Putnam Problems for October 28$^{\text{th}}$, 2003

A recurrent topic on the Putnam exam is binomial coefficients. Binomial coefficients are the numbers $\binom{n}{k}$ (read "n choose k") defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. (Recall: $n!$, read "n factorial", is the product $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.) These numbers are sometimes denoted by ${}_nC_k$, and they correspond to "the number of ways to select k objects out of n" or, equivalently, the number of subsets of $\{1, 2, \ldots, n\}$ with k elements. You're expected to know a few facts about binomial coefficients, including:

(A) **Pascal's Recursion:** $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

(B) **Binomial Theorem:** $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$.

Warm-up problems:

1. Prove (A).
2. Use (A) to prove (B) by induction on n.
3. Use (B) to show the following:
   - $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.
   - $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n-1} \mp \binom{n}{n} = 0$.
   - $1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$. [One-word hint: derivative!]
4. Find $1\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + 7\binom{n}{3} + \cdots + (2n+1)\binom{n}{n}$.
5. Find $\binom{n}{0} + \binom{n}{1}/2 + \binom{n}{2}/3 + \cdots + \binom{n}{n}/(n+1)$.
6. Find $\binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots + (-1)^n\binom{m}{n}$.

Now try the following:

1. Show that the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$ is

$$\sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

   (From 53$^{\text{rd}}$ Putnam, 1992.)

2. Find $\binom{n}{1}1^2 + \binom{n}{2}2^2 + \binom{n}{3}3^2 + \cdots + \binom{n}{n}n^2$. (From 23$^{\text{rd}}$ Putnam, 1962.)
3. Find $F_1\binom{n}{0} + F_2\binom{n}{1} + F_3\binom{n}{2} + \cdots + F_{n+1}\binom{n}{n}$, where $F_1, F_2, F_3, \ldots$ are the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, \ldots$.