Math 117
Answers to Exam 3
March 7, 2007

1. Evaluate the surface integral \( \iint_{\sigma} x + y + z \, dS \) where \( \sigma \) is the portion of the plane \( x + y + z = 3 \) that lies in the first octant.
   \[
   \iint_{\sigma} x + y + z \, dS = \int_0^3 \int_{0}^{3-x} 3\sqrt{3} \, dx \, dy = 27\sqrt{3}/2
   \]

2. Use Green’s theorem to evaluate the line integral \( \oint_C e^{x^3} \, dx - 2x^2 \, dy \) where \( C \) is the circle \( x^2 + y^2 = 9 \) oriented counterclockwise.
   \[
   \oint_C e^{x^3} \, dx - 2x^2 \, dy = \iint_R -4x \, dA = 0
   \]

3. Find parametric equations for the following surfaces. Be sure to include the range of \( u \) and \( v \) values for each parametrization.

   NOTE: There is more than one correct answer.
   
   (a) \( \sigma \) is the portion of the sphere \( x^2 + y^2 + z^2 = 16 \) that lies BELOW the \( xy \)-plane.
   \[
x = 4 \sin u \cos v, \quad y = 4 \sin u \sin v, \quad z = 4 \cos u; \quad 0 \leq v \leq 2\pi, \quad \pi/2 \leq u \leq \pi
   \]
   
   (b) \( \sigma \) is the portion of the cone \( z = \sqrt{x^2 + y^2} \) that lies below the plane \( z = 4 \).
   \[
x = u, \quad y = v, \quad z = \sqrt{u^2 + v^2}; \quad 0 \leq u^2 + v^2 \leq 16
   
   OR
   \[
x = u \cos v, \quad y = u \sin v, \quad z = u; \quad 0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi
   \]

4. Use a line integral to find the area enclosed by the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \). (You may find the identities \( \cos^2 t = \frac{1}{2}(1 + \cos 2t) \) or \( \sin^2 t = \frac{1}{2}(1 - \cos 2t) \) useful.)
   \[
   \frac{1}{2} \oint_C x \, dy - y \, dx = 6\pi
   \]

5. (a) Is the vector field \( \mathbf{F}(x, y) = \sin(xy)\mathbf{i} + (y^2 + \sin(xy))\mathbf{j} \) conservative? Justify your answer.

   The vector field is not conservative because \( \frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x} \).

   (b) The vector field \( \mathbf{G}(x, y) = (2xye^{x^2})\mathbf{i} + (e^{x^2} + 8y)\mathbf{j} \) is conservative. Use the Fundamental Theorem of Line Integrals to evaluate \( \int_C \mathbf{G} \cdot d\mathbf{r} \) where \( C \) is a curve from \((0, 1)\) to \((-1, 0)\).
   \[
   \phi(x, y) = ye^{x^2} + 4y^2, \quad \text{so} \quad \int_C \mathbf{G} \cdot d\mathbf{r} = \phi(-1, 0) - \phi(0, 1) = -5.
   \]
6. Set up, but DO NOT EVALUATE, an iterated integral to find the surface area of the portion of the cylinder \( y^2 + z^2 = 4 \) that lies above the rectangle \( R = \{(x, y) \mid 0 \leq x \leq 2, \ 0 \leq y \leq 1 \} \) in the \( xy \)-plane.

\[
\int \int_{\sigma} 1 \, dS = \int_0^2 \int_0^1 \sqrt{\frac{y^2}{4-y^2} + 1} \, dy \, dx
\]

or

\[
\int \int_{\sigma} 1 \, dS = \int_0^2 \int_{\pi/3}^{\pi/2} \sqrt{2} \, du \, dv
\]

7. Evaluate the line integral \( \int_{C} \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x, y) = (e^{-y} \cos x) \mathbf{i} - (e^{-y} \sin x) \mathbf{j} \) and \( C \) is the line segment from \((\pi/2, 1)\) to \((-\pi/2, 0)\).

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = -1 - e^{-1}
\]

Hint: Use the Fundamental Theorem of Line Integrals.