A Tight Polyhedral Immersion of the Twisted Surface of Euler Characteristic -3

Davide P. Cervone

August 1998

Abstract

Early studies of tight surfaces showed that almost every surface can be immersed tightly in three-space. For these surfaces we can ask: How many significantly different tight immersions are there? If we consider two such immersions to be the same when they are image homotopic, then we can answer the question by determining the number of homotopy classes where tight immersions are possible. A complete description of all classes of immersions under image homotopy already exists, and tight examples are known in all but three of the classes where tight immersions are possible. In this paper we produce examples in two of those three missing classes, and conjecture that no tight immersion exists in the third.

1 Introduction

A surface in three-space is *tight* provided that any plane cuts it into at most two pieces (a more formal definition is given in section 2 below). Surfaces such as a round sphere and a torus of revolution are tight, but a banana is not since a single plane can cut off both ends at once, leaving the banana in three pieces. In his initial study of tight surfaces, Kuiper showed that the Klein bottle and the real projective plane can not be tightly immersed in three-space, while all other surfaces except the projective plane with one handle can be [15], [16], [17]. The fate of the latter surface has been resolved only recently: there is no *smooth* tight immersion of this surface [11] but, surprisingly, there is a *polyhedral* one [10].

Having determined which surfaces can be tightly immersed, it is natural to ask: How many different tight immersions are there for each surface? To answer this question, we need some notion of when two immersions are considered to be the same. A reasonable criterion is that two are the same if one can be smoothly deformed to the other by a series of intermediate immersions. This is the idea underlying *image homotopy* described in more detail below. Image homotopy is an equivalence relation on the set of immersions of surfaces, and so the equivalence classes under this relation represent the different possible types of immersions. For example, there is only one type of sphere, since a theorem of Smale [19] indicates that any immersed sphere can be deformed into the standard round sphere, which leads to the famous result that a sphere can be turned inside out via a smooth homotopy illustrated recently in [18]. On the other hand, there are two distinct types of tori: one the standard torus of revolution; the other a "twisted" torus with self-intersection that can not be eliminated by a deformation that is an immersion at each step. Every immersed torus is image homotopic to one of these two.

In [20], Pinkall describes all the classes of immersions under image homotopy. Indeed, he shows that these classes form a semi-group under the operation of connected sum. His key ingredient is a relationship between image homotopy and the idea of cobordism, namely, two immersions of a given surface are image homotopic if, and only if, they are cobordant. Wells [22] showed that the cobordism group for surfaces in three-space is isomorphic to $\mathbf{Z}_8$, the cyclic group of order 8, and Pinkall uses this to analyze the structure of the semi-group formed by the image homotopy classes.

Pinkall called the two classes of tori S and T (for "standard" and "twisted"). He found that there are two classes of immersed projective planes: right- and left-handed versions of the Boy surface, denoted B and $\bar{B}$. For the Klein bottle, there are three classes: the standard immersion with reflective symmetry, K_0, together with right- and left-handed "twisted" versions, K_+ and K_- (these are formed by moving a figure-8 around a circle so that it rotates 180 degrees by the time it comes back to its starting point, with the direction of rotation determining the handedness). Note that the twisted torus is formed similarly by moving a figure-8 around a circle but this time rotating a full 360 degrees before it comes back to its starting position. These surfaces are shown in figure 1.

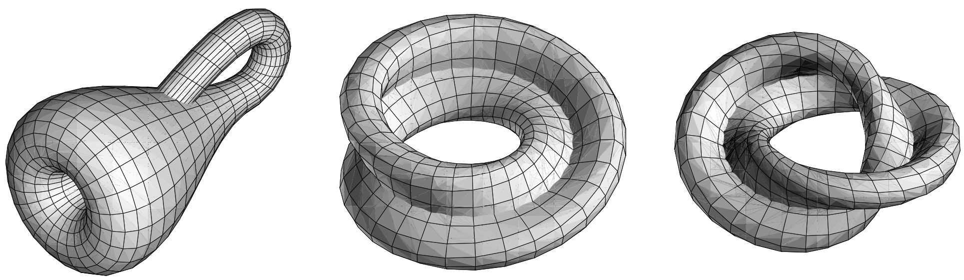

Figure 1: The standard Klein bottle, K_0 (left), the twisted Klein bottle, K_+ (middle), and the twisted torus, T (right).

We can interpret Pinkall's classification in terms of the cobordism group by breaking each element of $\mathbf{Z}_8$ into its different topological types. Each class in $\mathbf{Z}_8$ becomes a family of related classes under image homotopy, all formed from a basic surface by adding handles (i.e., by connected sum with some number of copies of S). Figure 2 shows this breakdown. Moving down a column corresponds to adding a handle. Two of the eight columns contain both orientable and non-orientable members; these form distinct families under image homotopy. This gives a total of ten families of surfaces formed by adding handles to one of the ten basic surfaces in the first row of the table. Every immersion is in one of these image homotopy classes.

-3	-2	-1	0	1	2	3	4
$K_- \# \bar{B}$	K_-	$\bar{B}$	S or K_0	B	K_+	$K_+ \# B$	T or $K_0 \# T$
$\#S$	$\#S$	$\#S$	$\#S$	$\#S$	$\#S$	$\#S$	$\#S$
$\#2S$	$\#2S$	$\#2S$	$\#2S$	$\#2S$	$\#2S$	$\#2S$	$\#2S$
$\#3S$	$\#3S$	$\#3S$	$\#3S$	$\#3S$	$\#3S$	$\#3S$	$\#3S$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 2: The cobordism group for immersed surfaces, $\mathbf{Z}_8$, is broken down by image homotopy into classes according to topological type. Two cobordism classes (0 and 4) include both orientable and non-orientable members, which form distinct columns under image homotopy. Moving down a column corresponds to adding a handle, while moving left or right corresponds to connected sum with $\bar{B}$ or B.

