-\frac{s-1}{(s-1)^2+25}\right\}$

- **Solution:**
- We use $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$ with $b=3$ and $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$ with $a=1, b=5$ so that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}+\frac{2}{s}-\frac{s-1}{(s-1)^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}+\mathcal{L}^{-1}\left\{\frac{2}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+5^2}\right\}$$
$$= \cos(3t)+2-e^t\cos(5t).$$

(2) Solve the following IVP using Laplace Transforms:

$$y'+4y=e^{-t}, \quad y(0)=0$$

- **Solution:**
- **Step 1:** Find the Laplace Transform of both sides (The going forwards to the s world part):
  - We have

$$\mathcal{L}\{y'\}+4\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$
$$\iff s\mathcal{L}\{y\}-y(0)+4\mathcal{L}\{y\} = \frac{1}{s-(-1)}$$
$$\iff s\mathcal{L}\{y\}-0+4\mathcal{L}\{y\} = \frac{1}{s+1}$$

- **Step 2:** Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\}(s+4) = \frac{1}{s+1}$$
$$\iff \mathcal{L}\{y\} = \frac{1}{(s+1)(s+4)}$$

- **Step 3:** Do partial fractions
  - We have that

$$\frac{1}{(s+1)(s+4)} = \frac{A}{(s+1)}+\frac{B}{(s+4)}$$

so that

$$1 = A(s+4)+B(s+1)$$

taking $s=-4$ we have

$$1 = B(-3) \implies B = \frac{-1}{3}$$

and taking $s=-1$ we have

$$1 = A3, \implies A = \frac{1}{3}$$

so that

$$\frac{1}{(s+1)(s+4)} = \frac{1/3}{(s+1)}-\frac{1/3}{(s+4)}$$

- **Step 4:** Take the inverse Laplace (The going back to t world part)

– Thus

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1/3}{(s+1)} - \frac{1/3}{(s+4)}\right\} \\
&= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-(-4)}\right\} \\
&= \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}.
\end{aligned}
$$

(3) Solve the following IVP using Laplace Transforms:

$$y' + y = e^{-2t}, \quad y(0) = 2$$

- **Solution:**
- **Step 1:** Find the Laplace Transform of both sides (The going forwards to the s world part):
  – We have

$$
\begin{aligned}
&\mathcal{L}\left\{y'\right\} + \mathcal{L}\left\{y\right\} = \mathcal{L}\left\{e^{-2t}\right\} \\
&\iff s\mathcal{L}\left\{y\right\} - y(0) + \mathcal{L}\left\{y\right\} = \frac{1}{s-(-2)} \\
&\iff s\mathcal{L}\left\{y\right\} - 2 + \mathcal{L}\left\{y\right\} = \frac{1}{s+2}
\end{aligned}
$$

- **Step 2:** Solve for $\mathcal{L}\left\{y\right\}$ using algebra: and get

$$
\begin{aligned}
&s\mathcal{L}\left\{y\right\} + \mathcal{L}\left\{y\right\} = 2 + \frac{1}{s+2} \\
&\iff \mathcal{L}\left\{y\right\}(s+1) = 2 + \frac{1}{s+2} \\
&\iff \mathcal{L}\left\{y\right\} = \frac{2}{s+1} + \frac{1}{(s+1)(s+2)}
\end{aligned}
$$

- **Step 3:** Do partial fractions
  – We have that

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

so that

$$1 = A(s+2) + B(s+1)$$

taking $s = -2$ we have

$$1 = B(-1) \implies B = -1$$

and taking $s = -1$ we have

$$1 = A, \implies A = 1$$

so that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

- **Step 4:** Take the inverse Laplace (The going back to t world part)

– Thus

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{s+1}+\frac{1}{(s+1)(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{s+1}+\frac{1}{(s+1)}-\frac{1}{(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{s-(-1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} \\
&= 2e^{-t}+e^{-t}-e^{-2t} \\
&= 3e^{-t}-e^{-2t}.
\end{aligned}
$$

(4) Solve the following IVP using Laplace Transforms:

$$y'+7y=1, \quad y(0)=3.$$

- **Solution:**
- **Step 1:** Find the Laplace Transform of both sides (The going forwards to the s world part):
  – We have

$$
\begin{aligned}
&\mathcal{L}\left\{y'\right\}+7\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{1\right\} \\
\iff\ & s\mathcal{L}\left\{y\right\}-y(0)+7\mathcal{L}\left\{y\right\}=\frac{1}{s} \\
\iff\ & s\mathcal{L}\left\{y\right\}-3+7\mathcal{L}\left\{y\right\}=\frac{1}{s}
\end{aligned}
$$

