Voting with Pulleys and Rubber Bands

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Union College Mathematics Department

Union College Undergraduate Mathematics Seminar
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Multicandidate voting

3 or more candidates run for office
**Multicandidate** voting: Set-up

A group must select one option from among several* alternatives:

- Candidates for president:
  - John McCain
  - Barack Obama
  - Ron Paul

- What to order for lunch: Pastrami, Cabbage, Rabbit, Salami

**“several” means** ≥ 3
**Multicandidate** voting: Set-up

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- Candidates for president:
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  - Ron Paul

- What to order for lunch: Pastrami, Cabbage, Rabbit, Salami

**General Assumptions:**

- Voters are treated equally
- More than 2 possible outcomes
- All possible outcomes are treated equally (no built-in bias favors one candidate)
Multicandidate voting: Set-up

In the US, a ballot usually only names a voter’s single most favored candidate.
**Multicandidate** voting: Set-up

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We will consider ballots that reveal each voter’s full *preference ranking*. . . . used in some other countries.

♦ Candidates for president: John McCain, Barack Obama, Ron Paul
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♦ Candidates for president: John McCain, Barack Obama, Ron Paul

Mei-Ling
  R
  B
  J
Multicandidate voting: Examples

1) **Borda Count**  Jean Charles de Borda (French Revolution)

- Each voter awards points to the candidates:  
  
<table>
<thead>
<tr>
<th>Alternative</th>
<th>Points</th>
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<tr>
<td>Ahmed</td>
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- For each alternative, sum the points awarded by all voters

- The winner is the alternative with the most points
Multicandidate voting: Examples

1) **Borda Count** Jean Charles de Borda (French Revolution)

Sample Profile:

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p's total points:  

\[ \_ \times 3 = \_ \]
\[ \_ \times 2 = \_ \]
\[ \_ \times 1 = \_ \]
\[ \_ \times 0 = \_ \]

SUM = \_
**Multicandidate** voting: Examples

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p’s total points:

\[
\begin{align*}
3 \times 3 &= 9 \\
0 \times 2 &= 0 \\
0 \times 1 &= 0 \\
4 \times 0 &= 0 \\
\text{SUM} &= 9
\end{align*}
\]
**Multicandidate** voting: Examples

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q's points: \(1 \times 3 = 3\)

r's points: \(1 \times 3 = 3\)

s's points: \(2 \times 3 = 6\)

5 × 2 = 10
1 × 1 = 1
0 × 0 = 0

0 × 2 = 0
6 × 1 = 6
0 × 0 = 0

2 × 2 = 4
0 × 1 = 0
3 × 0 = 0

SUM = 14
SUM = 9
SUM = 10

(p had 9 total)
## Multicandidate Voting: Examples

1) **Borda Count** Jean Charles de Borda (French Revolution)

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- q’s points: \(1 \times 3 = 3\)
- r’s points: \(1 \times 3 = 3\)
- s’s points: \(2 \times 3 = 6\)
- p’s points: \(5 \times 2 = 10\)
- \(0 \times 2 = 0\)
- \(2 \times 2 = 4\)
- \(1 \times 1 = 1\)
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- \(0 \times 1 = 0\)
- \(0 \times 0 = 0\)
- \(0 \times 0 = 0\)
- \(3 \times 0 = 0\)

**SUM** = 14

(p had 9 total)

**Borda winner is q**
**Multicandidate** voting: Examples

2) **Hare**  

Step 1  
Is some alternative the 1\textsuperscript{ST} choice of a majority of voters?  
If so, they win. If not go to step 2.

Step 2 Eliminate the alternative(s) having the fewest 1\textsuperscript{ST} choice votes.

Step 3 “Squeeze up” to close the gaps left by the eliminations. Then, go to step 1.

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Same Profile:

\[
\begin{array}{cccc}
3 & 1 & 1 & 2 \\
p & q & r & s \\
q & s & s & q \\
r & r & q & r \\
s & p & p & p \\
\end{array}
\]

p has a plurality of 1\(^{\text{st}}\) choice votes: 3 of 7. But no alternative has a majority. Proceed to step 2.
Multicandidate voting: Examples

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p has a **plurality** of 1\textsuperscript{ST} choice votes: 3 of 7. But no alternative has a **majority**.

