Differential Equations Exercises

CHAPTER 1

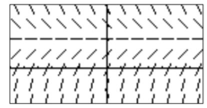
Introduction

1.1. Problems

(1) What does it mean to be a solution to a differential equation?
(2) Check if the function $y(t) = t + 1$ a particular solution to the following differential equation:
$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$
(3) Check if the function $y(x) = x + x \ln x$ solves the following Initial Value Problem (IVP):
$$x\frac{dy}{dx} = x + y, \quad y(1) = 3.$$
(4) Find the equilibrium solutions to the equation
$$\frac{dy}{dt} = y^2 + 2y$$
(5) Find the equilibrium solutions to the equation
$$\frac{dy}{dt} = y^4 t - 3y^3 t + 2y^2 t$$
(6) Classify the following equations as ODEs or PDEs.
   (a) $\frac{dy}{dt} = 2yt$
   (b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$
      - *Fun Fact:* This particular PDE is a very famous PDE and is called the *heat equation.* It models the flow of heat in a medium over time.
   (c) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
      - *Fun Fact:* This particular PDE is a very famous PDE and is called the *wave equation.* This PDE along with boundary conditions, describes the amplitude and phase of the wave.
   (d) $x\frac{d^2 y}{dx^2} = y\frac{dy}{dx} + x^2 y$
   (e) $2y'' - y' + y = 0$
(7) Classify the order of the following differential equations. Also classify if it is linear or nonlinear.
   (a) $\frac{dy}{dt} = 2yt$
   (b) $y\frac{d^2 y}{dt^2} = \cos t$
   (c) $ty''' - y'' - 2y = 0$
   (d) $\frac{dy^6}{dt^6} - 2\frac{dy}{dt} + y = t^2$
   (e) $\cos y + y' = t$
   (f) $6y''' - y^2 = y^{(5)}$
   (g) $\frac{d^2 y}{dt^2} = \frac{y}{y+t}$

1.2. Problems

(A)



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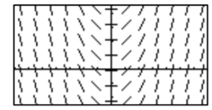

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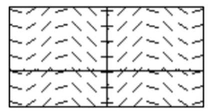

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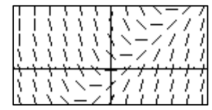
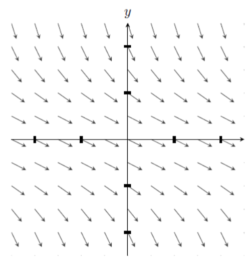
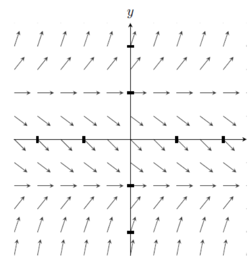


(1) Match the following slope fields with their equations
  (a) $\frac{dy}{dt} = \sin t$
  (b) $\frac{dy}{dt} = t - y$
  (c) $\frac{dy}{dt} = 2 - y$
  (d) $\frac{dy}{dt} = t$



(I)



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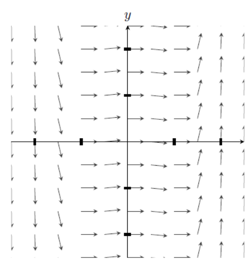
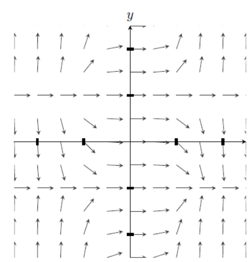


(III)

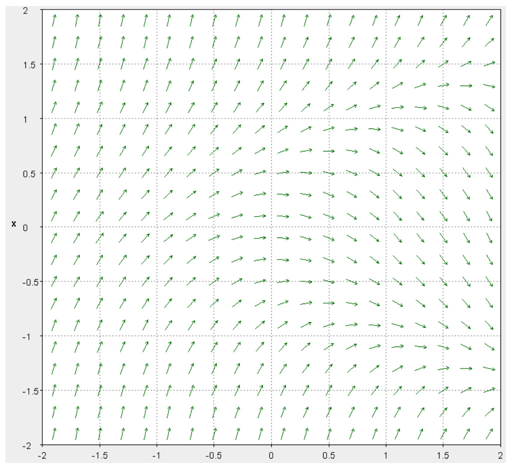


(IV)

(2) Match the following slope fields with their equations
  (a) $\frac{dy}{dt} = t^4(y^2 - 1)$
  (b) $\frac{dy}{dt} = t^3(t^2 - 1)$
  (c) $\frac{dy}{dt} = (y - 1)(y + 1)$
  (d) $\frac{dy}{dt} = -\sqrt{1 + y^4}$

(3) Suppose the following ODE

$$\frac{dy}{dt} = y^2 - t$$



has the following Slope Field:

(a) Suppose $y(t)$ is a solution to this ODE and also you know that $y(-1) = 1$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t \to \infty} y(t) = ?$$

