Putnam Problems for October 21^{st}, 2003

1. A *composition* of a positive integer n is a way of writing n as a sum of positive integers, taking order into account. For instance, $n = 3$ has four compositions:

$$\begin{array}{l} 3 \\ 2+1 \\ 1+2 \\ 1+1+1 \end{array}$$

   Find a formula (in terms of n) for the number of compositions of n. [Note: if you *do not* take order into account, then each possible sum is called a *partition* of n. For example, $n = 3$ has 3 partitions, since $1+2$ and $2+1$ would be considered to be the same partition. Try to find the number of partitions of different numbers n. Can you spot any patterns?][1]

2. Consider the sequence (a_n) defined as follows: $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$, for all $n \geq 2$. Find a formula for a_n in terms of n. [Note: this is an example of a *recursive formula* or a *recurrence*—where one term of the sequence is calculated using the previous terms. What you're looking for in this problem—a formula that involves just n, and not the previous terms of the sequence—is called a *closed form* for a_n.]

3. How many slices of pizza can one obtain by making n straight cuts with a pizza knife? Or, more academically, what is the maximum number of regions determined by n lines in the plane? How many of these regions are bounded? How many are unbounded? Can you find the maximum number of regions in 3-space determined by n planes?

4. Consider a currency consisting of 2-cent and 5-cent coins. What amounts can be paid using these coins?

5. (*The December 31 Game*) Two players alternately name dates. On each move, a player can increase the month or the day of the month, but not both. The starting position is January 1, and the player who names December 31 wins. According to the rules, the first player can start by naming some day in January after the first, or the first of some month after January. For example (Jan. 10, Mar. 10, Mar. 15, Apr. 15, Apr. 25, Nov. 25, Nov. 30, Dec. 30, Dec. 31) is an instance of the game won by the first player. Derive a strategy that the first player can use to guarantee winning. Prove that your strategy works.

[1] Remember that you must prove all your answers, even when the problem doesn't explicitly say that.