- **Step 2:** Solve for $\mathcal{L}\left\{y\right\}$ using algebra: and get

$$
\begin{aligned}
&s\mathcal{L}\left\{y\right\}+7\mathcal{L}\left\{y\right\}=3+\frac{1}{s} \\
\iff\ & \mathcal{L}\left\{y\right\}(s+7)=3+\frac{1}{s} \\
\iff\ & \mathcal{L}\left\{y\right\}=\frac{3}{s+7}+\frac{1}{(s+7)\,s}
\end{aligned}
$$

- **Step 3:** Do partial fractions
  – We have that

$$\frac{1}{(s+7)\,s}=\frac{A}{(s+7)}+\frac{B}{s}$$

so that

$$1=As+B(s+7)$$

taking $s=0$ we have

$$1=B7 \implies B=\frac{1}{7}$$

and taking $s=-7$ we have

$$1=A(-7), \implies A=-\frac{1}{7}$$

so that

$$\frac{1}{(s+7)\,s} = \frac{-1/7}{(s+7)} + \frac{1/7}{s}$$

- **Step 4:** Take the inverse Laplace (The going back to t world part)
  - Thus

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{3}{s+7} + \frac{1}{(s+7)\,s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{3}{s+7} + \frac{-1/7}{(s+7)} + \frac{1/7}{s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{3}{s+7}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/7}{(s+7)}\right\} + \mathcal{L}^{-1}\left\{\frac{1/7}{s}\right\} \\
&= 3e^{-7t} - \frac{1}{7}e^{-7t} + \frac{1}{7} \\
&= \frac{20}{7}e^{-7t} + \frac{1}{7}.
\end{aligned}$$

6.3. Problems

(1) What is the correct form of the partial fractions?

(a) $\dfrac{5s-1}{(s-3)(s^2+2s+5)} =$

- **Solution:**
- We have

$$\frac{5s-1}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

(b) $\dfrac{s-2}{(s-2)^2(s+5)} =$

- **Solution:**
- We have

$$\frac{s-2}{(s-2)^2(s+5)} = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{s+5}.$$

(c) $\dfrac{s+1}{(s^2+9)(s^3+2)} =$

- **Solution:**
- We have

$$\frac{s+1}{(s^2+9)(s^3+2)} = \frac{As+B}{s^2+9} + \frac{Cs^2+Ds+E}{s^3+2}.$$

(d) $\dfrac{s}{(s+1)(s^2+10)s^3} =$

- **Solution:**
- We have

$$\frac{s}{(s+1)(s^2+10)s^2} = \frac{A}{s+1} + \frac{Bs+C}{s^2+10} + \frac{D}{s} + \frac{E}{s^2} + \frac{F}{s^3}$$

(2) Take the inverse Laplace Transforms of the following:

(a) $F(s) = \frac{1}{s^2-8s+7}$

- **Solution:**
- We first try to factor: and we get

$$\frac{1}{s^2-8s+7} = \frac{1}{(s-1)(s-7)}$$

amd then we use partial fraction

$$\frac{1}{(s-1)(s-7)} = \frac{A}{s-1} + \frac{B}{s-7}$$

so that

$$1 = A(s-7) + B(s-1)$$

and taking $s = 7$we have $1 = B6$ so that $B = \frac{1}{6}$

- And using $s=1$ we have $1=A(-6)$ so that $A=-\frac{1}{6}$ so that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-8s+7}\right\}=\mathcal{L}^{-1}\left\{\frac{-\frac{1}{6}}{s-1}+\frac{\frac{1}{6}}{s-7}\right\}$$
$$=-\frac{1}{6}e^{t}+\frac{1}{6}e^{7t}.$$

(b) $F(s)=\frac{s+7}{s^2+6s+13}$
- **Solution:**
- We first try to factor! and we can't factor s^2+6s+7. Thus whenever we can't factor the denominator, we complete the square.
- Recall the special number for s^2+bs+c is $\left(\frac{b}{2}\right)^2$, so that $\left(\frac{6}{2}\right)^2=3^2=9$ hence

$$\begin{aligned}s^2+6s+13&=s^2+6s+9+(-9+13)\\&=\left(s^2+6s+9\right)+4\\&=(s+3)^2+2^2\end{aligned}$$