Pinkall gives generating relations for the semi-group of image homotopy classes, e.g., $T \# \bar{B} = B \# B \# B$. In general, moving right or left in the table corresponds to connected sum with a right- or left-handed projective plane. For example, $K_+ = B \# B$, and so $T \# \bar{B} = K_+ \# B$ (using the relation above), which corresponds to moving from column 4 to column 3 by adding a left-handed Möbius band. Similarly, $T \# B = K_- \# \bar{B}$.

In [8], the author provides tight polyhedral examples in all the image homotopy classes for which tight immersions are possible except for three, $K_- \# \bar{B} \# S$, $K_+ \# B \# S$ and $T \# S$, and conjectures that examples of the first two exist, while the third does not. Notice that the first two are mirror images of each other, so an example of one immediately yields an example of the other. In this paper, we provide such an example, leaving $T \# S$ as the only remaining missing case. Haab [12] recently proved that no *smooth* tight immersion of this surface exists. We continue to conjecture that no such immersion is possible in the polyhedral case as well.

To find the example we are looking for, we will use the relationship $T \# \bar{B} = K_+ \# B$ from above. First we will produce a polyhedral version of T having certain geometric properties; in particular, it will have an attachment site where the connected sum with $\bar{B}$ can be performed without interfering with potential tightness. The result will not yet be tight, but can be made tight by cutting off two disks and adding a tube between these two disks. This will produce a tight polyhedral immersion of $T \# \bar{B} \# S$, which is equal to $K_+ \# B \# S$, one of the previously missing examples. A mirror version of this surface then provides the other example.

2 Definitions and Basic Results

Given a triangulated surface M, a *simplexwise-linear map* is a function $f\colon M \to \mathbf{R}^3$ that maps faces and edges as the convex linear combinations of their vertices (i.e., as planar triangles with straight edges). We assume that f is *non-degenerate*, meaning that it does not reduce the dimension of any simplex of M. The *star* of a vertex, v, is the union of the simplices that contain v, and the *valence* of v is the number of edges containing v.

A simplexwise-linear mapping $f\colon M \to \mathbf{R}^3$ is an *embedding* if it is a one-to-one map. It is an *immersion* if it is locally one-to-one; that is, for every point p of M, there is a neighborhood U_p of p where the restriction of f to U_p is one-to-one. (For smooth surfaces, there are additional requirements that guarantee the existence of a tangent plane at every point, but these are not necessary in the polyhedral case.) The interiors of faces are always immersed, and the interiors of edges are immersed provided the adjacent faces don't coincide, so f is an immersion provided the vertices are immersed. In a simplexwise-linear map, a small neighborhood of a vertex is effectively the same as the star of the vertex, so we have the following lemma:

Lemma 2.1 *A simplexwise-linear map $f\colon M \to \mathbf{R}^3$ is an immersion if, and only if, the star of every vertex of M is embedded by f.*

To relate one immersion to another, we could use the concept of regular homotopy, but this produces too fine a classification, since, for example, two smooth immersions f and g may not be regularly homotopic even though their images are identical as subsets of $\mathbf{R}^3$ (one immersion may be a reparameterization of the other, see [8]). The idea of image homotopy allows for these reparameterizations: f and g are image homotopic if there is a diffeomorphism $\phi\colon M \to M$ where f and $g \circ \phi$ are regularly homotopic.

For simplicial surfaces, the notion of diffeomorphism is replaced by that of a symmetry of M. A mapping $\phi\colon M \to M$ of a triangulated surface to itself is a *symmetry* of M if it is a bijection that preserves the dimension of simplices (i.e., it maps vertices to vertices, edges to edges, and faces to faces). Then two immersions $f, g\colon M \to \mathbf{R}^3$ of a triangulated surface are *image homotopic* if there is a symmetry ϕ and a homotopy $H\colon M \times [0,1] \to \mathbf{R}^3$ such that $H_t(p) = h(p,t)$ is an immersion for each t in $[0,1]$ and such that $H_0 = f$ and $H_1 = g \circ \phi$. (If f and g are immersions of different triangulations of M, then one must first pass to a common refinement of these triangulations.)

A mapping $f\colon M \to \mathbf{R}^3$ is said to be *tight* provided that the preimage of every half-space of $\mathbf{R}^3$ is connected in M; that is, every plane cuts the image of M into at most two pieces. This is also called the *two-piece property* and was developed independently of tightness, but was found to be equivalent to it. Several other interpretations of tightness can be found in the literature, e.g. [4]. Tightness is a property of the mapping f, not the surface itself, but it is common to speak of M in place of $f(M)$ and let the mapping be implied. In practice, this ambiguity is resolved naturally by the context.

A tight surface has the property that, for almost every direction, the height function in that direction induced on $f(M)$ has exactly one maximum and one minimum; that is, local extrema are also global extrema, for if there were two local maxima for a particular direction, then a plane slightly below the lower of the two would cut off both maxima, separating the surface into at least three parts. For smooth tight surfaces, this means that all the positive curvature must be on the convex envelope (the surface of the convex hull), while all the points inside the convex hull have negative (or zero) curvature.

