

Math 113 Test 1 Practice Solutions

- You may use one 3x5 card with notes (both sides, if you wish). Calculators or computers are not allowed.
 - The actual test will not be this long.
 - Apart from mathematical expressions, the solutions (and any additional comments) are typeset in a sans-serif typeface (like this).
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(1) Write a function that expresses the surface area of a cube as a function of its volume.

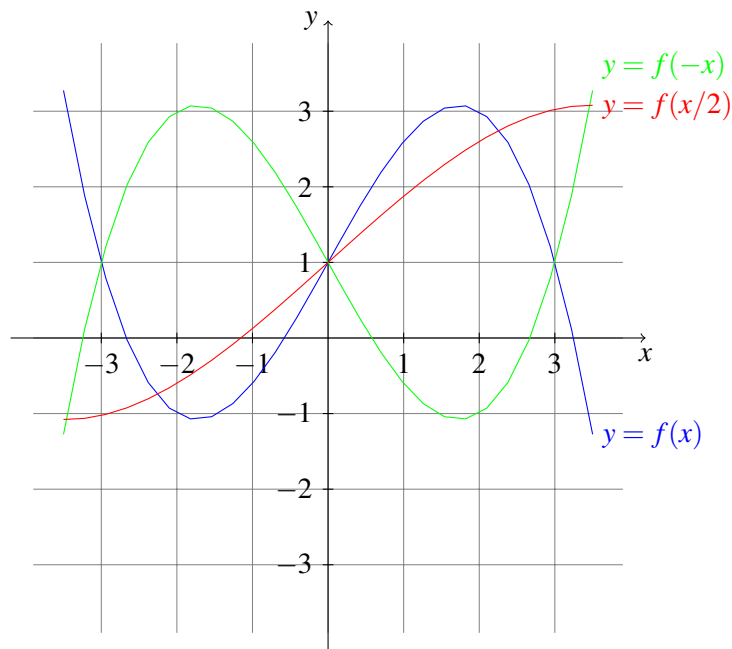
If the sides have length x , then the volume is $V = x^3$ (so $x = V^{1/3}$), and each face has area x^2 for a total area of $A = 6x^2 = 6(V^{1/3})^2 = 6V^{2/3}$. So the area, as a function of volume, is $A(V) = 6V^{2/3}$.

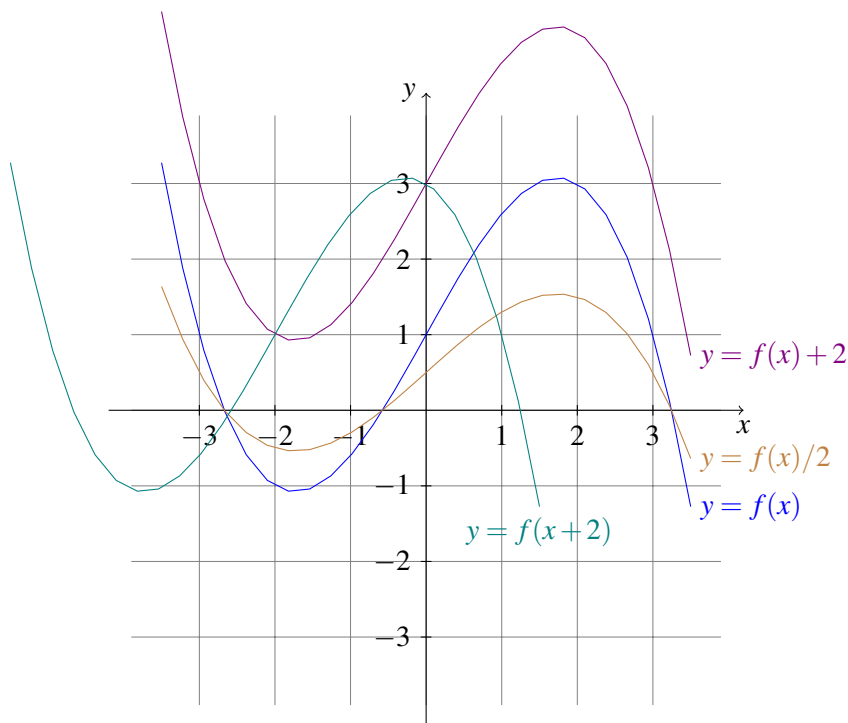
(2) Suppose the graph of $y = f(x)$ is below. On the same set of axes, sketch the graphs of

- $y = f(-x)$, (reflect across y -axis)
- $y = f(x/2)$, (scale by factor of 2 horizontally)
- $y = f(x)/2$, (scale by factor of $1/2$ vertically)
- $y = f(x) + 2$, (translate up 2 units)
- $y = f(x+2)$. (translate left 2 units)

(Be sure to label your graphs.)