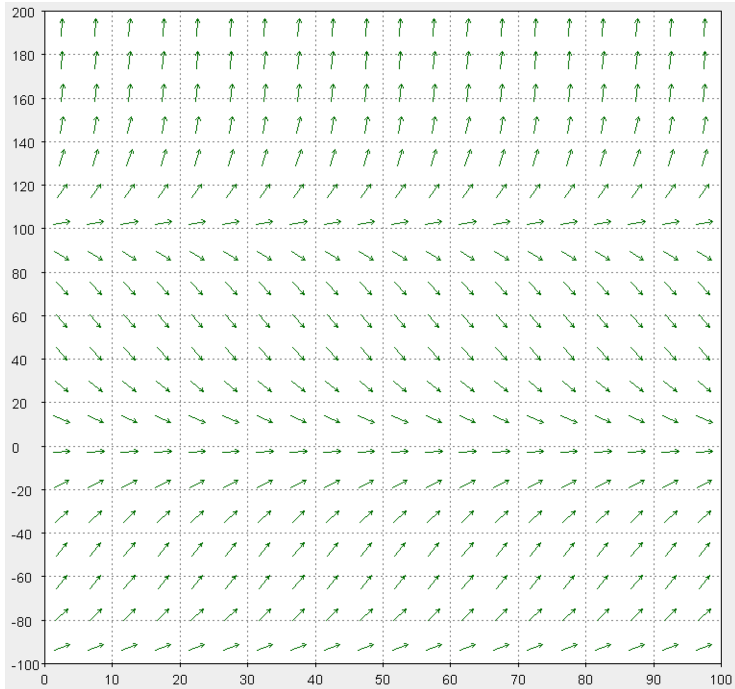
(b) Suppose $y(t)$ is a solution to this ODE and also you know that $y(1) = 0$. Then based on the slope field, what is your prediction for the long term behavior of $y(t)$, that is, what is your prediction of

$$\lim_{t \to \infty} y(t) = ?$$

(4) Let $P(t)$ represent the population of the Phan fish breed. Suppose you come up with the following differential equation that models $P(t)$:

$$\frac{dP}{dt} = P(P - 100)(P + 100)/100000$$

Its Slope Field is given by:



(a) Suppose that the population of the Phan fish is 80 at time $t = 0$. What is the long term behavior for the population of the Phan fish? Will it keep increasing/decreasing, stabilize to a certain number, or go extinct?

CHAPTER 2

First Order Differential Equations

2.1. Problems

(1) Use a computer app to draw the direction field for the given differential equations. Use the direction field to describe the long term behavior of the solution for large t. (Meaning use the direction field to predict $\lim_{t\to\infty} y(t)$ for different starting points). Find the general solution of the given differential equations, and use it to determined how solutions behave as $t \to \infty$.
   (a) $y' + 3y = t + e^{-2t}$
   (b) $y' + y = te^{-t} + 1$
   (c) $ty' - y = t^2 e^{-t}$
   (d) $2y' + y = 3t$

(2) Find the particular solution to given initial value problem.
   (a) $y' - y = 2te^{2t}$, $y(0) = 1$
   (b) $ty' + 2y = \sin t$, $y(\pi/2) = 1$, $t > 0$

(3) Consider the following initial value problem:

$$ty' + (t+1)\,y = 2te^{-t}, \quad y(1) = a, \quad t > 0$$

   where a is any real number.
   (a) Find the particular solution that solves this IVP.

2.2. Problems

(1) Find the general solutions for the following differential equations. Find the *explicit* solutions if you can. If you can't solve for y exactly, then leave it as an *implicit* solution:
   (a) $y' = ky$ where k is a parameter.
   (b) $y' = \dfrac{x^2}{y}$
   (c) $\dfrac{dy}{dx} = \dfrac{3x^2 - 1}{3 + 2y}$
   (d) $xy' = \frac{\left(1-y^2\right)^{1/2}}{y}$
   (e) $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$
   (f) $\dfrac{dy}{dx} = \dfrac{x}{\cos\left(y^2\right) y}$

(2) Consider the ODE
$$\frac{dy}{dt} = \frac{4y}{t}.$$
   (a) What kind of differential equation is this? Is it Linear? Is it separable?
   (b) If the ODE is both separable and Linear. Then use both methods to solve this equation. And check to make sure you get the same answer.

(3) Find the general solution to the following differential equation:
$$\frac{dy}{dt} = (y + 1)(y - 2).$$
(Hint: Use Partial fractions!)

2.3. Problems

(1) First check each if the following differential equations are homogeneous. Then find the general solutions for the following differential equations.
   (a) $x^2\dfrac{dy}{dx} = -\left(y^2 - yx\right).$
   (b) $\dfrac{dy}{dx} = \dfrac{x + 3y + 2\frac{y^2}{x}}{3x + y}.$
   (c) $\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{x^2 - y^2}{2xy}.$
(2) Consider the following homogeneous equation:
$$\frac{dy}{dx} = \frac{y - x}{y + x}.$$
   (a) Use the substitution $v = \frac{y}{x}$ to rewrite the equation only in terms of v and x.
   (b) Solve for the general solution.
(3) Consider the following homogeneous equation:
$$\frac{dy}{dx} = \frac{-y^2 - yx}{x^2}.$$
   (a) Use the substitution $v = \frac{y}{x}$ to rewrite the equation only in terms of v and x.
   (b) Solve for the general solution.
(4) Using the given substitution. Solve the differential equation:
   (a) Rewrite $\dfrac{dy}{dx} + xy = x^2y^2$ using the substitution $u = \frac{1}{y}$, only in terms of u, x.
   (b) Rewrite $\dfrac{dy}{dx} + y = \dfrac{x}{y^2}$ using the substitution $u = y^3$, only in terms of u, x.

2.4. Problems

(1) Initially, a tank contains 100 L of water with 10 kg of sugar in solution. Water containing sugar flows into the tank at the rate of 2 L/min, and the well-stirred mixture in the tank flows out at the rate of 5 L/min. The concentration $c(t)$ of sugar in the incoming water varies as $c(t) = 2 + \cos(3t)$ kg/L. Let $Q(t)$ be the amount of sugar (in kilograms) in the tank at time t (in minutes). Write the Initial Value Problem that $Q(t)$ satisfies?

(2) Initially, a tank contains 500 L (liters) of pure water. Water containing 0.3kg of salt per liter is entering at a rate of 2 L/min, and the mixture is allowed to flow out of the tank at a rate of 1 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms (kg). What is the IVP that $Q(t)$ satisfies?