- Hence

$$\frac{s+7}{s^2+6s+13}=\frac{s+7}{(s+3)^2+2^2},$$

- Now we try to use Formula #9 and #10

$$\mathcal{L}\left\{e^{at}\sin bt\right\}=\frac{b}{(s-a)^2+b^2}\text{ and }\mathcal{L}\left\{e^{at}\cos bt\right\}=\frac{s-a}{(s-a)^2+b^2}$$

and get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+7}{(s+3)^2+2^2}\right\}&=\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2^2}\right\}+\mathcal{L}^{-1}\left\{\frac{4}{(s+3)^2+2^2}\right\}\\&=\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}\\&=e^{-3t}\cos 2t+2e^{-3t}\sin 2t.\end{aligned}$$

(c) $F(s)=\frac{2s-1}{s^2-8s+18}$
- **Solution:**
- We first try to factor! and we can't factor $s^2+8s+11$. Thus whenever we can't factor the denominator, we complete the square.
- Recall the special number for s^2+bs+c is $\left(\frac{b}{2}\right)^2$, so that $\left(\frac{-8}{2}\right)^2=4^2=16$ hence

$$\begin{aligned}s^2+6s+13&=s^2-8s+16+(18-16)\\&=\left(s^2-8s+16\right)+2\\&=(s-4)^2+\left(\sqrt{2}\right)^2\end{aligned}$$

- Hence

$$\frac{2s-1}{s^2-8s+18}=\frac{2s-1}{(s-4)^2+\left(\sqrt{2}\right)^2},$$

- Now we try to use Formula #9 and #10

  $$\mathcal{L}\left\{e^{at}\sin bt\right\}=\frac{b}{(s-a)^2+b^2}\text{ and }\mathcal{L}\left\{e^{at}\cos bt\right\}=\frac{s-a}{(s-a)^2+b^2}$$

  and note that we need a $2(s-4)=2s-8$ in the numerator, and thus since $-8+8=0$ then

  $$2s-1=2s-\mathbf{8}+\mathbf{8}-1=2(s-4)\,\mathbf{+8}\,-1$$

  so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s-1}{(s-4)^2+\left(\sqrt{2}\right)^2}\right\}&=\mathcal{L}^{-1}\left\{\frac{2(s-4)}{(s-4)^2+\left(\sqrt{2}\right)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{\mathbf{+8}-1}{(s-4)^2+\left(\sqrt{2}\right)^2}\right\}\\&=2\mathcal{L}^{-1}\left\{\frac{(s-4)}{(s-4)^2+\left(\sqrt{2}\right)^2}\right\}+\frac{7}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-4)^2+\left(\sqrt{2}\right)^2}\right\}\\&=2e^{4t}\cos\left(\sqrt{2}t\right)+\frac{7}{\sqrt{2}}e^{4t}\sin\left(\sqrt{2}t\right).\end{aligned}$$

(3) Solve the following IVP using Laplace Transforms:

$$y''+4y=8,\quad y(0)=11,y'(0)=5.$$

- **Solution:**
- **Step 1:** Find the Laplace Transform of both sides (The going forwards to the s world part):
  - We have

$$\begin{aligned}&\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\mathcal{L}\{8\}\\\iff\ & s^2\mathcal{L}\{y\}-sy(0)-y'(0)+4\mathcal{L}\{y\}=\frac{8}{s}\\\iff\ & s^2\mathcal{L}\{y\}-11s-5+4\mathcal{L}\{y\}=\frac{8}{s}\end{aligned}$$

- **Step 2:** Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\begin{aligned}&s^2\mathcal{L}\{y\}+4\mathcal{L}\{y\}=11s+5+\frac{8}{s}\\\iff\ &\left(s^2+4\right)\mathcal{L}\{y\}=11s+5+\frac{8}{s}\\\iff\ &\mathcal{L}\{y\}=\frac{11s+5}{s^2+4}+\frac{8}{s\left(s^2+4\right)}\end{aligned}$$