An analogous idea for polyhedral surfaces uses the position of a vertex relative to its neighbors. A vertex v of M is a *local extreme vertex* if $f(v)$ is a vertex of the convex hull of the image of the star of v (i.e., it is an isolated local maximum for the height function on $f(M)$ in some direction). A vertex is a *(global) extreme vertex* if its image is a vertex of the convex hull of $f(M)$. A local extreme vertex corresponds to a point of positive curvature in a smooth surface, while a vertex that lies in the interior of the convex hull of its neighbors corresponds to a point of negative curvature. Note that v will not be an extreme vertex (local or global) if it lies in the interior of the convex hull of some subset of its adjacent vertices; for example, if v lies on the line segment between two of its neighbors, then v can not be locally or globally extreme.

With these definitions, we can characterize tight immersions for polyhedral surfaces as follows:

Lemma 2.2 *A simplexwise-linear map $f\colon M \to \mathbf{R}^3$ of a closed, compact, connected surface M is tight if, and only if,*

i) every local extreme vertex is a global extreme vertex,

ii) every edge of the convex hull of $f(M)$ is contained in $f(M)$, and

iii) every vertex of the convex hull of $f(M)$ is the image of a single vertex of M.

This lemma can be found in the literature ([4] or [13], for example) as a result for embedded surfaces, without the third condition. See [8] for an example of why this condition is needed for immersions.

Given a surface, M, one way to generate a tight polyhedral immersion of $M \# S$ is the following. Suppose we have a polyhedral decomposition of M

with the following properties: it contains a large, convex, planar polygon at the top and another one parallel to it at the bottom; the rest of the surface lies in between and has no interior vertices that are local extrema; the double curve does not contain any vertices; and the convex envelope of M intersects M only in the two planar polygons. Then if we remove these two polygons from both M and its convex envelope and glue the remainders along their common boundaries (this is called the *mod-2 sum* of M and its convex envelope), the resulting surface will be a tight immersion of $M \# S$.

To see that this surface is $M \# S$, note that the convex envelope is a topological sphere, and so when we remove two polygons the remainder is a tube. Attaching this to M adds a handle to it, so the result should be $M \# S$. (The two disks are each of Euler characteristic 1, so their removal lowers the Euler characteristic of M by 2. The tube is of Euler characteristic 0, so the resulting surface has Euler characteristic 2 less than the Euler characteristic of M, hence the surface is M plus a handle. The handle is untwisted, so it is $M \# S$ rather than $M \# T$.) If M is orientable, however, we need to be a bit more careful about this claim. It is possible that adding the tube attaches the inside to the outside of the surface, making it the non-orientable surface with the same Euler characteristic as $M \# S$. To prevent this, we need to be sure that the orientation of the two planar polygons are such that their normals point in opposite directions. This guarantees that the orientability is preserved during the mod-2 sum.

To see that the resulting surface is an immersion, first note that we have added no new vertices during this construction, and since the double curve of M did not include any vertices this means that the star of each vertex is embedded in M. Since the convex envelope did not intersect any faces of M other than the two removed polygons, adding this does not change the self-intersection of the surface, so the stars are embedded in $M \# S$ as well. Hence the surface is immersed.

To see that it is tight we need to verify the conditions of lemma 2.2. The interior vertices of $M \# S$ are all interior vertices of M, and since these are not locally extreme by hypothesis, condition *i* is satisfied. Since all of the edges of the convex envelope are part of the tube added to M in forming $M \# S$, condition *ii* is satisfied as well. Finally, since the self-intersection is unaltered and no vertex of M is on the double curve of M by assumption, condition *iii* is satisfied. Thus the resulting surface is tight, as claimed. This proves the following:

Lemma 2.3 *Suppose M is a polyhedral surface in three-space such that*

i) M contains two convex polygons, P_1 and P_2, in parallel planes such that the remainder of M lies between these two planes;

ii) there are no locally extreme vertices of M between P_1 and P_2;

iii) no vertex of M lies in the double set of M;

iv) the intersection of M and its convex envelope is exactly the polygons P_1 and P_2; and

v) for orientable M, the orientation induces normals to P_1 and P_2 that point in opposite directions.

Then the mod-2 sum of M and its convex envelope is a tight polyhedral immersion of $M \# S$.

Given a surface M, the conditions of this lemma can be checked either by computer or by hand, thus it provides a method of constructing tight surfaces from simpler surfaces whose requisite properties are easily verified. We will use this process below on the surface $M = T \# \bar{B}$ to obtain a tight immersion of $T \# \bar{B} \# S$. Our first goal, then is to produce an appropriate immersion of $T \# \bar{B}$. We begin by searching for a suitable version of T.

3 The Twisted Torus

The key to our new example is the twisted torus, T. We need to find a surface that: is a torus; is in the "twisted" image homotopy class; is an immersion; has a planar polygon on top and another on bottom with no locally extreme vertices in between; and has a site where a projective plane can be attached without introducing local extreme vertices in the interior. The last condition requires particular care, so we will begin with it.

Brehm describes several nine-vertex immersions of the projective plane, all having three-fold rotational symmetry [6]. Removing a topological disk (a face and its three neighbors; see figure 3), the remainder is an immersed Möbius band. Note that the three interior vertices are not locally extreme since each lies in the interior of a triangle formed by three of its neighbors.

