Putnam Practice Problems for October 7, 2003

Those taking the Putnam Exam are generally expected to know some "basic" inequalities. Here are a few

(1) $\sqrt{ab} \le \dfrac{a+b}{2}, \quad 0 < a, b$ and the generalization

(2) $(a_1 a_2 \cdots a_n)^{1/n} \le \dfrac{a_1 + a_2 + \cdots + a_n}{n}.$

(3) $\dfrac{a+b}{2} \le \sqrt{\dfrac{a^2+b^2}{2}}, \quad 0 < a, b$ and the generalization

(4) $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \le \sqrt{\dfrac{a_1^2 + \cdots + a_n^2}{n}}$

(5) $(a_1 b_1 + \cdots + a_n b_n)^2 \le (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$

Warm-up Practice problems

(1) Prove formula (1)

(2) Now prove formula (2) (This is the *arithmetic mean - geometric mean* inequality.)

(3) Use (5) to deduce (4) (Inequality (5) is the *Cauchy-Schwarz inequality*, and (4) is *arithmetic mean - quadratic mean* inequality).

Now try the following:

(1) For $a, b, c \ge 0$, prove $(a+b)(b+c)(c+a) \ge 8abc$.

(2) If $a_i > 0$ for $i = 1, \ldots, n$, then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \ge 2^n.$$

(3) Prove the following inequality

$$n\left[(n+1)^{1/n} - 1\right] < 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} < n - (n-1)n^{-1/(n-1)}.$$