- **Step 3:** Do partial fractions
  - On the term

  $$\frac{8}{s\left(s^2+4\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}$$

  so that

  $$8=A\left(s^2+4\right)+(Bs+C)\,s$$

  and multiplying the RHS out we get

  $$8=As^2+4A+Bs^2+Cs$$

  and combining we get

  $$0s^2+0s+8=(A+B)\,s^2+Cs+4A$$

so that we have

$$\begin{aligned}0 =& A + B\\0 =& C\\8 =& 4A\end{aligned}$$

hence

$$A = 2, B = -2 = C = 0$$

so that

$$\frac{8}{s\left(s^2+4\right)} = \frac{2}{s} + \frac{-2s}{s^2+4}$$

- **Step 4:** Take the inverse Laplace (The going back to t world part)
  - Thus

$$\begin{aligned}y &= \mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\}\\&= \mathcal{L}^{-1}\left\{\frac{11s+5}{s^2+4} + \frac{2}{s} + \frac{-2s}{s^2+4}\right\}\\&= \mathcal{L}^{-1}\left\{\frac{9s+5}{s^2+4} + \frac{2}{s}\right\}\\&= 9\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{5}{2}\mathcal{L}\left\{\frac{2}{s^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s}\right\}\\&= 9\cos(2t) + \frac{5}{2}\sin(2t) + 2.\end{aligned}$$

(4) Solve the following IVP using Laplace Transforms:

$$y'' - 4y' + 5y = 2e^t, \quad y(0) = 3, y'(0) = 1.$$

- **Solution:**
- **Step 1:** Find the Laplace Transform of both sides (The going forwards to the s world part):
  - We have

$$\begin{aligned}&\mathcal{L}\left\{y''\right\} - 4\mathcal{L}\left\{y'\right\} + 5\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{2e^t\right\}\\&\iff \left[s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0)\right]\\&- 4\left[s\mathcal{L}\left\{y\right\} - y(0)\right] + 5\mathcal{L}\left\{y\right\} = \frac{2}{s-1}\\&\iff \left[s^2\mathcal{L}\left\{y\right\} - 3s - 1\right]\\&- 4\left[s\mathcal{L}\left\{y\right\} - 3\right] + 5\mathcal{L}\left\{y\right\} = \frac{2}{s-1}\end{aligned}$$

- **Step 2:** Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\begin{aligned} s^2\mathcal{L}\{y\} - 3s - 1 - 4s\mathcal{L}\{y\} + 12 + 5\mathcal{L}\{y\} &= \frac{2}{s-1} \\ \iff \mathcal{L}\{y\}\left(s^2 - 4s + 5\right) - 3s + 11 &= \frac{2}{s-1} \\ \iff \mathcal{L}\{y\}\left(s^2 - 4s + 5\right) &= 3s - 11 + \frac{2}{s-1} \\ \iff \mathcal{L}\{y\} &= \frac{3s-11}{s^2-4s+5} + \frac{2}{(s-1)(s^2-4s+5)} \end{aligned}$$

- **Step 3:** Do partial fractions
  - On the term
$$\frac{2}{(s-1)(s^2-4s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2-4s+5)}$$
    so that
$$2 = A\left(s^2 - 4s + 5\right) + (Bs + C)(s-1).$$
  - Using $s = 1$ give us that
$$2 = A(1 - 4 + 5) \implies 2 = A2 \implies A = 1$$
  - and multiplying the RHS out we get
$$2 = As^2 - 4As + 5A + Bs^2 - Bs + Cs - C$$
    and combining we get
$$0s^2 + 0s + 2 = (A + B)s^2 + (-4A - B + C)s + (5A - C)$$
    so that we have
$$\begin{aligned} 0 &= A + B \\ 0 &= -4A - B + C \\ 2 &= 5A - C \end{aligned}$$
    but we already know that $A = 1$ hence $B = -1$ and hence
$$C = 5A - 2 = 3.$$
  - Thus
$$\frac{2}{(s-1)(s^2-4s+5)} = \frac{1}{(s-1)} + \frac{-s+3}{(s^2-4s+5)}$$

- **Step 4:** Take the inverse Laplace (The going back to t world part)
  - Thus

$$\begin{aligned} y &= \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} \\ &= \mathcal{L}^{-1}\left\{\frac{3s-11}{s^2-4s+5} + \frac{1}{(s-1)} + \frac{-s+3}{(s^2-4s+5)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\} + e^t \end{aligned}$$