We will actually draw a second set of axes, to prevent the graphs from getting too crowded.





(3) Simplify the following expressions.

(a) $e^{\ln 5 + \ln x}$

$$= e^{\ln 5} e^{\ln x} = \boxed{5x}$$

(b) $\ln(15x) - \ln(15/x)$

$$= \ln\left(\frac{15x}{15/x}\right) = \ln(x^2) = \boxed{2\ln x}$$

(c) $e^{2\ln x - 3\ln y}$

$$= e^{\ln x^2 - \ln y^3} = e^{\ln\left(\frac{x^2}{y^3}\right)} = \boxed{\frac{x^2}{y^3}}$$

Of course (as with most of these) there is more than one way to get this. E.g., we could have alternately had

$$e^{2\ln x - 3\ln y} = \frac{e^{2\ln x}}{e^{3\ln y}} = \frac{(e^{\ln x})^2}{(e^{\ln y})^3} = \boxed{\frac{x^2}{y^3}}$$

(d) $\ln(e^x e^y)$

$$= \ln(e^{x+y}) = \boxed{x+y}$$

(4) After taking a 40mg oral dose of fluoxetine (commonly known as Prozac), it is rapidly taken into the blood to a concentration of 21 ng/ml. The concentration decays exponentially, with a half-life of 1.5 days (meaning that, if the concentration today is c , the concentration in 1.5 days will be $c/2$). Give a formula for the concentration $F(t)$ at time t , t in days after dosage. (Expressions like $27\ln 2$ can be left in that form.)

Since this is exponential decay, we immediately have $F(t) = Ae^{kt}$, with $k < 0$. (Or, if you prefer, $F(t) = Ae^{-kt}$, with $k > 0$.) We know that $F(0) = 21$ and that 1.5 days later we have half as much,

so $F(1.5) = 10.5$.

$$F(t) = Ae^{kt}$$

$$F(0) = 21 = Ae^0 = A, \text{ so } F(t) = 21e^{kt}$$

$$F(1.5) = 21/2 = 21e^{k \cdot 1.5}$$

$$k = \ln(1/2)/1.5 = -\frac{\ln 2}{1.5} \qquad \left(\ln \frac{1}{x} = -\ln x\right)$$

$$\boxed{F(t) = 21e^{-\frac{\ln 2}{1.5}t}}$$

Another correct solution would be $F(t) = 21\left(\frac{1}{2}\right)^{\frac{t}{1.5}}$.

(5) State the limit definition of the derivative and use it to calculate the derivative of $f(x) = 2/x$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{2x - 2(x+h)}{(x+h)x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{(x+h)xh} = \lim_{h \rightarrow 0} \frac{-2}{(x+h)x} \\ &= \boxed{-\frac{2}{x^2}} \end{aligned}$$

(6) Evaluate the following limits or show that they do not exist. If the limit diverges to ∞ or $-\infty$, indicate this.

$$(a) \lim_{x \rightarrow -\infty} \frac{3x^3 - 4x + 1}{5x^2 - 8x + 2}$$

$$= \lim_{x \rightarrow -\infty} \frac{3x^3}{5x^2} = \lim_{x \rightarrow -\infty} \frac{3}{5}x = \boxed{-\infty}$$

(Since $x \rightarrow \pm\infty$ and this is a rational function, we only needed to consider the highest order term on the top and bottom.)

$$(b) \lim_{x \rightarrow 3^+} \frac{x^2 - 9}{x^2 - x - 6}$$

$$= \lim_{x \rightarrow 3^+} \frac{(x+3)(x-3)}{(x-3)(x+2)} = \lim_{x \rightarrow 3^+} \frac{(x+3)}{(x+2)} = \boxed{\frac{6}{5}}$$

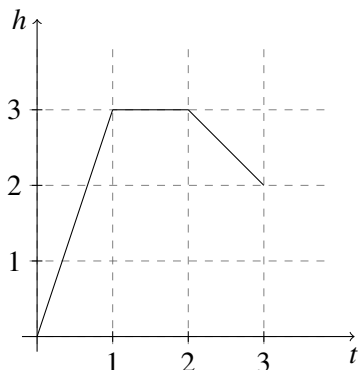
(Since the top and the bottom of the original fraction both go to 0, we know $(x-3)$ is a factor of both.)