(3) Initially, a tank contains 400 L of water with 10 kg of salt in solution. Water containing 0.1 kg of salt per liter (L) is entering at a rate of 1 L/min, and the mixture is allowed to flow out of the tank at a rate of 2 L/min. Let $Q(t)$ be the amount of salt at time t measured in kilograms. What is the IVP that $Q(t)$ satisfies?

(4) Consider a pond that initially constains 10 million gal of pure water. Water containing a polluted chemical flows into the pond at the rate of 6 million gal/year, and the mixture in the pond flows out at the rate of 5 million gal/year. The concentration $\gamma(t)$ of chemical in the incoming water varies as $\gamma(t) = 2 + \sin 2t$ grams/gal. Let $Q(t)$ be the amount of chemical at time t measured by millions of grams. What is the IVP that $Q(t)$ satisfies?

(5) A tank contains 200 gal of liquid. Initially, the tank contains pure water. At time $t = 0$, brine containing 3 lb/gal of salt begins to pour into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to drain away at the same rate. How many minutes must elapse before there are 100 lb of salt in the tank?

(6) A huge tank initially contains 10 gallons (gal) of water with 6 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the well-stirred mixture is allowed to flow out of the tank at a rate of 2 gal/min. What is the amount of the salt in the tank after 10 min?

(7) Initially a tank holds 40 gallons of water with 10 lb of salt in solution. A salt solution containing $\frac{1}{2}$b of salt per gallon runs into the tank at the rate of 4 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $y(t)$ be the amount of salt in the tank after t minutes. Then what is $y(20)$.

2.5. Problems

(1) A detective is called to the scene of a crime where a dead body has just been found.
- She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 80°F. The detective checks the programmable thermostat and finds that the room has been kept at a constant 68°F for the past 3 days.
- After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5° F. This last temperature reading was taken exactly one hour after the first one.
- The next day the detective is asked by another investigator, "What time did our victim die?" Assuming that the victim's body temperature was normal (98.6°) prior to death, what is her answer to this question? Newton's Law of Cooling can be used to determine a victim's time of death.

2.6. Problems

(1) What is the largest open interval in which the solution to the IVPs in part (a) and part (b) are guaranteed to exist by the Existence and Uniqueness Theorem?

(a) The IVP given by:

$$\begin{cases} \left(t^2+t-2\right) y' + e^t y = \frac{(t-4)}{(t-6)} \\ y(-3) = -1. \end{cases}$$

(b) The IVP given by:

$$\begin{cases} \left(t^2+t-2\right) y' + e^t y = \frac{(t-4)}{(t-6)} \\ y(5) = 47. \end{cases}$$

(2) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-3)\, y' + y = \frac{(t-3)\cdot\ln(t-1)}{t-10} \\ y(6) = -7. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

(3) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-1)\, y' + \sqrt{t+2}y = \frac{3}{t-3} \\ y(2) = -5. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

(4) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} t^2 y' + \ln|t-4|\, y = \frac{t-1}{\sin t} \\ y(5) = 9. \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

(5) Consider the IVP below

$$\frac{dy}{dt} = y^{1/5}, \quad y(0) = 0.$$

(a) Is this a Linear or nonlinear equation? Can you use Theorem 1 from Section 2.6?

(b) Using Theorem 2 from Section 2.6 (the general theorem), can you guarantee that there is a unique solution to this IVP? Why?

2.7. Problems

(1) Consider the following differential equation:

$$\frac{dy}{dt} = (y+2)(y-1)(y+5)$$

(a) Draw a Phase Line. Classify the Equilibrium solutions.
(b) Draw all possible sketch of solutions of this differential equation.
(c) Consider the IVP

$$\frac{dy}{dt} = (y+2)(y-1)(y+5), \quad y(0) = 3.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(2) Consider the following differential equation:

$$\frac{dy}{dt} = y(y-3)^2(y+4)$$

(a) Draw a Phase Line. Classify the Equilibrium solutions.
(b) Draw all possible sketch of solutions of this differential equation.
(c) Consider the IVP

$$\frac{dy}{dt} = y(y-3)^2(y+4), \quad y(0) = -5.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(d) Consider the IVP

$$\frac{dy}{dt} = y(y-3)^2(y+4), \quad y(0) = 1.$$

Let $y(t)$ be the unique solution that solves this IVP. Draw a sketch of $y(t)$ and use it to find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$?

(3) Let $y(t)$ be the unique solution to the IVP given by

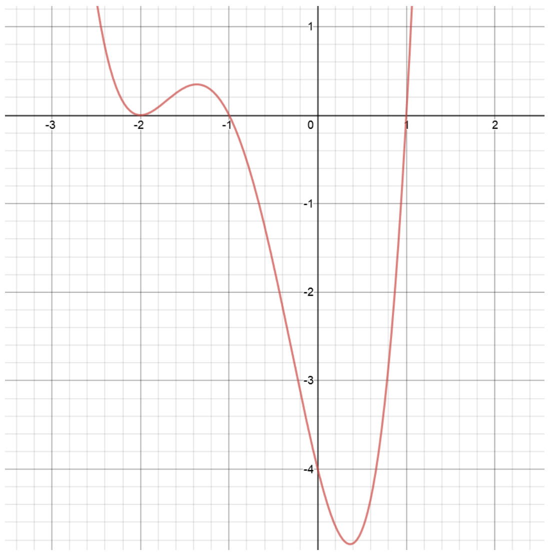
$$\frac{dy}{dt} = y^2 \sin y, \quad y(0) = 1.$$

Draw a Phase Line for the ODE to find out $\lim_{t\to\infty} y(t)$ for the unique solution of the IVP above.