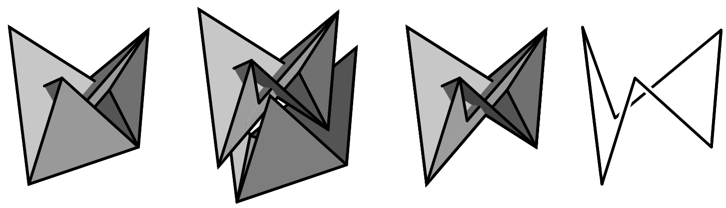

Figure 3: A polyhedral immersion of the real projective plane with nine vertices (left) has its base triangle and three neighboring triangles removed (middle left). The remainder is an immersed Möbius band (middle right) with its boundary formed by six edges (right).

The boundary of this band has edges that follow an "up-down-up-down-up-down" pattern, which is the same pattern found in the boundary of a neighborhood of a monkey saddle. This suggests that if we have a surface containing a monkey saddle, a small neighborhood of the saddle point can be removed and replaced by a suitably scaled copy of this Möbius band. If done carefully, the six boundary vertices of the band will not be locally extreme (figure 4). One way to do this is to place the three interior vertices of the Möbius band near the central vertex being removed from the monkey saddle, then place the other six vertices on the straight line segments from the boundary of the monkey saddle to the three interior vertices. This guarantees that the new vertices are not locally extreme.

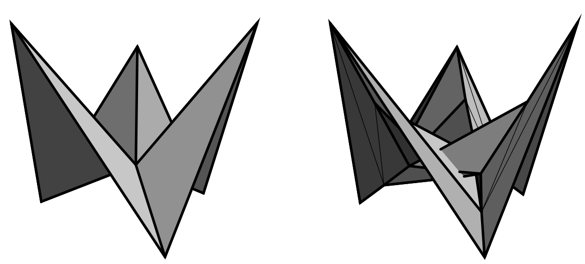

Figure 4: A neighborhood of a monkey saddle (left) can be replaced by a copy of the Möbius band from Brehm's model of the projective plane (right). If done carefully, none of the vertices will be locally extreme.

To use this idea, we need to find an immersion of T containing a monkey saddle. Since we want to have a model with no locally extreme interior vertices, the monkey saddle must be the only interior critical point; that is, we need a surface with exactly three critical points.

We begin the search for this surface by describing the levels sets for it. Since we know there must be a monkey saddle, we can start with the level containing it; the level set will have three curves crossing at one point. This gives us six segments emanating from a central point. Figure 5 shows these segments, together with a portion of a level just below the critical level and one just above it.

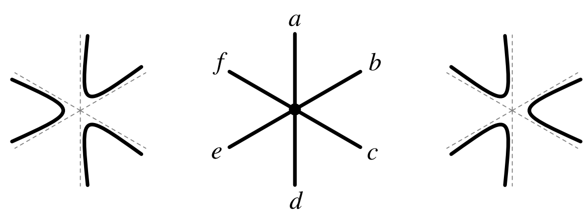

Figure 5: The level set in a neighborhood of a monkey saddle consists of six curves emanating from a point (center). This configuration breaks up into three curves in levels just below the critical level (left) or above it (right).

To complete the critical level we need to attach these segments in pairs by curves. In doing so, however, we need to be sure that the surface we create is a torus. When we connect the segments in the critical level, we also connect them in the nearby levels, and since we are allowing no other saddles, the number of components present in the nearby levels can not change as we move to higher and lower levels, except by passing through a maximum or a minimum. Since we are allowed only one of each, we must have exactly one component in the level above (for the maximum) and one in the level below (for the minimum). Thus we can not attach segment a to segments b or f as this would produce at least two components in one of the nearby levels.

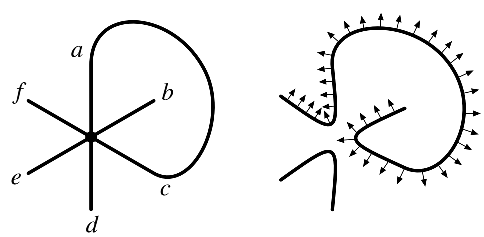

Figure 6: The level set can not contain a curve from a to c since the levels just above the critical level would attach the outside to the inside of the surface, making it non-orientable.

Since the torus is orientable, we can't attach segment a to segments c or e.

The orientation of the torus induces an orientation on the level curves in the non-critical levels; but attaching a to c or e would attach the outside of the surface to the inside (figure 6), violating orientability.

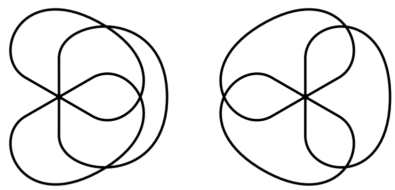

Figure 7: The two forms for the critical level of a torus containing a monkey saddle and no other saddles.

Thus, the only possible connection is a to d. Similarly, b must connect to e and c to f. Up to symmetry, this yields only two possible configurations, as shown in figure 7. Both represent critical levels of tori with exactly three critical points. In order for these tori to be immersed, however, we need to be able to transform the immersed circles in the level sets on either side of the critical level into *embedded* circles (and then down to a point to form the maximum and minimum points). Such a transformation is possible only if the circles have turning number equal to 1 or -1. This is not the case for any of the circles produced in either configuration; however, by adding small loops to the curves at suitable locations, it is possible to modify the curves so that they have appropriate turning numbers.

