All of the following can be proven using a technique called mathematical induction, but some can also be done using other methods. Even if you know about induction, try another approach. We’ll discuss your ideas, and I’ll give you a brief overview of mathematical induction, which is often helpful on the Putnam exam.

1. Show that \(1 + 2 + 3 + 4 + \cdots + (n-1) + n = \frac{n(n+1)}{2}\), for all positive integers \(n\).

2. Find a formula for the sum of the first \(n\) positive odd numbers. Prove your formula.

3. Show that \(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}\), for all positive integers \(n\).

4. Prove that \(9^n - 5^n\) is a multiple of 4, for all nonnegative integers \(n\).

5. For which values of \(n\) does the inequality \(2^n < n!\) hold? Prove your assertion.

6. Prove that every positive integer can be expressed as a product of prime numbers.

7. Find a formula for \(\prod_{k=2}^{n} (1 - \frac{1}{k^2})\), for \(n \geq 2\), and prove your formula\(^1\).

8. Let \(k\) be a fixed positive integer. The \(n\)th derivative of \(\frac{1}{x^k - 1}\) has the form \(\frac{P_n(x)}{(x^k - 1)^{n+1}}\) where \(P_n(x)\) is a polynomial. Find \(P_n(1)\). (63\(^{rd}\) Putnam, 2002.)

\(^1\)Note on notation: Just like \(\sum_{k=1}^{n} f(k)\) denotes the sum \(f(1) + f(2) + \cdots + f(n)\), the notation \(\prod_{k=1}^{n} f(k)\) represents the product \(f(1)f(2)\cdots f(n)\). For example, \(n!\) can be written as \(\prod_{k=1}^{n} k\).