(4) Consider the differential equation

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ is given by the following graph (in y versus $f(y)$):



(a) Draw the Phase Line and classify the Equilibrium solutions.

2.8. Problems

(1) Determine whether each of the following equations are exact. If it is exact, find the solution. Implicit solutions are fine.

(a) $(2x+3)+(2y-2)\,y'=0$

(b) $(2x+4y)+(2x-2y)\,y'=0$

(c) $\left(3x^2-2xy+2\right)dx+\left(6y^2-x^2+3\right)dy=0$

(d) $\left(2xy^2+2y\right)+\left(2x^2y+2x\right)y'=0$

(e) $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$

(f) $(e^x\sin y+3y)\,dx-(3x-e^x\sin y)\,dy=0$

(g) $\left(\dfrac{y}{x}+6x\right)dx+(\ln x-2)\,dy=0, x>0$

(2) Find the implicit particular solution to the initial value problem

$$\left(9x^2+y-1\right)dx-(4y-x)\,dy=0,\quad y(1)=0.$$

(3) Find the values of b for which the given equation is exact.

$$\left(ye^{2xy}+x\right)dx+bxe^{2xy}dy=0.$$

2.9. Problems

(1) Find the approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$ and 0.4 using Euler's Method with $h = 0.1$.

$$\frac{dy}{dt} = t + y, \quad y(0) = 1.$$

(2) Find the approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$ and 0.4 using Euler's Method with $h = 0.05$.

$$\frac{dy}{dt} = t + y^2, \quad y(0) = 1.$$

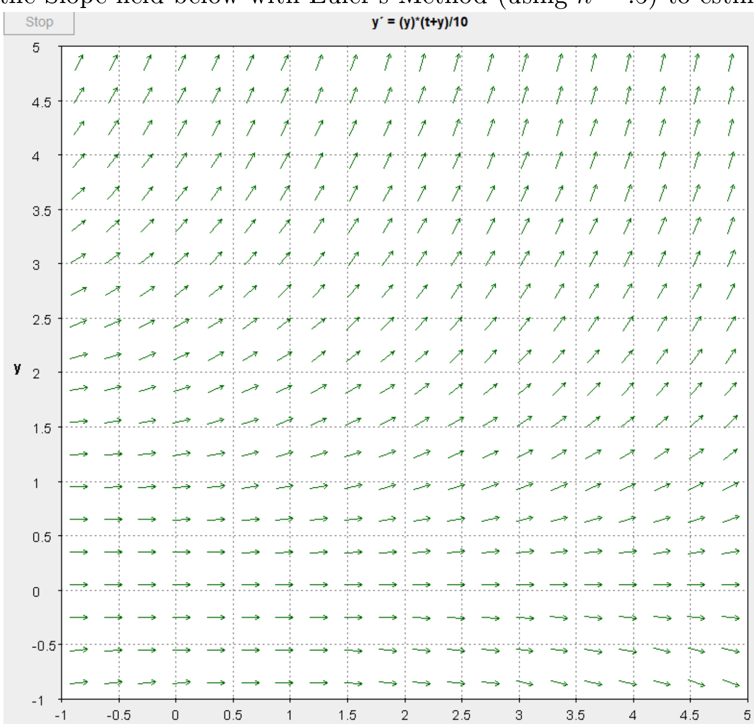
(3) Find the approximate value of $y(2)$ using Euler's Method with $h = 0.5$ for the solution of the following IVP

$$\frac{dy}{dt} = y(3 - ty), \quad y(0) = 0.5.$$

(4) Consider the solution $y(t)$ to the IVP:

$$\frac{dy}{dt} = y(t + y)/10, \quad y(0) = 1.$$

Use the Slope field below with Euler's Method (using $h = .5$) to estimate the value of $y(3)$:



(a)

CHAPTER 3

Second Order Linear Equations

3.1. Problems

No Homework

3.2. Problems

(1) Check if the following functions are solutions to the given EQ?
   (a) Check directly if $y_1 = 2e^{5t}$ is a solution or not to $y'' - 6y' + 5y = 0$?
   (b) Check directly if $y_2 = 2e^{t}$ is a solution or not to $y'' - 6y' + 5y = t$?

(2) Recall from the *Lecture Notes*, that if $y(t) = e^{rt}$ is a solution to the ODE given by

$$ay'' + by' + cy = 0$$

for constant a, b, c where $a \neq 0$, then the exponent r in front the t must be a solution to the *characteristic EQ* $ar^2 + br + c = 0$.
   (a) By yourself, rederive that if $y(t) = Ae^{rt}$ is a **solution** to the equation above then the number r must satisfy the *characteristic EQ* $ar^2 + br + c = 0$ or $A = 0$. (**Hint:** How do we check something is a solution? Well you just plug it to the LHS and RHS and check if they are equal!)

(3) Use the method given in Section 3.2 to find the general solution to

$$y'' + 5y' - 6y = 0$$

(4) Use the method given in Section 3.2 to find the general solution to

$$y'' - 7y' = 0$$

(5) Use the method given in Section 3.2 to find the particular solution to the IVP

$$y'' + y' - 20y = 0, \quad y(0) = 18, y'(0) = 9$$

3.3. Problems

(1) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} (t-3)\,y'' + \sin t y' + y = \frac{\ln(t-1)}{t-10} \\ y(15) = -7, y'(15) = 10 \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

(2) What is the largest open interval in which the solution of the initial value problem

$$\begin{cases} t^2 y'' + e^t y' + (t-1)\,y = \sqrt{t+2} \\ y(-1) = 1, y'(-1) = 5 \end{cases}$$

is guaranteed to exist by the Existence and Uniqueness Theorem?