In doing so, the first configuration in figure 7 only produces immersions that are image homotopic to the standard torus. Luckily, however, the second one generates immersions homotopic to the twisted torus, as we will see below. Also, the three-fold symmetry it exhibits will make producing the polyhedral version easier, especially in light of the fact that the Möbius band from Brehm's model exhibits this symmetry already.

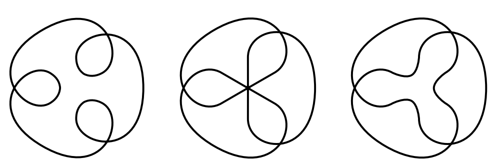

Figure 8: The level curves just below and above the critical level containing the monkey saddle have turning numbers 4 and 2.

After the connections are made, the level below the critical one has turn-

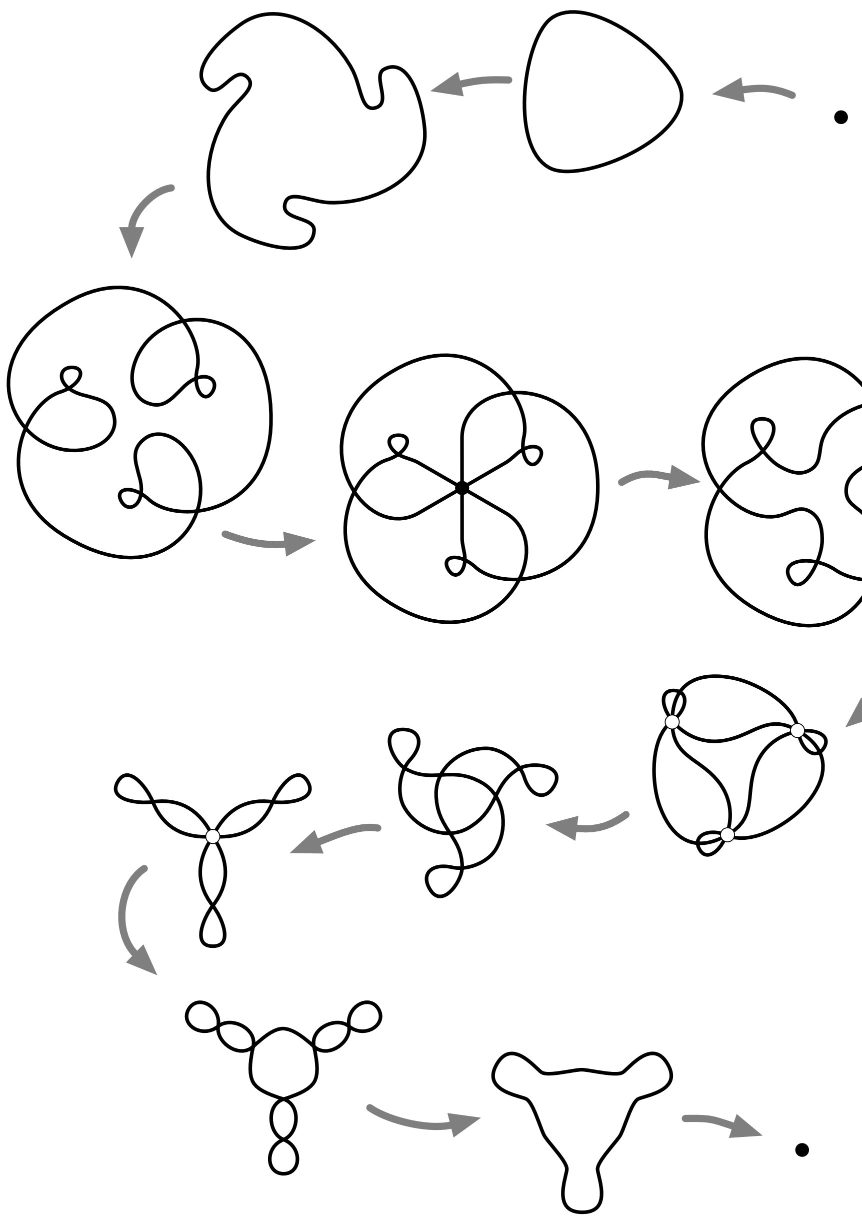

Figure 9: The complete sequence of level sets for an immersed torus in the non-standard image homotopy class. The critical points are shown as solid dots. The four triple points are shown as hollow dots.

ing number 4, while the one above has turning number 2 (figure 8). By adding three loops, we can reduce the turning numbers by 3 each, giving turning numbers of 1 and -1 respectively. Since three loops are added, we can maintain the three-fold symmetry as well. Figure 9 shows a complete sequence of level sets for the immersed torus so formed, and the small loops can be seen in the 4th, 5th and 6th levels. (This sequence is similar to one described in [5].)

To verify that this is a torus in the twisted class, T, we need one more result of Pinkall's, namely that a torus is image homotopic to T if, and only if, it has two cycles whose neighborhoods form twisted bands and that are not in the same homology class (in $\mathbf{Z}_2$ homology) [21]. A band is twisted if its boundary curves are linked, so, for example, a figure-8 strip is twisted (figure 10).

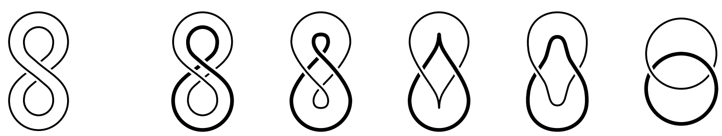

Figure 10: A strip in the form of a figure-8 is twisted, since its boundary curves are linked.

