Putnam Problems for November 11^{th}, 2003

1. Prove that

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx.$$

2. Let f be a continuous function. Prove that

$$\int_{x=0}^{x=1}\int_{y=x}^{y=1}\int_{z=x}^{z=y} f(x)f(y)f(z)\,dz\,dy\,dx = \frac{1}{6}\left(\int_0^1 f(t)\,dt\right)^3.$$

   [Hint: let F be an antiderivative of f, and calculate the two sides of the above identity in terms of F.]

3. Let f be a differentiable function such that $f'(a-x) = f'(x)$ for all x in the interval $[0, a]$. Evaluate $\int_0^a f(x)\,dx$, and give an example of such a function f.

4. Let f be a real valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

   Prove that $f(x)$ has a limit as $x \to +\infty$, and that this limit is $\leq 1 + \frac{\pi}{4}$.