Test 2 Practice — October 2009
Math 113

- You may use one 3x5 card with notes (both sides, if you wish). Calculators or computers are not allowed.
- The actual test will be comparable in length to test 1.

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(1) Find all points on the curve $x^2 + \frac{y^2}{4} = 25$ where the tangent line has slope -1.

We first do implicit differentiation:

$$\frac{d}{dx}(x^2 + y^2/4) = \frac{d}{dx}(25)$$
$$2x + \frac{2y}{4}\frac{dy}{dx} = 0$$

We could solve for $\frac{dy}{dx}$ at this point. However, in this problem, we are looking for points where $\frac{dy}{dx} = -1$, so we can just plug in -1 for $\frac{dy}{dx}$ now:

$$2x + \frac{2y}{4}(-1) = 0$$
$$2x = y/2$$
$$y = 4x$$

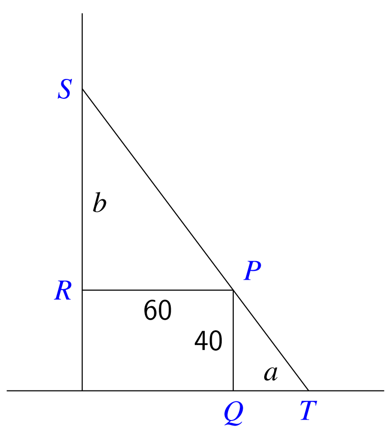
So, we are looking for points on the curve where $y = 4x$. We do this by substituting $y = 4x$ into the original equation.

$$x^2 + (4x)^2/4 = 25$$
$$5x^2 = 25$$
$$x = \pm\sqrt{5}$$

We can now use $y = 4x$ to get the corresponding y-coordinates. So, the points on the curve where $\frac{dy}{dx} = -1$ are $(\sqrt{5}, 4\sqrt{5})$ and $(-\sqrt{5}, -4\sqrt{5})$.

(2) A train is traveling west at 120 feet per second down a straight track. There is a utility pole 40 feet from the track, and the train's headlight casts a shadow of the pole on a wall that is 60 feet west of the pole and perpendicular to the track. How fast is the shadow moving when the front of the train is 30 feet east of the pole?

(Everything in blue here would not necessarily be part of what you would write when solving the problem; it's just to help explain the work that is being done.)



T is the position of the train, P is the position of the pole, S is the position of the shadow, Q is the point on the track south of the pole, and R is the point on the wall west of the pole. The triangles PRS and TQP are similar, so

$$\frac{a}{40} = \frac{60}{b}$$
$$ab = 40 \cdot 60 = 2400$$
$$(ab)' = (2400)'$$
$$ab' + a'b = 0$$

Now we plug in what we know and solve for what we want. At the moment in question,

$$a = 30, \quad a' = -120, \quad b = \frac{2400}{a} = 80$$

and plugging these into $ab' + a'b = 0$ gives

$$30b' + (-120)(80) = 0$$
$$b' = \frac{120 \cdot 80}{30} = 4 \cdot 80 = 320$$

So the shadow is moving north at $\boxed{\text{320 feet per second}}$.

(3) Find the 2nd-order Taylor polynomial for $\ln x$ at $x = 1$.

We have

$$f(x) = \ln x \qquad f(1) = \ln 1 = 0$$
$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$
$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$

so the 2nd-order Taylor polynomial at $x = 1$ is

$$p_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$= \boxed{(x-1) - \frac{1}{2}(x-1)^2}.$$

(4) Identify where the function $f(x) = xe^x$ is increasing/decreasing and concave up/down, and use that information to sketch the graph.

Critical points:

$$f'(x) = xe^x + 1e^x = (x+1)e^x = 0$$
$$x + 1 = 0 \quad (e^x \text{ is never 0 so we can divide it out.})$$
$$x = -1$$

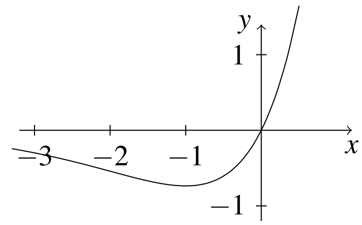
Possible inflection points:

$$\begin{aligned} f''(x) &= (x+1)e^x + 1e^x = (x+2)e^x = 0 \\ x+2 &= 0 \\ x &= -2 \end{aligned}$$

We now break up the domain, $(-\infty,\infty)$, at these points and identify where f' and f'' are $+/-$, keeping in mind that f' can only change sign at $x=-1$ and f'' can only change sign at $x=-2$. E.g., $f'(-2) = (-2+1)e^{-2} < 0$, so $f'(x) < 0$ on $(-\infty,-1)$. We fill in the rest of the table similarly.

	$(-\infty,-2)$	$(-2,-1)$	$(-1,\infty)$
f'	$-$ dec.	$-$ dec.	$+$ inc.
f''	$-$ conc. down	$+$ conc. up	$+$ conc. up

To get a sense of scale, our minimum at $x=-1$ is $f(-1) = -e^{-1} = -\frac{1}{e}$ which is between $-1/3$ and $-1/2$. Using this information, and the fact that $f(0) = 0e^0 = 0$, we can sketch the graph:



(5) You have 1 meter of wire. You are going to cut the wire into two pieces, and form two squares. What should the lengths be in order to maximize the sum of the areas of the two squares?