A cycle with trivial homology can't form such a band, and two distinct, non-trivial cycles on a torus must intersect, so a small neighborhood of two such cycles looks like that shown at the left of figure 11. These bands are untwisted, but they can be made twisted by putting a small loop in each (since a figure-8 is twisted), as shown at the right.

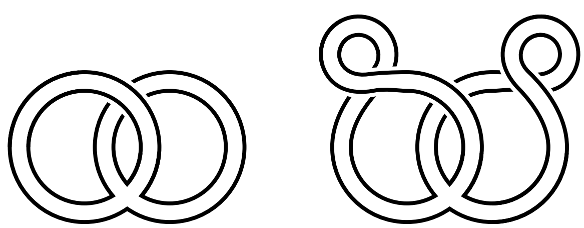

Figure 11: The neighborhoods of two intersecting cycles form two joined bands (left). Adding small loops makes the bands twisted (right) as required by a torus in the twisted image-homotopy class.

This figure can be deformed to have three-fold rotational symmetry, as shown in figure 12. Notice that the first six level sets from the sequence shown in figure 9, taken together, form exactly this same shape. Thus the torus whose level sets are shown in figure 9 is in the twisted class.

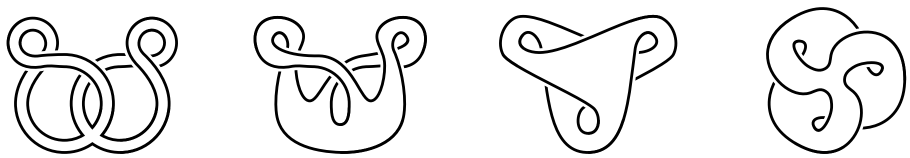

Figure 12: Two cycles whose neighborhoods form twisted bands can be deformed into a figure having three-fold rotational symmetry. The first six level sets from figure 9 form the shape at the right.

4 A Polyhedral Realization

At this point, we have a series of level sets for an immersion of the twisted torus that has a monkey saddle and no other interior critical points. Now we need to find a polyhedral realization of these level sets that has large, planar convex polygons at the top and bottom, with no locally extreme interior vertices. Ideally, we would like to have as few interior vertices as possible, as this will mean fewer vertices to check.

Using the three-fold rotational symmetry of the level sets, we can describe a polyhedral immersion with these properties by giving only one third of the vertices and faces, then making a copy of these rotated by $2\pi/3$ around the z-axis and another copy at $4\pi/3$. The result will be the complete surface. For example, in the list below, the vertex a_1 produces vertices a_2 and a_3 under these rotations, while the face $b_1\,a_2\,e_2$ generates faces $b_2\,a_3\,e_3$ and $b_3\,a_1\,e_1$. Vertex O and faces $a_1\,a_2\,a_3$ and $e_1\,e_2\,e_3$ are invariant under these rotations, so are not duplicated.

A view of the surface is shown in figure 13. The monkey saddle is at the center at vertex O, but it is hidden from view by other faces. Calculation of the level sets of this surface shows that they match the ones listed in figure 9 above, so it is a twisted torus. Note that the surface has a large triangular polygon at the top and a large hexagonal one at the bottom, and that the convex envelope is these two polygons together with the tube between them. A more schematic "fold-out" view of the surface is given in figure 14,

$O \to (0,0,0)$

$a_1 \to (10,0,-6)$	$d_1 \to b_1 + (0,0,14)$
$b_1 \to R(a_1)$	$e_1 \to (4d_1 + d_2)/5$
$c_1 \to (a_1+b_1)/2$	$f_1 \to (6,3,-3)$

$b_1\,a_2\,e_2$	$c_1\,e_2\,d_3$	$a_1\,c_1\,e_1$	$c_1\,b_1\,e_2$
$c_1\,e_3\,d_3$	$c_1\,f_1\,e_3$	$f_1\,c_1\,e_1$	$f_1 O\,e_3$
$f_1\,e_1\,O$	$a_1\,c_1\,a_2$	$c_1\,b_1\,a_2$	$d_1\,e_1\,e_3$
$a_1\,a_2\,a_3$	$e_1\,e_2\,e_3$		

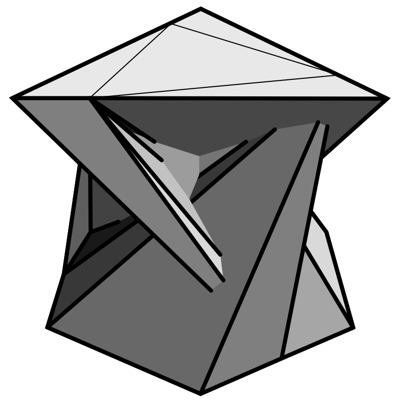

Figure 13: The polyhedral model of the twisted torus. Here $R(p)$ represents the rotation of point p about the z-axis by $\pi/3$. Since the surface has three-fold rotational symmetry, only one third of the vertices and faces are given; the others are obtained by rotating these through angles of $2\pi/3$ and $4\pi/3$ around the z-axis.

where the self-intersection is shown as a dotted line. The self-intersection does not contain any vertex of M, so the surface is immersed. Thus this surface satisfies almost all the properties that we required at the beginning of section 3; the only one remaining to be checked is that it contains no interior vertices that are locally extreme.