(3) Consider the equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p, q are continuous in some interval I. What are the 2 things you have to do by the General Solution Theorem in order to find the general solution to the ODE above

(4) Consider the equation

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

(a) Is the function $y_1(t) = t^{\frac{1}{2}}$ a solution to this ODE?
(b) Is the function $y_2(t) = t^{-1}$ a solution to this ODE?
(c) Use the General Solution Theorem to show that

$$y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{-1}$$

gives the general solution to the ODE above.

3.4. Problems

(1) Find the general solution of the following 2nd Order Linear ODEs with constant coefficients.
   (a) $y'' + 16y = 0$
   (b) $y'' - 4y' + 9y = 0$
   (c) $y'' - 4y' + 29y = 0$

(2) Find the particular solution to the following IVP:

$$y'' - 8y' + 17y = 0, \quad y(0) = -4, y'(0) = -1.$$

3.5. Problems

3.5.1. Part 1; Repeated roots.

(1) Find the general solution of the following 2nd Order Linear ODEs with constant coefficients.
  (a) $y'' + 14y' + 49y = 0$
  (b) $y'' - 18y' + 81y = 0$

(2) Find the particular solution to the following IVP:

$$y'' - 4y' + 4y = 0, \quad y(0) = 12, y'(0) = -3.$$

3.5.2. Part 2; the method of reduction of order.

(1) Suppose you know that $y_1(t) = t$ is a solution to

$$t^2y'' - 3ty' + 3y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1y_1 + c_2y_2$ the general solution of this ODE.

(2) Suppose you know that $y_1(t) = t^{-1}$ is a solution to

$$2t^2y'' + ty' - 3y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1y_1 + c_2y_2$ the general solution of this ODE.

(3) Suppose you know that $y_1(t) = t$ is a solution to

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1y_1 + c_2y_2$ the general solution of this ODE.

(4) Suppose you know that $y_1(t) = t^2$ is a solution to

$$t^2y'' - 3ty' + 4y = 0, \quad t > 0.$$

Find a second solution $y_2(t)$ that makes $y = c_1y_1 + c_2y_2$ the general solution of this ODE.

3.6. Problems

(1) Consider the following non-homogeneous 2nd order ODE:

$$y'' + y' - 2y = e^{3t}.$$

(a) Find the General Solution
(b) Find the particular solution to the IVP:

$$y'' + y' - 2y = e^{3t}, \quad y(0) = \frac{1}{10}, y'(0) = \frac{13}{10}.$$

(2) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 2y' + 2y = e^{2t}.$$

(3) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 4y' + 3y = 4e^{3t}.$$

(4) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' - 2y' + y = e^{t}.$$

(5) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + y' - 6y = 52\cos(2t).$$

(6) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + 2y' + 3y = \sin(t).$$

(7) Find the general solution to the following non-homogeneous 2nd order ODE:

$$y'' + 9y = 27t^2.$$

(8) For the following ODEs. Use the method of undertermined coefficients (MOUC) to make the correct guess for the y_p. You DO NOT have to solve for the coefficients, $A, B, C \ldots$. Simply make the correct guess for the y_p.
(a) $y'' - 2y' + y = te^{t}$,
(b) $y'' + y' - 2y = t^2 e^{t}$,
(c) $y'' + y' = t^2 + \cos t$,
(d) $y'' + y' - 6y = e^{5t} + \sin(3t)$,
(e) $y'' + y' - 2y = te^{t} + t^2$,

3.7. Problems

(1) A mass weighing 8 lb stretches a spring $\frac{1}{2}$ feet. The mass is pulled down an additional 1 feet. and then set in motion with an upward velocity of 2 ft/sec. Assume that there is no damping force and that the downward direction is the positive direction. The gravity constant g is 32 $\frac{\text{ft}}{s^2}$. The function $u(t)$ describing the displacement of the mass from the equilibrium position as a function of time t satisfies what initial value problem?

(2) A mass of 5 kg stretches spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass.

3.8. Problems

(1) A 64 lb mass stretches a spring 4 feet. The mass is displaced an additional 5 feet. and then released; and is in a medium with a damping coefficients $\gamma = 7\frac{\text{lb sec}}{\text{ft}}$. Suppose there is no external forcing. Formulate the IVP that governs the motion of this mass:

(2) A 32 lb mass stretches a spring 4 feet. The mass is displaced an additional 6 feet. and then released with an initial velocity of 3 $\frac{\text{ft}}{\text{sec}}$; and is in a medium with a damping coefficients $\gamma = 2\frac{\text{lb sec}}{\text{ft}}$. Suppose there is an external forcing due to wind given by $F(t) = 3\cos(3t)$. Formulate the IVP that governs the motion of this mass:

(3) Consider the following undamped harmonic oscillator with external forcing:

$$u'' + 5u = \sin(3t), \quad u(0) = 1, \ u'(0) = -1.$$

What is the natural frequency? What is the frequency for the external force? Will you get resonance? What is your guess for u_p? Solve the IVP.

(4) Consider the following undamped harmonic oscillator with external forcing:

$$u'' + 16u = 7\cos(4t), \quad u(0) = 0, \ u'(0) = 0.$$

What is the natural frequency? What is the frequency for the external force? Will you get resonance? What is your guess for u_p?

3.9. Problems

(1) Consider the following ODE

$$y'' + 16y = \frac{1}{\sin(4t)}.$$

(a) Find a particular solution to the ODE above using the method of variation of parameters.
(b) What is the general solution to the ODE above.