We just need to figure out $\mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\}$ and to do this, we try to factor the denominator. But we can't, thus we complete the square using the magic number $\left(\frac{b}{2}\right)^2=\left(\frac{-4}{2}\right)^2=4$

$$s^2-4s+5=\left(s^2-4s+4\right)+1=(s-2)^2+1$$

hence

$$\frac{2s-8}{s^2-4s+5}=\frac{2s-8}{(s-2)^2+1}$$

thus

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s-8}{(s-2)^2+1}\right\}&=\mathcal{L}^{-1}\left\{\frac{2(s-2)}{(s-2)^2+1}\right\}-\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+1}\right\}\\&=2e^{2t}\cos t-4e^{2t}\sin t,\end{aligned}$$

hence

$$\begin{aligned}y(t)&=\mathcal{L}^{-1}\left\{\frac{2s-8}{s^2-4s+5}\right\}+e^t\\&=2e^{2t}\cos t-4e^{2t}\sin t+e^t.\end{aligned}$$

6.4. Problems

(1) Take the Laplace transforms of the following functions

(a) $f(t) = u_7(t)e^{6(t-7)}$

- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $c = 7$ and $f(t-7) = e^{6(t-7)}$ hence $f(t) = e^{6t}$ so that $F(s) = \mathcal{L}\{e^{6t}\} = \frac{1}{s-6}$ hence

$$\begin{aligned}\mathcal{L}\left\{u_7(t)e^{6(t-7)}\right\} &= e^{-cs}F(s)\\ &= e^{-7s}\frac{1}{s-6}.\end{aligned}$$

(b) $f(t) = u_2(t)e^{-9(t-2)}$

- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $c = 2$ and $f(t-2) = e^{-9(t-2)}$ hence $f(t) = e^{-9t}$ so that $F(s) = \mathcal{L}\{e^{-9t}\} = \frac{1}{s+9}$ hence

$$\begin{aligned}\mathcal{L}\left\{u_2(t)e^{-9(t-2)}\right\} &= e^{-cs}F(s)\\ &= e^{-2s}\frac{1}{s+9}.\end{aligned}$$

(c) $f(t) = u_2(t)(t-2)^3$

- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $c = 2$ and $f(t-2) = (t-2)^3$ hence $f(t) = t^3$ so that $F(s) = \mathcal{L}\{t^3\} = \frac{6}{s^4}$ hence

$$\begin{aligned}\mathcal{L}\left\{u_2(t)(t-2)^3\right\} &= e^{-cs}F(s)\\ &= e^{-2s}\frac{6}{s^4}.\end{aligned}$$

(d) $f(t) = u_6(t)\sin(3(t-6))$

- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $c = 6$ and $f(t-6) = \sin(3(t-6))$ hence $f(t) = \sin(3t)$ so that $F(s) = \mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+3^2}$ hence

$$\begin{aligned}\mathcal{L}\{u_6(t)\sin(3(t-6))\} &= e^{-cs}F(s)\\ &= e^{-6s}\frac{3}{s^2+9}.\end{aligned}$$

(e) $f(t) = u_1(t)\cos(7(t-1))$

- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $c = 1$ and $f(t-1) = \cos(7(t-1))$ hence $f(t) = \cos(7t)$ so that $F(s) = \mathcal{L}\{\cos(7t)\} = \frac{s}{s^2+7^2}$ hence

$$\begin{aligned}\mathcal{L}u_1(t)\cos(7(t-1)) &= e^{-cs}F(s)\\ &= e^{-s}\frac{s}{s^2+7^2}.\end{aligned}$$

(f) $f(t) = \begin{cases} 5 & t < 7 \\ 8 & t \geq 7 \end{cases}$

- **Solution:**
- We first write $f(t)$ into a Heaviside function.
  - Note that
$$f(t) = 5+? \cdot u_7(t)$$
  - But since for $t \geq 7$ we
$$f(t) = 8, \quad t \geq 7$$
then
$$8 = 5+? \cdot 1, \quad t \geq 7$$
which means
$$? = 3$$
hence
$$f(t) = 5 + 3 \cdot u_7(t).$$
- We use the formula $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$, where $c = 7$ and get
$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{5\} + 3\mathcal{L}\{u_7(t)\} \\ &= \frac{5}{s} + 3\frac{e^{-7s}}{s}.\end{aligned}$$

(2) Take the inverse Laplace transforms of the following functions

(a) $F(s) = \frac{e^{-3s}}{s+1}$
- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, and note that $\frac{e^{-3s}}{s+1} = e^{-3s}F(s)$ hence $F(s) = \frac{1}{s+1}$ now
$$\begin{aligned}f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} \\ &= e^{-t}\end{aligned}$$
so that with $c = 3$
$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s+1}\right\} &= \mathcal{L}^{-1}\left\{e^{-3s}F(s)\right\} \\ &= u_3(t)f(t-3) \\ &= u_3(t)e^{-(t-3)}.\end{aligned}$$