Note that the lower hexagon is formed by the vertices a_k and b_k and the upper triangle by the vertices d_k, so these are not interior vertices. Vertex O is a monkey saddle and clearly not locally extreme (it lies inside the convex hull of its neighbors). By definition, vertex c_k lies on the line segment halfway between its neighbors a_k and b_k, so it is not locally extreme. Similarly, e_k lies on the line between its neighbors d_k and d_{k+1}, so is not an extreme vertex. This leaves only f_k to check; but it lies within the tetrahedron formed by its four neighbors, so it is not extreme either. Thus this model satisfies all the required properties.

5 Putting it All Together

Given the polyhedral model developed in the previous section, we obtain the surface $M = K_+ \# B = T \# \bar{B}$ by replacing a neighborhood of the vertex O

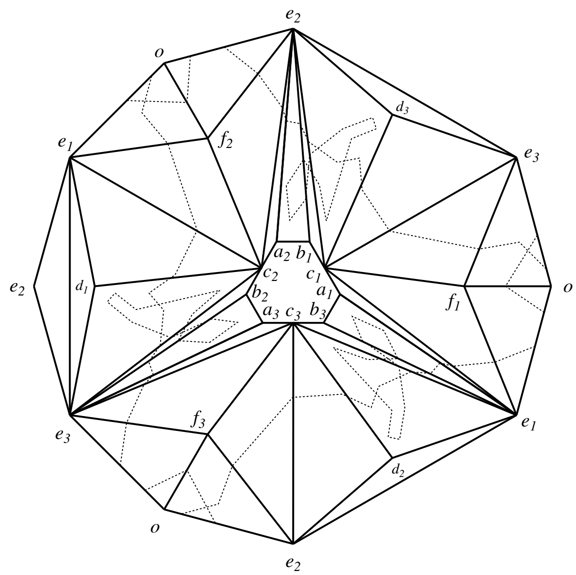

Figure 14: A schematic version of the twisted torus that exhibits its threefold symmetry (but the subdivision of the base hexagon is not shown). Moving the triangles containing vertex O from the two locations at the left to the copy of O at the right would form a more traditional rectangular decomposition of the torus, with vertices e_2, e_1, f_2 and e_2 along the top and bottom, and e_2, e_3, f_3 and e_2 along the left and right. The dotted line represents the self-intersection that occurs in the model.

by a copy of the Möbius band central to one of Brehm's projective planes, as shown in figure 4. That is, in the list of vertices and faces from figure 13, we replace O and the two faces containing O by

$g_1 \to (2,2,0)$	$f_1\,h_1\,e_3$	$h_1\,i_3\,e_3$	$f_1\,e_1\,h_1$
$h_1 \to (g_1+f_1)/2$	$e_1\,i_1\,h_1$	$h_1\,i_1\,g_1$	$i_1\,g_3\,g_1$
$i_1 \to (g_1+e_1)/2$	$g_1\,g_3\,h_3$	$h_1\,g_2\,i_3$	

This does not change properties *i* or *iv* from lemma 2.3, and one can check that *iii* still holds as well. Since M is non-orientable, condition *v* does not apply. For *ii*, we have already checked a_k, b_k, c_k, d_k, and e_k. By definition, h_k lies on the line segment between its neighbors g_k and f_k, and i_k lies between g_k and e_k. Again, f_k lies within the tetrahedron formed by its four neighbors (though O has been replaced by g_k). Finally, g_k lies within the tetrahedron formed by four of its neighbors (h_1, i_1, g_2 and g_3) as well. Thus no interior vertex of M is locally extreme.

We have checked all the properties required by lemma 2.3, so we can conclude that the mod-2 sum of M and its convex envelope is a tight immersion of $K_+ \# B \# S$. For completeness, the generating vertices and faces are given below, including the ones that form the outer handle. Again, due to the symmetry, only one third of the vertices and faces are given, the others being rotations by $2\pi/3$ and $4\pi/3$ of the ones shown.

$$\begin{aligned} O &\to (0,0,0) \\ a_1 &\to (10,0,-6) \\ b_1 &\to R(a_1) \\ c_1 &\to (a_1+b_1)/2 \\ d_1 &\to b_1+(0,0,14) \\ e_1 &\to (4d_1+d_2)/5 \\ f_1 &\to (6,3,-3) \\ g_1 &\to (2,2,0) \\ h_1 &\to (g_1+f_1)/2 \\ i_1 &\to (g_1+e_1)/2 \end{aligned}$$

$b_1\,a_2\,e_2$	$c_1\,e_2\,d_3$	$a_1\,c_1\,e_1$
$c_1\,b_1\,e_2$	$c_1\,e_3\,d_3$	$c_1\,f_1\,e_3$
$f_1\,c_1\,e_1$	$f_1\,h_1\,e_3$	$h_1\,i_3\,e_3$
$f_1\,e_1\,h_1$	$e_1\,i_1\,h_1$	$h_1\,i_1\,g_1$
$i_1\,g_3\,g_1$	$g_1\,g_3\,h_3$	$h_1\,g_2\,i_3$
$a_1\,c_1\,d_1$	$b_1\,d_1\,c_1$	$b_1\,a_2\,d_1$
$a_1\,d_1\,e_3$	$a_1\,e_3\,d_3$	

Note that we don't really need to check the handedness of the projective plane that we used in our sum with T, since $T \# \bar{B} = K_+ \# B$ while $T \# B = K_- \# \bar{B}$ and these two are mirror images of each other. Thus if we had chosen the wrong handedness, we could simply take a mirror image of the resulting surface to obtain the desired tight immersion of $K_+ \# B \# S$.

