Math 113-03
Exam 2
November 10th, 2008

1. (21 points) Evaluate the following integrals.

(a) \[ \int_0^1 4x^3 + e^x \, dx \]
\[ = \left[ x^4 + e^x \right]_0^1 \]
\[ = 1 + e - 1 = e \]

(b) \[ \int \sqrt{x} + \sin(2x) \, dx \]
\[ = \int x^{1/2} + \sin(2x) \, dx \]
\[ = \frac{2}{3} x^{3/2} - \frac{1}{2} \cos(2x) + C \]

(c) \[ \int \frac{x}{4 + x^2} \, dx \]
\[
\begin{align*}
   \text{Let } u &= 4 + x^2 \\
   du &= 2x \, dx \\
   &= \frac{1}{2} \int \frac{1}{u} \, du \\
   &= \frac{1}{2} \ln |u| + C \\
   &= \frac{1}{2} \ln (4 + x^2) + C
\end{align*}
\]

2. (6 points) State the Fundamental Theorem of Calculus.

If \( f \) is continuous on \([a, b]\) and \( F \)
is an antiderivative of \( f \), then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]
3. (12 points) (a) Find the numerical value of the sum \( \sum_{k=1}^{100} (3 + 2k) \).

\[
= \sum_{k=1}^{100} 3 + 2 \sum_{k=1}^{100} k = 300 + 2 \left( \frac{100 \cdot (101)}{2} \right) = 10,400
\]

(b) Express the sum \( \sum_{k=3}^{n} 4k \) in closed form.

\[
= \sum_{k=1}^{n} 4k - (4 + 8) = 4 \left( \frac{n(n+1)}{2} \right) - 12 = 2n^2 + 2n - 12
\]

4. (12 points) (a) Find an approximation to the integral \( \int_{0}^{2} x^2 \, dx \) by dividing the interval \([0,2]\) into 4 subintervals of equal length and computing the Riemann sum \( \sum_{k=1}^{4} f(x_k^*) \Delta x \). Use the left endpoint of the \( k \)th interval for \( x_k^* \).

\( \Delta x = \frac{2}{4} = \frac{1}{2} \)

\( x_0^* = 0, x_1^* = \frac{1}{2}, x_2^* = 1, x_3^* = \frac{3}{2} \)

\[
\frac{1}{4} \sum_{k=1}^{4} f(x_k^*) \Delta x = \left( 0^2 + \left( \frac{1}{2} \right)^2 + 1^2 + \left( \frac{3}{2} \right)^2 \right) \left( \frac{1}{2} \right) = \frac{7}{4}
\]

(b) Sketch a graph of \( f(x) = x^2 \) together with the rectangles whose areas are represented in the sum in part (a).
5. (28 points) Evaluate the following integrals.

(a) \( \int_0^2 xe^x \, dx \)

\( u = x \) \quad \text{dv} = e^x \, dx \\
\text{du} = dx \quad \text{v} = e^x \\
= xe^x \bigg|_0^2 - \int_0^2 e^x \, dx = xe^x - e^x \bigg|_0^2 = 2e^2 - e^2 - (0 - 1) = e^2 + 1 \\

(b) \( \int_0^1 x\sqrt{3 + x^2} \, dx \)

\( u = 3 + x^2 \) \\
\text{du} = 2x \, dx \\
when \( x = 0, \, u = 3 \) \\
when \( x = 1, \, u = 4 \)

\( = \int_3^4 \frac{1}{2} u^{1/2} \, du \)

\( = \frac{1}{2} (\frac{3}{5}) u^{3/2} \bigg|_3^4 \)

\( = \frac{1}{3} (8 - 3\sqrt{3}) \)

(c) \( \int \tan^{-1}(6x) \, dx \)

\( u = \tan^{-1}(6x) \) \quad \text{dv} = dx \\
\text{du} = \frac{6}{1 + 36x^2} \, dx \quad \text{v} = x \\
= x \tan^{-1}(6x) - \int \frac{6x}{1 + 36x^2} \, dx \)

\( \text{I} \) \quad \text{u = 1 + 36x^2} \\
\text{du = 72x dx} \\
\Rightarrow \quad x \tan^{-1}(6x) - \frac{1}{12} \int \frac{du}{u} \\
= x \tan^{-1}(6x) - \frac{1}{12} \ln|u| + C \\
= x \tan^{-1}(6x) - \frac{1}{12} \ln(1 + 36x^2) + C \\

(d) \( \int x^2 \sin(3x) \, dx \)

\( u = x^2 \) \quad \text{dv = sin(3x) dx} \\
\text{du = 2x dx} \quad \text{v = -1/3 cos(3x)} \\
= -\frac{x^2}{3} \cos(3x) + \frac{2}{3} \int x \cos(3x) \, dx \\
\text{I} \) \quad \text{u = x} \\
\text{dv = cos(3x)} \\
\text{du = dx} \quad \text{v = 1/3 sin(3x)} \\
= -\frac{x^2}{3} \cos(3x) + \frac{2}{3} \left[ \frac{x}{3} \sin(3x) - \frac{1}{9} \cos(3x) \right] + C \\
= -\frac{x^2}{3} \cos(3x) + \frac{2x}{9} \sin(3x) + \frac{2}{27} \cos(3x) + C
6. (7 points) Set up, but DO NOT EVALUATE, a definite integral to find the area of the region bounded by $y = 4 + x^2$ and $y = 5x$.

\[ 4 + x^2 = 5x \]
\[ x^2 - 5x + 4 = 0 \]
\[ (x - 4)(x - 1) = 0 \]
\[ x = 1 \text{ or } 4 \]

\[ \text{area} = \int_{1}^{4} 5x - (4 + x^2) \, dx \]

7. (7 points) Set up, but DO NOT EVALUATE, a definite integral to find the volume of the solid of revolution obtained by revolving the region bounded by $y = x^2 + 1$ and $y = 2$ about the line $x = 3$.

\[ \text{volume} = \int_{1}^{2} \pi \left( (3 + \sqrt{y} - 1)^2 - (3 - \sqrt{y} - 1)^2 \right) \, dy \]

8. (7 points) A cone-shaped reservoir is 6 feet in diameter across the top and 5 feet deep. The reservoir is filled to the top with water. Set up, but DO NOT EVALUATE, a definite integral to find the work required to pump all of the water to a drain that is at a height 5 feet above the top of the tank. The density of water is 62.4 lb/ft$^3$.

\[ \text{Work} = \int_{0}^{5} \rho \left( \frac{3}{5} x \right)^2 (10 - x) \, dx \]

\[ W = \frac{3}{5} \]