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Now, back to step 1!

Alternative *s* gets 4 of the 1\textsuperscript{ST} place votes – a majority of the 7 votes cast.
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Proceed to step 2.

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**Hare winner is s**
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**Borda winner is q**  **Hare winner is s**
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**Borda winner is q**  **Hare winner is s**

3) **Plurality Rule**  The winner is the alternative with the greatest number of 1<sup>st</sup> place votes
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**Borda winner is q**  **Hare winner is s**

3) **Plurality Rule**  The winner is the alternative with the greatest number of 1ST place votes

**Plurality winner is p**

Same election: 3 different voting rules ⇒ 3 different winners
Multicandidate voting

How about real life?

Does the choice of voting rule really make a difference?
Multicandidate voting

How about real life?

Does the choice of voting rule really make a difference?

Yes... especially when the election is close.
Who remembers a recent presidential election in which a razor-thin margin in a southern state made a critical difference?
Multicandidate voting

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Almost certainly, Gore.
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Using Hare?
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Almost certainly, Gore.

Using Hare?

Almost certainly, Gore.
Hex-Mean voting rule

- Three alternatives: p, q, r

- 6 possible rankings:
  - p > q > r
  - p > r > q
  - q > p > r
  - q > r > p
  - r > p > q
  - r > q > p

- Label each hex vertex with a ranking, as in the sketch

- What is the labeling pattern?
Hex-Mean voting rule

- Three alternatives: p, q, r
- 6 possible rankings:
  - p > q > r
  - p > r > q
  - q > p > r
  - q > r > p
  - r > p > q
  - r > q > p
- Label each hex vertex with a ranking, as in the sketch
- What is the labeling pattern?
- Adjacent rankings differ by one pairwise reversal
**Hex-Mean voting rule**

- Each voter chooses a vertex
Hex-Mean voting rule

- Each voter chooses a vertex
- $O$ = mean location of all votes
Hex-Mean voting rule

- Each voter chooses a vertex
- \( \mathcal{O} \) = mean location of all votes
- How do we find the “mean” of points in the plane? We’ll come back to that.
- Where is \( \mathcal{O} \)?
**Hex-Mean voting rule**

- Each voter chooses a vertex
- $\mathbf{O}$ = mean location of all votes
- How do we find the “mean” of points in the plane? We’ll come back to that.
- Where is $\mathbf{O}$?
- The winning ranking is that of the vertex **closest** to the mean
Hex-Mean voting rule

- Each voter chooses a vertex
- $\bigcirc = \text{mean location of all votes}$
- How do we find the “mean” of points in the plane? We’ll come back to that.
- Where is $\bigcirc$?
- The winning ranking is that of the vertex closest to the mean:
  $r > q > p$
- The Hex-Mean winner is $r$
- Who cares?
Hex-Mean voting rule

- **Theorem** The Hex-Mean rule is the same as the Borda Count.
The Mean

2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)
The Mean
2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)

  1. **Average Coordinate Method**

![Diagram showing points on a coordinate plane]
The Mean

2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)

  1. **Average Coordinate Method**

- Find the average x coordinate
The Mean
2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)

  1. **Average Coordinate Method**

- Find the average x coordinate
- Find the average y coordinate
The Mean

2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)

  1. **Average Coordinate Method**

- Find the average x coordinate

- Find the average y coordinate

- Use these as the coordinates of the mean point O
The Mean

2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)

  2. Ideal Rubber Band Method
The Mean

2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)

  2. Ideal Rubber Band Method

- An i.r.b.:
  - will shrink to a point if you let go of both ends
  - Tension is proportional to stretch
The Mean

2 equivalent definitions

• Given three (blue) points in the plane (or on a number line, or in space)

  2. Ideal Rubber Band Method

• An i.r.b.
  ♦ will shrink to a point if you let go of both ends
  ♦ Tension is proportional to stretch

• Loop one end of an i.r.b. around a blue point, and the other end about a movable point ☝
The Mean

