Putnam Practice Problems for November 7, 2005

Those taking the Putnam Exam are generally expected to know some “basic” inequalities. Here are a few

1. $\sqrt{ab} \leq \frac{a + b}{2}$, $0 < a, b$ and the generalization

2. $(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$.

3. $\frac{a + b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}}$, $0 < a, b$ and the generalization

4. $a_1 + a_2 + \cdots + a_n \leq \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}$

5. $(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$

Warm-up Practice problems

1. Prove formula (1)

2. Now prove formula (2) (This is the arithmetic mean - geometric mean inequality.)

3. Use (5) to deduce (4) (Inequality (5) is the Cauchy-Schwarz inequality, and (4) is arithmetic mean - quadratic mean inequality).

Now try the following:

1. For $a, b, c \geq 0$, prove $(a + b)(b + c)(c + a) \geq 8abc$.

2. (Putnam 2003) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

   \[(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.\]

3. Prove the following inequality

   \[n \left[ (n + 1)^{1/n} - 1 \right] < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < n - (n - 1)n^{-1/(n-1)}.\]