(b) $F(s) = \frac{e^{-5s}}{s-7}$
- **Solution:**
- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, and note that $\frac{e^{-5s}}{s-7} = e^{-5s}F(s)$ hence $F(s) = \frac{1}{s-7}$ now
$$\begin{aligned}f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-7}\right\} \\ &= e^{7t}\end{aligned}$$

so that with $c = 5$

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-5s}\frac{1}{s-7}\right\} &= \mathcal{L}^{-1}\left\{e^{-5s}F(s)\right\} \\ &= u_5(t)f(t-5) \\ &= u_5(t)e^{7(t-5)}.\end{aligned}$$

(c) $F(s) = \frac{2e^{-2s}}{s^2+4}$

- **Solution:**
- We use the formula $\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-cs}F(s)$, and note that $\frac{2e^{-2s}}{s^2+4} = e^{-2s}F(s)$ hence $F(s) = \frac{2}{s^2+4}$ now

$$\begin{aligned}f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} \\ &= \sin(2t)\end{aligned}$$

so that with $c = 2$

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-2s}\frac{2}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\} \\ &= u_2(t)f(t-2) \\ &= u_2(t)\sin(2(t-2)).\end{aligned}$$

(d) $F(s) = \frac{se^{-9s}}{s^2+7}$

- **Solution:**
- We use the formula $\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-cs}F(s)$, and note that $\frac{se^{-9s}}{s^2+7} = e^{-9s}F(s)$ hence $F(s) = \frac{s}{s^2+7}$ now

$$\begin{aligned}f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\sqrt{7}\right)^2}\right\} \\ &= \cos\left(\sqrt{7}t\right)\end{aligned}$$

so that with $c = 9$

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-9s}\frac{2}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{e^{-9s}F(s)\right\} \\ &= u_9(t)f(t-9) \\ &= u_9(t)\cos\left(\sqrt{7}(t-9)\right).\end{aligned}$$

(e) $F(s) = \frac{(s+2)e^{-3s}}{(s+2)^2+16}$

- **Solution:**

- We use the formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, and note that $\frac{(s+2)e^{-3s}}{(s+2)^2+16} = e^{-3s}F(s)$ hence $F(s) = \frac{(s+2)}{(s+2)^2+16}$ now

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{(s-(-2))}{(s-(-2))^2+4^2}\right\} \\ &= e^{-2t}\cos(4t) \end{aligned}$$

so that with $c = 3$

$$\begin{aligned} \mathcal{L}^{-1}\left\{e^{-3s}\frac{(s+2)}{(s+2)^2+16}\right\} &= \mathcal{L}^{-1}\left\{e^{-3s}F(s)\right\} \\ &= u_3(t)f(t-3) \\ &= u_3(t)e^{-2(t-3)}\cos(4(t-3)). \end{aligned}$$

(3) Take the inverse Laplace transforms of

$$F(s) = \frac{e^{-3s}}{s^2-3s+2}.$$

- **Solution:**
- We first try to factor the denominator and get

$$\frac{e^{-3s}}{s^2-3s+2} = \frac{e^{-3s}}{(s-1)(s-2)}$$

and then we do partial fractions on

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

so that

$$1 = A(s-2) + B(s-1)$$

so that $A = -1, B = 1$ and hence

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{-\frac{e^{-3s}}{s-1} + \frac{e^{-3s}}{s-2}\right\} \\ &= -u_3(t)e^{(t-3)} + u_3(t)e^{2(t-3)}. \end{aligned}$$

(4) Take the inverse Laplace transforms of

$$F(s) = \frac{se^{-9s}}{s^2+6s+11}.$$

- **Solution:**
- We first try to factor the denominator but we can't!
- Thus we will try to complete the square with the special number $\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$ hence

$$\begin{aligned} s^2+6s+11 &= \left(s^2+6s+9\right)+2 \\ &= (s+3)^2 + \left(\sqrt{2}\right)^2. \end{aligned}$$

- And get

$$\frac{se^{-9s}}{s^2+6s+11} = e^{-9s}\frac{s}{(s+3)^2+\left(\sqrt{2}\right)^2}$$