6 Conclusion

The process of building up a tight example in stages may seem a complicated approach at first, and one might ask whether it wouldn't be easier simply to start with level sets of the surface $T \# \bar{B}$ and produce a corresponding polyhedral immersion directly. The answer is that while technically it is possible to do so, the hardest part of the process is to go from the level sets to a polyhedral realization that can be made tight. There is no procedure to follow to accomplish this (note that we didn't discuss how this was done, but simply verified the result), however, the fewer vertices involved, the easier it is to do, in general. In our model, almost all the vertices are on the large planar polygons (where condition *ii* of lemma 2.3 does not apply), with only the central vertex, O, and the vertices f_k in the interior. Replacing O by the nine vertices of the Brehm projective plane would have seriously complicated the interior of the polyhedral surface if we hadn't had a previous plan for how to handle these vertices. Localizing the Möbius band in the neighborhood of O made it possible to reduce the surface to something that could be developed by hand.

As with the results listed in [8], the model described here does not provide a smooth example in these homotopy classes, since the smoothing algorithm of [14] does not apply to this surface (due to the high valence of some of its vertices). In light of the difference between the smooth and polyhedral results for the projective plane with one handle ([11] and [10]), there is no guarantee that this model has a tight smoothing. It would be interesting to find such a smoothing, or to show that none exists.

The only remaining surface for which a polyhedral tight immersion is possible, but for which no example is known, is $T \# S$. One may ask why we can not apply the process of lemma 2.3 to the polyhedral model of T generated in section 3, since we could apply it to $T \# \bar{B}$. The reason is that the two convex polygons do not satisfy condition v of that lemma; any orientation of the surface induces normals that both point in the same direction; adding the handle formed by the convex envelope would attach the inside to the outside of the surface making it non-orientable, resulting in $K_0 \# S$ not $T \# S$. This is not a problem for $T \# \bar{B}$, since it is non-orientable already. Haab has shown that no *smooth* tight immersion of $T \# S$ exists [12], and we conjecture that no polyhedral one does either, though a proof seems difficult to obtain. Haab's results rely fundamentally on the smoothness of the immersion, so they are unlikely to be useful in the polyhedral situation.

References

[1] T. F. Banchoff, "Tightly embedded 2-dimensional polyhedral manifolds", *Amer. J. Math.* **87** (1965), 462–472.

[2] ———, "The two-piece property and tight n-manifolds-with-boundary in $\mathbf{E}^n$", *Trans. Amer. Math. Soc.* **161** (1971) 259–267.

[3] ———, "Tight polyhedral Klein bottles, projective planes and Möbius bands", *Math. Ann.* **207** (1974) 233–243.

[4] T. F. Banchoff and W. Kühnel, "Tight submanifolds, smooth and polyhedral", in *Tight and taut submanifolds*, edited by T. C. Cecil and S.-s. Chern, Cambridge U. Press, 1997, 51–118.

[5] T. F. Banchoff and F. Takens, "Height functions with three critical points", *Ill. J. Math.* **76** (1975) 325–335.

[6] U. Brehm, "How to build minimal polyhedral models of the Boy surface", *Math. Intel.* **12** (1990) 51–56.

[7] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Mathematics **107**, Pitman Publ., Boston, London, Melbourne, 1985.

[8] D. P. Cervone, "Tight immersions of simplicial surfaces in three space", *Topology* **35** (1996) 863–873.

[9] ———, "Tightness for smooth and polyhedral immersions of the real projective plane with one handle", in *Tight and taut submanifolds*, edited by T. E. Cecil and S.-s. Chern, Cambridge U. Press, 1997, 119–133.

[10] ———, "A tight polyhedral immersion in three-space of the projective plane with one handle", to appear in *Pacific J. Math.* See also the URL ⟨`http://www.geom.umn.edu/locate/rp2-handle`⟩.

[11] F. Haab, "Immersions tendues de surfaces dans $\mathbf{E}^3$", *Comment. Math. Helv.* **67** (1992) 182–202.

[12] ———, "Surfaces tendues dans $\mathbf{E}^3$: des théorèms de structure", in preparation.

[13] W. Kühnel, “Tight and 0-tight polyhedral embeddings of surfaces”, *Invent. Math.* **58** (1980) 161–177.

[14] W. Kühnel and U. Pinkall, “Tight smoothing of some polyhedral surfaces”, in *Global Differential Geometry and Global Analysis* (Berlin 1984), edited by D. Ferus et al., Lecture Notes in Mathematics **1156**, Springer, Berlin, 1985, 227–239.

[15] N. H. Kuiper, “On surfaces in Euclidean three space”, *Bull. Soc. Math. Belg.* **12** (1960) 5–22.

[16] ———, “Convex immersions of closed surfaces in $\mathbf{E}^3$”, *Comm. Math. Helv.* **35** (1961) 85–92.

[17] ———, “There is no tight continuous immersion of the Klein bottle into $\mathbf{R}^3$”, IHES preprint (1983).

[18] D. Maxwell, S. Levy, and T. Munzner, *Outside In*, video (1994), distributed by A. K. Peters.

[19] S. Smale, “A classification of immersions of the two-sphere”, *Trans. Amer. Math. Soc.* **90** (1959) 281–290.

[20] U. Pinkall, “Regular homotopy classes of immersed surfaces”, *Topology* **24** (1985) 421–434.

[21] ———, “Tight surfaces and regular homotopy”, *Topology* **25** (1986) 475–481.

[22] R. Wells, “Cobordism groups of immersions”, *Topology* **5** (1966) 281–294.