Let x be the length of the first piece (so $1-x$ is the length of the other piece). Forming these pieces of wire into squares gives squares of size $x/4$ and $(1-x)/4$. We wish to maximum the sum of the areas

$$A(x) = (x/4)^2 + \left(\frac{1-x}{4}\right)^2 = \frac{1-2x+2x^2}{16}, \quad 0 \le x \le 1.$$

Critical points:

$$\begin{aligned} A'(x) &= (-2+4x)/16 = 0 \\ x &= 1/2 \end{aligned}$$

Now, plugging the endpoints and critical points into $A(x)$, we get

$$\begin{aligned} A(0) &= 1/16 \\ A(1/2) &= 1/32 \\ A(1) &= 1/16 \end{aligned}$$

So letting one length of wire be 1m while the other is 0m maximizes the area. (Cutting the wire in half actually minimizes the sum of the areas.)

(6) (a) For $a > 0$, find a formula for $\frac{d}{dx}(a^x)$.

Noting that $a^x = e^{(\ln a)x}$, we have

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{(\ln a)x}) = e^{(\ln a)x}\ln a = \boxed{a^x \ln a}$$

(b) For (fixed) $x > 0$, find $\lim_{r\to 0} \frac{x^r - 1}{r}$. (Be careful: since this is a limit as $r \to 0$, we are treating r as our variable and x as a constant.)

As $r \to 0$, the top and bottom both approach 0, so we use l'Hôpital's rule. Again, keep in mind that r is the variable and x is acting like a constant, so the derivative of x^r will be $x^r \ln x$.

$$\begin{aligned} \lim_{r\to 0} \frac{x^r - 1}{r} &\overset{\text{l'H}}{=} \lim_{r\to 0} \frac{x^r \ln x - 0}{1} \\ &= \boxed{\ln x} \qquad \text{(since } x^r \to 1 \text{ as } r \to 0\text{)} \end{aligned}$$

(So, we could alternatively define $\ln x = \lim_{r\to 0} \frac{x^r - 1}{r}$.)

(7) A spigot on a tank of water is opened, and water drains out at a rate of $12 - t$ gallons/minute (where $t = 0$ is the moment the spigot was opened), until the tank is empty after 12 minutes. How much water was in the tank?

Let $V(t)$ be the volume of whater in the tank. We know that $V(12) = 0$ and $V'(t) = -(12-t) = t - 12$ (remember: $12 - t$ is how fast it's *leaving* the tank, so $-(12-t)$ is the rate of change of V), and we are trying to find $V(0)$.

Approach 1: We can integrate $V'(t)$ to get the total change in $V(t)$ between $t = 0$ and $t = 12$:

$$\begin{aligned} V(12) - V(0) &= \int_0^{12} V'(t)\,dt \\ &= \int_0^{12} (t - 12)\,dt \\ &= [t^2/2 - 12t]_0^{12} \\ &= (72 - 144) - (0 - 0) = -72 \end{aligned}$$

So

$$V(0) = V(12) + 72 = 72 \text{ gallons.}$$

Approach 2: Since $V'(t) = t - 12$, $V(t)$ is an antiderivative of $t - 12$. So

$$\begin{aligned} V(t) &= \int (t - 12)\,dt \\ &= t^2/2 - 12t + C \end{aligned}$$

We solve for C by plugging in a known value of V:

$$\begin{aligned} V(12) = 0 &= 72 - 144 + C \\ C &= 72 \\ V(0) &= 0 - 0 + C = 72 \text{ gallons.} \end{aligned}$$

(8) Find the following integrals. (Note: $\int \cos x\,dx = \sin x + C$; $\int \cos x\,dx = \sin x$ would not receive full credit.)

(a) $\int (x^3 - 3x + \frac{3}{x} - \frac{1}{x^3})\,dx$

$= \boxed{\frac{x^4}{4} - \frac{3x^2}{2} + 3\ln x + \frac{x^{-2}}{2} + C}$

(b) $\int \left(\frac{1}{\sqrt{1-x^2}} - \frac{\pi}{x^2}\right) dx$

$= \boxed{\sin^{-1}(x) + \pi x^{-1} + C}$

(c) $\int \frac{1 + \sin t}{\cos t}\,dt$

$$= \int \left(\frac{1}{\cos t} + \frac{\sin t}{\cos t} \right) dt$$
$$= \int (\sec t + \tan t) dt$$
$$= \boxed{\ln|\sec t + \tan t| + \ln|\sec t| + C}$$

(d) $\int_0^5 \sqrt[3]{y}(y^2+1)\, dy$

$$= \int_0^5 (y^{7/3} + y^{1/3}) dy$$
$$= \left[\frac{3}{10} y^{10/3} + \frac{3}{4} y^{4/3} \right]_0^5$$
$$= (\frac{3}{10}(5)^{10/3} + \frac{3}{4}(5)^{4/3}) - (0+0)$$
$$= \boxed{\frac{3}{10}(5)^{10/3} + \frac{3}{4}(5)^{4/3}}$$

(9) Are the following correct or not?

(a) $\int \frac{x^2}{1+x^2}\, dx = x - \tan^{-1} x + C$

We can check whether a proposed antiderivative is correct by differentiating.

$$\frac{d}{dx}[x - \tan^{-1} x] = 1 - \frac{1}{1+x^2}$$
$$= \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}$$
$$= \frac{x^2}{1+x^2}$$

So $\boxed{\text{yes}}$ this formula is correct.

(b) $\int \frac{x}{1+x^2}\, dx = \ln(1+x^2) + C$

$$\frac{d}{dx}[\ln(1+x^2)] = \frac{1}{1+x^2} \cdot 2x$$
$$= \frac{2x}{1+x^2}$$
$$\neq \frac{x}{1+x^2}$$

So $\boxed{\text{no}}$ this formula is not correct (but it's close: we could fix it by multiplying by $1/2$).