Math 235
Written Assignment 3

Solutions to the problems below are due at the beginning of class on Wednesday, April 22nd. Please follow the guidelines in the course policy handout when completing this assignment.

1. A positive integer n is called *square-full* if $p^2|n$ for every prime factor p of n.

Prove that if n is square-full, then it can be written in the form $n = a^2b^3$ for some positive integers a and b. (*Hint:* Use the canonical prime factorization of n.)

2. Write a proof (different from the one presented in class) of Theorem 3.4 by assuming that there is a largest prime p and using the integer $N = p! + 1$ to produce a contradiction.

3. Prove that there are infinitely many primes of the form $6n + 5$ where n is an integer in two ways:

(a) By using a proof similar to the proof of Theorem 3.6 on pp. 53-54 of your textbook.

(b) By using Dirichlet's Theorem (Theorem 3.7).

4. Prove the following statements about congruences.

(a) If $a \equiv b \pmod{n}$ and the integers a, b, n are all divisible by $d > 0$, then $a/d \equiv b/d \pmod{n/d}$.

(b) If $a \equiv b \pmod{n}$, then $\gcd(a, n) = \gcd(b, n)$.

5. (a) Use congruences to prove that the integer $53^{103} + 103^{53}$ is divisible by 39.

(b) Use congruences to prove that for all $n \in \mathbb{N}$, $13|3^{n+2} + 4^{2n+1}$.