(2) Find the general solution to

$$t^2y'' - 4ty' + 6y = t^3, \quad t > 0$$

given that

$$y_1(t) = t^2, \ y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

(3) Find the general solution to

$$t^2y'' - 3ty' + 3y = 8t^3, \quad t > 0$$

given that

$$y_1(t) = t, \ y_2(t) = t^3$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

(4) Find the general solution to

$$2t^2y'' + ty' - 3y = 2t^{5/2}, \quad t > 0$$

given that

$$y_1(t) = t^{-1}, \ y_2(t) = t^{3/2}$$

forms a fundamental set of solution for the corresponding homogeneous differential equation.

CHAPTER 4

Higher Order Linear Equations

4.1. Problems

(1) What is the largest interval for which there exists a unique solution by the Existence and Uniqueness Theorem for the following IVP:

$$\begin{cases} (t-5)\,y^{(4)} - \frac{\ln(t+7)}{t} y'' + e^t y = \frac{t^2+1}{(t-1)} \\ y(2) = -1 \\ y'(2) = 1 \\ y''(2) = 2 \\ y'''(2) = 5. \end{cases}$$

(2) Find general solution of

$$y''' + 10y'' + 7y' - 18y = 0.$$

(Hint: $r^3 + 10r^2 + 7r - 18 = (r-1)(r+2)(r+9)$)

(3) Find general solution of

$$y^{(4)} - 10y''' + 36y'' - 54y' + 27y = 0.$$

(Hint: $r^4 - 10r^3 + 36r^2 - 54r + 27 = (r-1)(r-3)^3$)

(4) Find general solution of

$$y^{(5)} - 4y^{(4)} + 13y''' - 36y'' + 36y' = 0.$$

(Hint: $r^5 - 4r^4 + 13r^3 - 36r^2 + 36r = r(r-2)^2\left(r^2+9\right)$)

(5) Find general solution of

$$y^{(4)} + 11y'' + 18y = 0.$$

(Hint: $r^4 + 11r^2 + 18 = \left(r^2+2\right)\left(r^2+9\right)$)

(6) Find general solution of

$$y^{(6)} + 32y^{(4)} + 256y'' = 0.$$

(Hint: $r^6 + 32r^4 + 256r^2 = r^2\left(r^2+16\right)^2$)

4.2. Problems

(1) Consider
$$y''' - 4y'' - 11y' + 30y = 4e^{-3t} + \cos t.$$
Find the general form of y_p. (Hint: $r^3 - 4r^2 - 11r + 30 = (r+3)(r-2)(r-5)$)

(2) Consider
$$y^{(4)} + 8y''' + 16y'' = t + e^t.$$
Find the general form of y_p(Hint: $r^4 + 8r^3 + 16r^2 = r^2(r+4)^2$)

(3) Consider
$$y^{(4)} - 10y''' + 36y'' - 54y' + 27y = 2te^t + \cos(3t),$$
and suppose you know that $y_h = c_1e^t + c_2e^{-3t} + c_3te^{-3t} + c_4t^2e^{-3t}$. Find the general form of y_p

(4) Consider
$$y^{(4)} - 2y''' = 2t + 1.$$
Find the general form of y_p. (Hint: $r^4 - 2r^3 = r^3(r-2)$)

CHAPTER 5

Systems of First Order Linear Equations

5.1. Problems

(1) Show that the functions

$$x_1(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$$
$$x_2(t) = \frac{1}{3}e^t - \frac{4}{3}e^{-2t}$$

solve the following system of first order differential equations IVP

$$\begin{cases} x_1' = x_2 & x_1(0) = 1 \\ x_2' = 2x_1 - x_2 & x_2(0) = -1 \end{cases}.$$

(2) Show that the functions

$$x_1(t) = \sin(2t)$$
$$x_2(t) = \cos(2t)$$

solve the following system of first order differential equations IVP

$$\begin{cases} x_1' = 2x_2 & x_1(0) = 0 \\ x_2' = -2x_1 & x_2(0) = 1 \end{cases}.$$

(3) Turn the following second order ODE

$$y'' + y' + 2y = t^2$$

into a system of first differential equations.

(4) Turn the following second order ODE

$$y'' - 2y' + 10y = 0.$$

into a system of first differential equations.

5.2. Problems

(1) Let $A = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, find $A\mathbf{x}$, $4\mathbf{x}$, and $\mathbf{x} + \mathbf{y}$.

(2) Turn the following system of first order equations into matrix-product form:

(a) Given by:

$$\begin{cases} x_1' = 3x_2 \\ x_2' = 9x_1 - 3x_2 \end{cases}$$

(b) Given by:

$$\begin{cases} x_1' = -x_1 + 2x_2 \\ x_2' = 7x_1 + 5x_2 \end{cases}$$

(3) Turn the following vector valued ODE into a system of first order equations.

(a) Given by:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 3 & 17 \end{pmatrix} \mathbf{x}$$

where $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

(b) Given by:

$$\boldsymbol{Y}' = \begin{pmatrix} 0 & 4 \\ 7 & 3 \end{pmatrix} \boldsymbol{Y}$$

where $\boldsymbol{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

(4) Find the equilibrium solutions of the following system:

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

(5) Find the equilibrium solutions of the following system:

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x}.$$

(6) Check if the following vector functions satisfy the following differential equations.