$$(c) \lim_{x \rightarrow 4^-} \frac{x^2 - 9}{x^2 - 7x + 12}$$

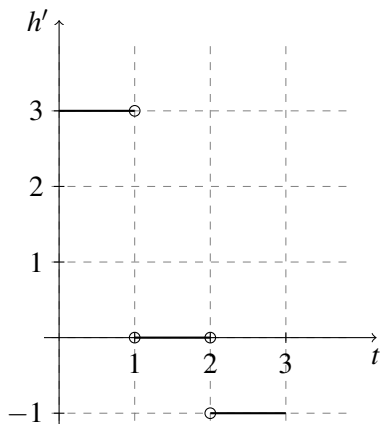
$$\begin{aligned} &= \lim_{x \rightarrow 4^-} \frac{\overbrace{x^2 - 9}^+}{\underbrace{(x-4)}_- \underbrace{(x-3)}_+} \\ &= \boxed{-\infty} \end{aligned}$$

(Plugging $x = 4$ into the original fraction gives us nonzero on top and 0 on the bottom, so we know the fraction “blows up,” and it’s just a matter of analyzing signs to see whether it is going to ∞ or $-\infty$. Note that we never need to factor the top in this problem—we know that it is positive when x is near 4.)

- (7) Suppose the following graph gives the height $h(t)$ of a balloon, in km, as a function of time (t is measured in hours past noon). Assume the balloon only moves up and down.



- (a) Sketch the graph of the balloon’s velocity. What are the units?



Since the units for h are km and units for t are hours, the units for the velocity, $h'(t)$, are km/hour (or simply km/h, but that might look a bit confusing since we are using h for the height of the balloon).

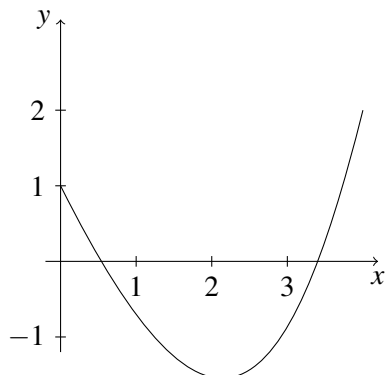
Of course, this is not realistic—a velocity instantly jumping from 3km/h to 0km/h would essentially require an infinite acceleration.

- (b) What is the balloon’s average velocity between noon and 3pm?

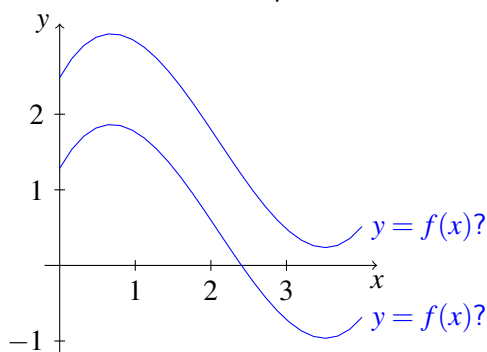
Between $t = 0$ and $t = 3$, the balloon went up 2km. So

$$\text{avg. velocity} = \frac{\Delta h}{\Delta t} = \boxed{\frac{3}{2} \text{ km/hour}}.$$

- (8) Suppose the following is the graph of $y = f'(x)$. Sketch two possibilities for what $y = f(x)$ might look like.



Reading off f' (which tells us about the slope of $y = f(x)$) from the graph, we see that the slope of the graph of $y = f(x)$ is initially 1, drops to 0 at $x \approx 0.6$, drops until it is about -1.5 at $x \approx 2.1$, and so on. We sketch possibilities for f by starting at an arbitrary point and obeying this information about the slope.



- (9) Calculate the derivatives of the following functions. (Remember that simplifying or rewriting the function another way can make this easier sometimes.)

(a) $T(x) = 8x^4 - 8x^2 + 1$

$$T'(x) = 32x^3 - 16x$$

(b) $v(t) = e^t(\sqrt{t} + \cos t)$

$$v'(t) = e^t(\sqrt{t} + \cos t)' + (e^t)'(\sqrt{t} + \cos t)$$

$$= e^t\left(\frac{1}{2\sqrt{t}} - \sin t\right) + e^t(\sqrt{t} + \cos t)$$

- (10) Find the line tangent to $y = \cos x$ at $x = \pi/6$.

Let $y = f(x) = \cos x$. We have $f'(x) = -\sin x$. The tangent line at $x = \pi/6$ is

$$y = f(\pi/6) + f'(\pi/6)(x - \pi/6)$$

$$y = \cos(\pi/6) - \sin(\pi/6)(x - \pi/6)$$

$$y = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \pi/6)$$

(11) Find the inverse of the function $f(x) = \ln(x+1)$.

(We set $y = f(x)$, and solve for x in terms of y to get $x = f^{-1}(y)$. Then we just rewrite f^{-1} as a function of x .)

$$y = \ln(x+1)$$

$$e^y = x+1$$

$$x = e^y - 1 = f^{-1}(y)$$

So $\boxed{f^{-1}(x) = e^x - 1}$.

(12) Sketch the graph of a function that is differentiable at $x = 0$, continuous but not differentiable at $x = 1$, and discontinuous at $x = 2$.

