Math 313/513, Homework 6 (due Tues. Feb. 28)

Name:____________________________________ 313 or 513 (circle)

Reading

- Read sections 5.1–5.2

Book problems

- Math 313:

1. Find bases for the column space and null space of the following matrices:

$$A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \\ -3 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

2. Find an example of a 2×2 Markov (stochastic) matrix M such that M has a zero entry somewhere, but M^2 has all strictly positive entries.

3. Consider a car rental company servicing only Philadelphia, Baltimore, and Newark. The total number of cars is constant. Each day, 90% of the cars that start at any location also end up there. 5% of the cars that begin in Philly end up in Baltimore. 1% of the cars that begin in Baltimore end up in Philly. Also, 2% of the cars that begin in Newark end up in Baltimore.

   Set up the Markov matrix for this problem (make sure it is in fact a Markov matrix!). Find the steady state vector exactly. In the long run, what fraction of the cars end up in each of the three cities?

4. Let A be an invertible $n \times n$ matrix. Find a formula for $\det(A^{-1})$ in terms of $\det(A)$.

5. Strang, 5.1, #3

6. Strang, 5.1, #12

7. Strang, 5.1, #15

8. Compute the determinants of the following matrices in any way you wish (by hand). Hint: for some, it might be better to row-reduce; for others, it might be better to expand down a particular column or across a particular row.

$$\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{vmatrix}, \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ -2a+d & -2b+e & -2c+f \end{vmatrix}$$

- Math 513: all of the above, plus:
  - Prove carefully the following parts of the Fundamental Theorem of Linear Algebra:
    a. If $A\vec{x} = \vec{b}$ is solvable for every $\vec{b}$, then A is invertible.
    b. If the determinant of A is not zero, then the columns of A are linearly independent.

MATLAB assignment
This week, let's look at how many iterations it takes for a Markov chain to get "close" to its steady state.

Goal: write a function `lastname_markov` that takes in as arguments a stochastic matrix M and an initial state $\vec{x}_0$ (a probability vector) and returns the number of iterations k necessary for the kth state $\vec{x}_k = M^k\vec{x}_0$ to be within 0.01 (in Euclidean norm, explained below) of the true steady state. Some hints to get you started:

- To find the true steady state, you'll need to find a solution to $(M - I)\vec{x} = \vec{0}$ whose entries are nonnegative and sum to 1. To solve this system in MATLAB, it may be helpful to use `rref` to put a matrix in reduced echelon form. Feel free to investigate other ways of solving a system that has a free variable.
- To create the $n \times n$ identity, use `eye(n)`
- The difference of two vectors in Euclidean norm is the square root of $(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$ (dot product).
- You may wish to use a while loop so that you can iterate until x_k is close enough to the true steady state.
- Consider adding in a mechanism that prevents an infinite loop in case something goes wrong.
- You should test out your code on some examples, like those we did in class and the above problem on car rentals.

Include your comments, and submit your code to Blackboard. Please remember to name your file in the form `lastname_markov.m`