(a) Where

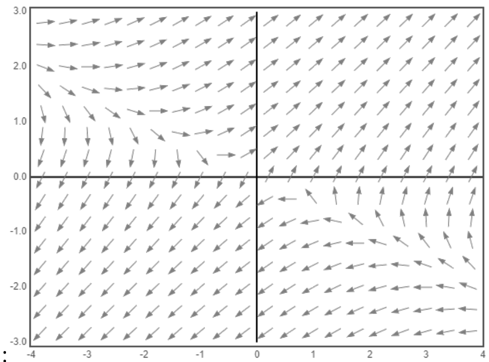
$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} \quad \mathbf{x}(t) = \begin{pmatrix} 4e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

(b) Where

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$

(7) Consider the following differential equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \mathbf{x},$$



with corresponding direction field:

(a) Using the direction field draw a Phase portrait. (A Phase portrait is graph a several possible different solutions of the ODE)

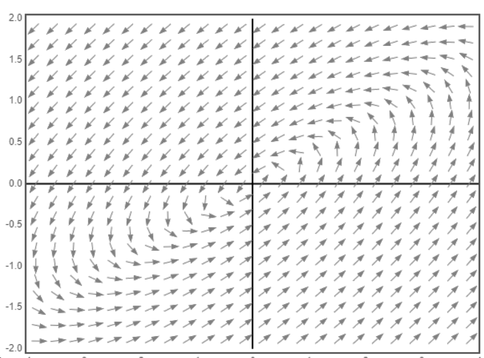
(b) Consider the following IVP:

$$\mathbf{x}' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 1.5 \end{pmatrix}$$

draw the unique soluton to this IVP and use it to **predict** long term behavior of $\lim_{t\to\infty} x_1(t)$ and $\lim_{t\to\infty} x_2(t)$.

(8) Consider the following differential equation

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$



with corresponding direction field:

(a) Using the direction field draw a Phase portrait. (A Phase portrait is graph a several possible different solutions of the ODE)

(b) Consider the following IVP:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2.5 \\ 0 \end{pmatrix}$$

draw the unique soluton to this IVP and use it to **predict** long term behavior of $\lim_{t\to\infty} x_1(t)$ and $\lim_{t\to\infty} x_2(t)$.

(c) Try to draw what the graphs of $x_1(t)$ and $x_2(t)$ are individually as functions of t, for the same IVP above.

5.3. Problems

(1) Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

(a) Show that the two functions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

are solutions to the system above.

(b) Solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

(2) Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}$$

(a) Check that the two functions are solutions to the system. If they are not solutions, then stop and do not do parts (b) and (c).

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} 4e^{-3t} \\ 4e^{-3t} \end{pmatrix}$$

(b) Are $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ linearly independent? If they are not, then stop and do not move on to part (c).

(c) Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}.$$

(3) Consider the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}$$

(a) Check that the two functions are solutions to the system. If they are not solutions, then stop and do not do parts (b) and (c).

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{pmatrix} \text{ and } \mathbf{x}^{(2)}(t) = \begin{pmatrix} 2e^{-3t} + e^{-4t} \\ 2e^{-3t} + 2e^{-4t} \end{pmatrix}$$

(b) Are $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ linearly independent? If they are not, then stop and do not move on to part (c).

(c) Find the general solution of

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}.$$

5.4. Problems

(1) Find the general solution of the given system of differential equations:

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

(c) $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \mathbf{x}$

(d) The system

$$x_1' = 3x_1 + 4x_2$$
$$x_2' = x_1$$

(2) Solve the following initial value problems and find $x_1(t)$ and $x_2(t)$.

(a) Where the IVP is given by

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \mathbf{x} \quad , \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b) Where the IVP is given by

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \mathbf{x} \quad , \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

5.5. Problems

(1) Sketch the Phase portrait for the following systems and classify the equilibrium solution for the following systems.

(a) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$ and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = -1, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -2, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -3, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

(c) $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = 5, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(d) The system

$$x_1' = 3x_1 + 4x_2$$
$$x_2' = x_1$$

and assume you know that the associated eigenvalues and eigenvector are

$$\lambda_1 = 4, \qquad \mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -1, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(2) Suppose A is a matrix and consider the following system

$$\mathbf{x}' = A\mathbf{x}.$$

(a) Suppose the matrix A has the following associated eigenvalues with corresponding eigenvectors

$$\lambda_1 = -4, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\lambda_2 = -2, \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} .$$

Draw the Phase portrait for $\mathbf{x}' = A\mathbf{x}$.

(b) Consider the same matrix as in Part (a). Draw the trajectory curve for $t \geq 0$ of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

(c) Consider the same matrix as in Part (a). Draw the trajectory curve for $t \geq 0$ of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

(d) Consider the same matrix as in Part (a). Draw the trajectory curve for $t \geq 0$ of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

(e) Consider the same matrix as in Part (a). Draw the trajectory curve for $t \geq 0$ of the solution of the following IVP:

$$\mathbf{x}' = A\mathbf{x} \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

5.6. Problems

(1) Consider the following system
$$\mathbf{x}' = \begin{pmatrix} 0 & 9 \\ -9 & 0 \end{pmatrix} \mathbf{x}$$
   (a) Find the general solution
   (b) Solve the IVP:
$$\mathbf{x}' = \begin{pmatrix} 0 & 9 \\ -9 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
   and find $x_1(t)$ and $x_2(t)$.
   (c) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
   (d) Classify the Equilibrium solution
   (e) Draw the Phase Portrait

(2) Consider the following system
$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$
   (a) Find the general solution
   (b) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
   (c) Classify the Equilibrium solution
   (d) Draw the Phase Portrait

(3) Consider the following system
$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$
   (a) Find the general solution
   (b) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
   (c) Classify the Equilibrium solution
   (d) Draw the Phase Portrait

(4) Consider the following system
$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$$
   (a) Find the general solution
   (b) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
   (c) Classify the Equilibrium solution
   (d) Draw the Phase Portrait

(5) Draw the Phase portrait of the following system
$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

(6) Draw the Phase portrait of the following system
$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{x}.$$

5.7. Problems

(1) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution
(b) Solve the IVP

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(c) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
(d) Classify the Equilibrium solution
(e) Draw the Phase Portrait

(2) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution
(b) Solve the IVP

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

(c) Determine the direction of the oscillations in the phase plane (do solutions go clockwise or counterclockwise)
(d) Classify the Equilibrium solution
(e) Draw the Phase Portrait

(3) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution
(b) Draw the Phase Portrait

(4) Consider the following system

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

(a) Find the general solution
(b) Draw the Phase Portrait

CHAPTER 6

The Laplace Transform

6.1. Problems

(1) Use the definition of Laplace transform to find the Laplace transform of $f(t) = 1$. That is, find $\mathcal{L}\{1\}$.