2 equivalent definitions

• Given three (blue) points in the plane (or on a number line, or in space)

2. Ideal Rubber Band Method

• An i.r.b.
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• Loop one end of an i.r.b. around a blue point, and the other end about a movable point ○

• Repeat with the other blue points
The Mean
2 equivalent definitions

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   2. Ideal Rubber Band Method

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- Loop one end of an i.r.b. around a blue point, and the other end about a movable point ○

- Repeat with the other blue points

- Release ○ and let it reach equilibrium
  - rubber band forces cancel out exactly
The Mean

2 equivalent definitions

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• Loop one end of an i.r.b. around a blue point, and the other end about a movable point ○

• Repeat with the other blue points

• Release ○ and let it reach equilibrium – rubber band forces cancel out exactly

• The two methods always agree, producing the same point ○
• **Theorem**  The Hex-Mean rule is the same as the Borda Count

• And the mean can be found using rubber bands

• Putting these together we get...
Physical model for Borda count

- Tie 3 i.r.b.s around r>p>q and a movable point O
- Tie 5 i.r.b.s around q>r>p & O
- Release and let it reach equilibrium – rubber band forces cancel out exactly
Physical model for Borda count

- Tie 3 i.r.b.s around \( r>p>q \) and a movable point \( \mathcal{O} \)
- Tie 5 i.r.b.s around \( q>r>p \) & \( \mathcal{O} \)
- Release and let it reach equilibrium – *rubber band forces cancel out exactly*
- The vertex closest to \( \mathcal{O} \) (green line) tell us the Borda winner

**Conclusion** Borda count = voting with rubber bands on the hexagon (3 alternatives)
Physical model for Borda count

- How about **four** alternatives?

- There are **24** possible rankings of four alternatives
Physical model for Borda count

• How about **four** alternatives?

• There are **24** possible rankings of four alternatives

• A hexagon has only **6** vertices.
Physical model for Borda count

- How about four alternatives?
- There are 24 possible rankings of four alternatives
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• Then vote with i.r.b.s; choose vertex closest to O
Physical model for Borda count

• **Conclusion** Borda count = voting with rubber bands on the hexagon (3 alternatives)

• With rubber bands, greater distance = harder pull

• Is there an alternative, with greater distance = same pull?
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- **Yes.** Replace rubber bands with weights and strings
An Alternative to the Mean

- Choose 3 points on the plane
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- Drill a hole through at each point, and pass a string through each hole
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- Attach a unit weight to each end below the table

- Tie all other ends to one movable point
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An Alternative to the Mean

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• Drill a hole through at each point, and pass a string through each hole

• Attach a unit weight \( \square \) to each end below the table

• Tie all other ends to one movable point \( \square \)

• Release, allow \( \square \) to reach equilibrium

• This point is called the median centre . . .

• . . . and it is different from the mean
A New Voting Rule

- Each voter chooses a vertex
A New Voting Rule

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- □ = mediancentre of all votes
A New Voting Rule

- Each voter chooses a vertex
- $\Box = \text{mediancentre of all votes}$
- The winning ranking is that of the vertex closest to the MC
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We call this new voting rule the MCBorda rule
A New Voting Rule

- Each voter chooses a vertex
- $\square = \text{median centre of all votes}$
- The winning ranking is that of the vertex closest to the MC
- We call this new voting rule the MC$^c$Borda rule
- It is so new that we are still learning about its basic properties
3 BIG Questions
1) How does the median centre differ from the mean?
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2) How does the M\textsuperscript{C}Borda voting rule differ from the Borda count?
3 **BIG** Questions

1) How does the median centre differ from the mean?

2) How does the M^C^Borda voting rule differ from the Borda count?

3) How are the answers to the previous two questions linked?
3 BIG Questions

1) How does the mediancentre* differ from the mean?

2) How does the M^cBorda voting rule differ from the Borda count?

3) How are the answers to the previous two questions linked?

* And how is the mediancentre related to the median?
3 BIG Questions

1) How does the median centre differ from the mean?

WE’LL EXPERIMENT . . .

. . . USING DAVIDE CERVONE’S SOFTWARE