- Then we need to separate, so that we can use formulas $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$ and $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$,

$$\begin{aligned} G(s) = \frac{s}{(s+3)^2+\left(\sqrt{2}\right)^2} &= \frac{s+3}{(s+3)^2+\left(\sqrt{2}\right)^2} + \frac{-3}{(s+3)^2+\left(\sqrt{2}\right)^2} \\ &= \frac{s+3}{(s+3)^2+\left(\sqrt{2}\right)^2} - \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+3)^2+\left(\sqrt{2}\right)^2} \end{aligned}$$

- So that

$$\begin{aligned} g(t) =& \mathcal{L}^{-1}\{G(s)\} \\ =& \mathcal{L}^{-1}\left\{\frac{s-(-3)}{(s-(-3))^2+\left(\sqrt{2}\right)^2}\right\} - \frac{3}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s-(-3))^2+\left(\sqrt{2}\right)^2}\right\} \\ =& e^{-3t}\cos\left(\sqrt{2}t\right) - \frac{3}{\sqrt{2}}e^{-3t}\sin\left(\sqrt{2}t\right). \end{aligned}$$

- And to finish off, remeber we actualy want to

$$\mathcal{L}^{-1}\left\{\frac{se^{-9s}}{s^2+6s+11}\right\} = \mathcal{L}^{-1}\left\{e^{-9s}G(s)\right\}$$

hence we use the formula $\mathcal{L}\{u_c(t)g(t-c)\} = e^{-cs}G(s)$ so that

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{se^{-9s}}{s^2+6s+11}\right\} &= u_9(t)g(t-9) \\ &= u_9(t)e^{-3(t-9)}\cos\left(\sqrt{2}(t-9)\right) - u_9(t)\frac{3}{\sqrt{2}}e^{-3(t-9)}\sin\left(\sqrt{2}(t-9)\right). \end{aligned}$$

6.5. Problems

(1) Find the solution to the following IVP using Laplace Transforms

$$y' + 9y = u_5(t), \quad y(0) = -2.$$

- **Solution:**
- **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{u_5(t)\}$$

so that

$$s\mathcal{L}\{y\} - y(0) + 9\mathcal{L}\{y\} = \frac{e^{-5s}}{s}.$$

hence

$$s\mathcal{L}\{y\} + 2 + 9\mathcal{L}\{y\} = \frac{e^{-5s}}{s}.$$

- **Step2:** Solve for $\mathcal{L}\{y\}$, we have

$$\mathcal{L}\{y\}(s+9) = -2 + \frac{e^{-5s}}{s}$$
$$\mathcal{L}\{y\} = \frac{-2}{s+9} + \frac{e^{-5s}}{s(s+9)}$$

- **Step3:** We do partial fractions on

$$\frac{1}{s(s+9)} = \frac{1/9}{s} - \frac{1/9}{s+9}$$

so we have

$$\frac{e^{-5s}}{s(s+9)} = \frac{1}{9}\frac{e^{-5s}}{s} - \frac{1}{9}\frac{e^{-5s}}{s+9}$$

- **Step4:** Take the inverse Laplace transform: Using $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, and get

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{-2}{s+9} + \frac{1}{9}e^{-5s}\frac{1}{s} - \frac{1}{9}\frac{e^{-5s}}{s+9}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{e^{-5s}\frac{1}{s}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s+9}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{1}{s-(-9)}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{e^{-5s}\frac{1}{s}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{e^{-5s}\frac{1}{s-(-9)}\right\} \\ &= -2e^{-9t} + \frac{1}{9}u_5(t) - \frac{1}{9}u_5(t)e^{-9(t-5)}. \end{aligned}$$

(2) Find the solution to the following IVP using Laplace Transforms

$$y' + y = u_7(t)e^{-2(t-7)}, \quad y(0) = 1.$$

- **Solution:**
- **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\left\{u_7(t)e^{-2(t-7)}\right\}$$

so that

$$s\mathcal{L}\{y\} - y(0) + \mathcal{L}\{y\} = \frac{e^{-7s}}{s-(-2)}.$$

hence

$$s\mathcal{L}\left\{y\right\}-1+\mathcal{L}\left\{y\right\}=\frac{e^{-7s}}{s+2}.$$

