Math 660: Principal curvatures

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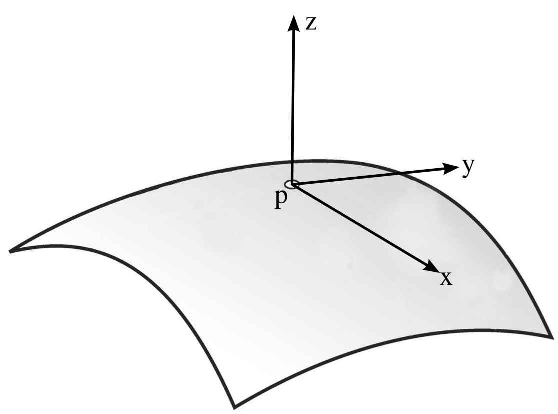
Abstract

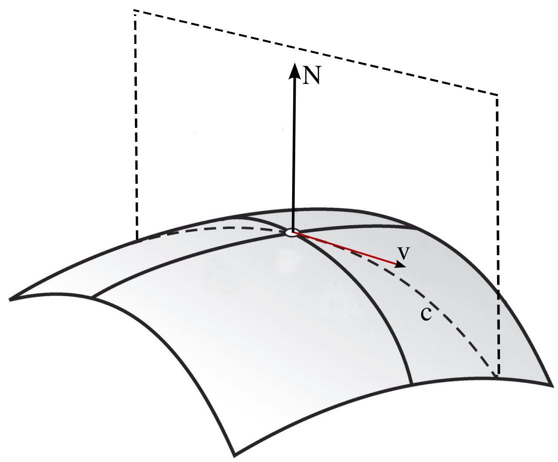
Our goal is to explain the idea of principal curvatures of surfaces in $\mathbb{R}^3$ as simply as possible, without referring to the shape operator or covariant derivative. These notes were used in a Riemannian geometry course at UPenn for students who had not previously studied differential geometry of curves and surfaces.

Let's assume we have a surface M in $\mathbb{R}^3$ that is given by the graph of a smooth function $z = f(x, y)$. Assume that M passes through the origin, p, and its tangent plane there is the $\{z = 0\}$ plane[1]. Let $N = (0, 0, 1)$, a unit normal to M at p.



Let v be a unit vector in T_pM, say $v = (v_1, v_2, 0)$. Let c be the parameterized curve given by slicing M through the plane spanned by v and N:

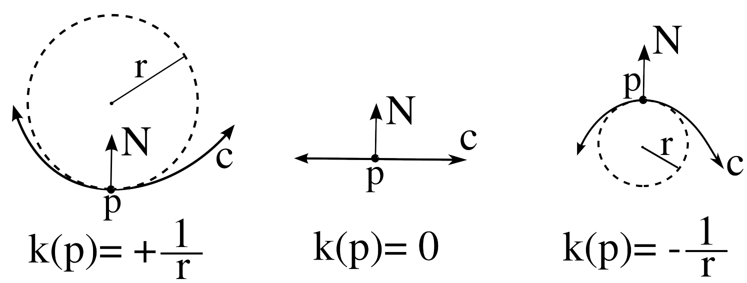
[1]In general, if M is any surface in $\mathbb{R}^3$, and if $p \in M$, then we may apply rigid motions to assume without loss of generality that p is the origin and T_pM is the $\{z = 0\}$ plane. Near p, M is locally a graph $z = f(x, y)$.



That is,

$$c(t) = (v_1 t, v_2 t, f(v_1 t, v_2 t)).$$

Being a plane curve, c has a *signed curvature* κ_v at p with respect to the unit normal N: κ_v is the reciprocal of the radius of the *osculating circle* to c at p, taken with sign as in the examples below:



More rigorously, κ_v is defined by the formula:

$$c''(s)|_{s=0} = \kappa_v N(p), \tag{1}$$

where $s = s(t)$ represents arc length with $s(0) = 0$ (i.e., $s(t) = \int_0^t |c'(t)|dt$). (For instance, one can readily verify that a circle of radius r has signed curvature $1/r$ at each point with respect to the inward-pointing unit normal).

Exercise 1.

a. *Prove that in the present case, you may without loss of generality replace $c''(s)|_{s=0}$ with $c''(t)|_{t=0}$.*

b. *Using formula (1) prove that*

$$\kappa_v = f_{xx}(0,0)v_1^2 + 2f_{xy}(0,0)v_1 v_2 + f_{yy}(0,0)v_2^2,$$

where f_{xx}, etc. are the partial derivatives of f.

c. Conclude that κ_v is given by $(\operatorname{Hess} f)(v,v)$. *That is,*

$$\kappa_v = [v_1 \;\; v_2] \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Now we have that κ_v is given by a quadratic form (the Hessian). The *principal curvatures* of the surface at p will be the largest and smallest possible values λ_1, λ_2 of κ_v (as v ranges over the possible unit tangent vectors), and the corresponding unit tangent vectors v will be called the *principal directions*, e_1 and e_2. It is a standard fact from linear algebra that λ_1 and λ_2 are the eigenvalues of the Hessian matrix, with eigenvectors e_1 and e_2. Since the Hessian is symmetric, λ_1 and λ_2 are real and e_1 and e_2 are orthogonal.

The *Gauss curvature* of M at p is the number $\lambda_1\lambda_2$ (i.e., the determinant of the Hessian) and the *mean curvature* of M at p is $\lambda_1 + \lambda_2$ (i.e., the trace of the Hessian).

Exercise 2. *Find the principal curvatures, principal directions, Gauss curvature, and mean curvature at the origin for*

1. *the graph of $z = x^2 + y^2$,*
2. *the graph of $z = x^2$,*
3. *the graph of $z = xy$, and*
4. *a sphere of radius r passing through the origin, tangent to the $\{z = 0\}$ plane.*

Warning: these formulas for the principal, Gauss, and mean curvatures are only valid at p, since their derivation relied on the fact that $f_x(0) = f_y(0) = 0$.