(2) Use the definition of Laplace transform to find the Laplace transform of $f(t) = t$. That is, find $\mathcal{L}\{t\}$.

(3) Use the definition of Laplace transform to find the Laplace transform of $f(t) = t^2$. That is, find $\mathcal{L}\left\{t^2\right\}$.

(4) Use the properties of Laplace transform and the following facts

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s}, s > 0 \\
\mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a}, s > a, \\
\mathcal{L}\{t\} &= \frac{1}{s^2}, s > 0, \\
\mathcal{L}\left\{t^2\right\} &= \frac{2}{s^3}, s > 0, \\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2}, s > 0, \\
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2}, s > 0,
\end{aligned}$$

to compute the Laplace transforms of the following functions.

(a) $\mathcal{L}\left\{2e^{5t} + 7\cos(3t) + 2t\right\} =$

(b) $\mathcal{L}\left\{-7e^{-9t} - 5t^2 - 5\sin(3t)\right\} =$

(c) $\mathcal{L}\left\{-5\sin(\sqrt{7}t) + 2 + 5t\right\} =$

(d) $\mathcal{L}\left\{4e^{-t} - 6e^{3t} + \cos(3t)\right\} =$

6.2. Problems

(1) Use the table of Laplace Transforms to help you compute the following inverse Laplace transforms.
   (a) $\mathcal{L}^{-1}\left\{\frac{5}{s-6}\right\}$
   (b) $\mathcal{L}^{-1}\left\{\frac{5}{7-s}+\frac{1}{s+3}\right\}$
   (c) $\mathcal{L}^{-1}\left\{\frac{3}{s+9}-\frac{10}{s^2}\right\}$
   (d) $\mathcal{L}^{-1}\left\{\frac{3}{s^2+7}+\frac{2}{(s-5)^3}\right\}$
   (e) $\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+36}\right\}$
   (f) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}+\frac{2}{s}-\frac{s-1}{(s-1)^2+25}\right\}$

(2) Solve the following IVP using Laplace Transforms:
$$y'+4y=e^{-t},\quad y(0)=0$$

(3) Solve the following IVP using Laplace Transforms:
$$y'+y=e^{-2t},\quad y(0)=2$$

(4) Solve the following IVP using Laplace Transforms:
$$y'+7y=1,\quad y(0)=3.$$

6.3. Problems

(1) What is the correct form of the partial fractions?
  (a) $\dfrac{5s-1}{(s-3)(s^2+2s+5)} =$
  (b) $\dfrac{s-2}{(s-2)^2(s+5)} =$
  (c) $\dfrac{s+1}{(s^2+9)(s^3+2)} =$
  (d) $\dfrac{s}{(s+1)(s^2+10)s^3} =$
(2) Take the inverse Laplace Transforms of the following:
  (a) $F(s) = \frac{1}{s^2-8s+7}$
  (b) $F(s) = \frac{s+7}{s^2+6s+13}$
  (c) $F(s) = \frac{2s-1}{s^2-8s+18}$
(3) Solve the following IVP using Laplace Transforms:
$$y'' + 4y = 8, \quad y(0) = 11, y'(0) = 5.$$
(4) Solve the following IVP using Laplace Transforms:
$$y'' - 4y' + 5y = 2e^t, \quad y(0) = 3, y'(0) = 1.$$

6.4. Problems

(1) Take the Laplace transforms of the following functions
   (a) $f(t) = u_7(t)e^{6(t-7)}$
   (b) $f(t) = u_2(t)e^{-9(t-2)}$
   (c) $f(t) = u_2(t)\,(t-2)^3$
   (d) $f(t) = u_6(t)\sin(3\,(t-6))$
   (e) $f(t) = u_1(t)\cos(7\,(t-1))$
   (f) $f(t) = \begin{cases} 5 & t < 7 \\ 8 & t \geq 7 \end{cases}$

(2) Take the inverse Laplace transforms of the following functions
   (a) $F(s) = \frac{e^{-3s}}{s+1}$
   (b) $F(s) = \frac{e^{-5s}}{s-7}$
   (c) $F(s) = \frac{2e^{-2s}}{s^2+4}$
   (d) $F(s) = \frac{se^{-9s}}{s^2+7}$
   (e) $F(s) = \frac{(s+2)e^{-3s}}{(s+2)^2+16}$

(3) Take the inverse Laplace transforms of

$$F(s) = \frac{e^{-3s}}{s^2 - 3s + 2}.$$

(4) Take the inverse Laplace transforms of

$$F(s) = \frac{se^{-9s}}{s^2 + 6s + 11}.$$

6.5. Problems

(1) Find the solution to the following IVP using Laplace Transforms

$$y' + 9y = u_5(t), \quad y(0) = -2.$$

(2) Find the solution to the following IVP using Laplace Transforms

$$y' + y = u_7(t)e^{-2(t-7)}, \quad y(0) = 1.$$

(3) Find the solution to the following IVP using Laplace Transforms

$$y'' + 9y = u_3(t)\sin\left(2\left(t-3\right)\right), \quad y(0) = 0, y'(0) = 0.$$

CHAPTER 7

Series Solutions of Second Order Linear Systems

7.1. Problems

Read Chapter 6 Lecture Notes.