- **Step2:** Solve for $\mathcal{L}\left\{y\right\}$, we have

$$\mathcal{L}\left\{y\right\}(s+1)=1+\frac{e^{-7s}}{s+2}$$

$$\mathcal{L}\left\{y\right\}=\frac{1}{s+1}+\frac{e^{-7s}}{(s+1)(s+2)}$$

- **Step3:** We do partial fractions on

$$\frac{1}{(s+1)(s+2)}=\frac{1}{s+1}+\frac{-1}{s+2}$$

so we have

$$\frac{e^{-7s}}{(s+1)(s+2)}=\frac{e^{-7s}}{s+1}+\frac{-e^{-7s}}{s+2}$$

- **Step4:** Take the inverse Laplace transform: Using $\mathcal{L}\left[u_c(t)f(t-c)\right]=e^{-cs}F(s)$, and get

$$\begin{aligned}y&=\mathcal{L}^{-1}\left\{\frac{1}{s+1}+\frac{e^{-7s}}{s+1}+\frac{-e^{-7s}}{s+2}\right\}\\&=\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}+\mathcal{L}^{-1}\left\{e^{-7s}\frac{1}{s-(-1)}\right\}-\mathcal{L}^{-1}\left\{e^{-7s}\frac{1}{s-(-2)}\right\}\\&=e^{-t}+u_7(t)e^{-(t-7)}-u_7(t)e^{-2(t-7)}.\end{aligned}$$

(3) Find the solution to the following IVP using Laplace Transforms

$$y''+9y=u_3(t)\sin\left(2\left(t-3\right)\right),\quad y(0)=0,y'(0)=0$$

- **Solution:**
- **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\left\{y''\right\}+9\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{u_3(t)\sin\left(2\left(t-3\right)\right)\right\},y(0)=0,y'(0)=0.$$

and recall $\mathcal{L}\left[u_c(t)f(t-c)\right]=e^{-cs}F(s)$, hence $a=3$, $f(t-5)=\sin\left(2\left(t-5\right)\right)$ hence $f(t)=\sin 2t$ and $\mathcal{L}\left\{\sin 2t\right\}=\frac{2}{s^2+4}$ hence

$$\mathcal{L}\left\{u_3(t)\sin\left(2\left(t-3\right)\right)\right\}=\frac{2e^{-3s}}{s^2+4}$$

so that

$$\begin{aligned}s^2\mathcal{L}\left\{y\right\}-sy(0)-y'(0)+9\mathcal{L}\left\{y\right\}&=\frac{2e^{-3s}}{s^2+4},\implies\\\left(s^2+9\right)\mathcal{L}\left\{y\right\}&=\frac{2e^{-3s}}{s^2+4},\implies\\\mathcal{L}\left\{y\right\}&=\frac{2e^{-3s}}{\left(s^2+9\right)\left(s^2+4\right)}\end{aligned}$$

- **Step2:** We do partial fractions on

$$\frac{2}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$

hence

$$2 = (As+B)\left(s^2+4\right) + (Cs+D)\left(s^2+9\right), \implies$$

$$0\cdot s^3 + 0\cdot s^2 + 0\cdot s + 2 = (A+C)\,s^3 + (B+D)\,s^2 + (4A+9C)\,s + (4B+9D)$$

hence

$$\begin{aligned} A+C &= 0 \\ B+D &= 0 \\ 4A+9C &= 0 \\ 4B+9D &= 2 \end{aligned}$$

and get

$$A = 0 \quad B = -\frac{2}{5}, \quad C = 0, \quad D = \frac{2}{5}$$

hence

$$\frac{2}{(s^2+9)(s^2+4)} = -\frac{2}{5}\frac{1}{s^2+9} + \frac{2}{5}\frac{1}{s^2+4}$$

- **Step3:** Take the inverse Laplace transform: Using $\mathcal{L}\left[u_c(t)f(t-c)\right] = e^{-cs}F(s)$, and $\mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}$ and $\mathcal{L}\left\{\cos(at)\right\} = \frac{a}{s^2+a^2}$ we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{-\frac{2}{5}\frac{e^{-3s}}{s^2+9} + \frac{2}{5}\frac{e^{-3s}}{s^2+4}\right\} \\ &= -\frac{2}{5}\frac{1}{3}\mathcal{L}^{-1}\left\{e^{-3s}\frac{3}{s^2+3^2}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{e^{-3s}\frac{2}{s^2+2^2}\right\} \\ &= -\frac{2}{15}u_3(t)\sin\left(3\left(t-3\right)\right) + \frac{1}{5}u_3(t)\sin\left(2\left(t-3\right)\right). \end{aligned}$$

CHAPTER 7

Series

7.1. Problems

TBA