**Instructions:**
1. Read all directions.
2. In keeping with the Union College policy on academic honesty, you should neither accept nor provide unauthorized assistance in the completion of this work.

Name: S1M J.B & S.A.  
Date: ____________

**Direction 1: Circle the correct answer**

1. What is the current density in a wire of radius 1mm with a current of $I = 3.14A$? (Assume the current density is uniform).
   - a. $3.14 \times 10^6 \text{ A/m}^2$
   - b. $1 \times 10^6 \text{ A/m}^2$
   - c. $3.14 \times 10^{-6} \text{ A/m}^2$
   - d. $1 \times 10^{-6} \text{ A/m}^2$

2. Two capacitors with capacitance $C_1$ and $C_2$ are connected in series. If $C_1 < C_2$, what can we conclude about the equivalent capacitance of the two capacitors, $C_{eq}$?
   - a. $C_2 < C_{eq}$
   - b. $C_1 < C_{eq} < C_2$
   - c. $C_{eq} < C_1$
   - d. none of the above.

3. The SI unit/s for magnetic flux is/are
   - a. Weber
   - b. Tesla meter squared
   - c. both a & b
   - d. none

4. The total magnetic flux through a closed surface _______.
   - a. is not equal to the volume integral of divergence of the magnetic field.
   - b. is equal to zero.
   - c. is dependent of the shape of the Gaussian surface.
   - d. all of the above
   - e. none of the above

5. Which one of the following is true?
   - a. A moving charge can experience force due to a magnetic field.
   - b. A moving charge can experience force due to an electric field.
   - c. A moving charge generates magnetic field in its vicinity.
   - d. all of the above.
   - e. none of the above.

6. According to Ampere’s Law ($\oint c \cdot d\vec{s} = \mu_o I_{nc} + \mu_o I_d$), which one of the following is true?
   - a. the integral is independent of the shape of the curve $c$ as long as it is closed.
   - b. the integral is equal to the curl of the magnetic field within the region bounded by the curve $c$.
   - c. the integral can be positive or negative.
   - d. all of the above.
   - e. none of the above.

7. The loop rule is an application of
   - a. conservation of momentum.
   - b. conservation of energy.
   - c. conservation of charge.
   - d. all of the above.
   - e. none of the above.
8. Suppose that the temperature, as a function of position \((x,y)\), is given by \(T(x,y) = xy^2 + 3y\).

a. In what direction, from the point \((-4,3)\), would one experience the greatest instantaneous cooling?

\[
\nabla T(x,y) = y^2 \hat{i} + (2xy + 3) \hat{j}
\]

\(\nabla T(-4,3) = 9 \hat{i} - 21 \hat{j}\)

Direction of greatest instantaneous cooling

\(-\nabla T(-4,3) = -9 \hat{i} + 21 \hat{j}\)

b. Specify a direction from the point \((-4,3)\) in which one would experience no instantaneous rate of change of temperature.

Perpendicular to \(\nabla T(-4,3)\):

\(21 \hat{i} + 9 \hat{j}\) or \(-21 \hat{i} - 9 \hat{j}\)

c. Compute the instantaneous rate of change of temperature with respect to distance traveled from the point \((-4,3)\) in the direction of the origin.

\[
\text{Vector from } (-4,3) \text{ to } (0,0) = 4 \hat{i} - 3 \hat{j}
\]

\[
\hat{a} = \frac{4 \hat{i} - 3 \hat{j}}{\sqrt{4^2 + (-3)^2}} = \frac{4}{5} \hat{i} - \frac{3}{5} \hat{j}
\]

\[
D_{\hat{a}} T(-4,3) = \nabla T(-4,3) \cdot \hat{a} = (9 \hat{i} - 21 \hat{j}) \cdot \left(\frac{4}{5} \hat{i} - \frac{3}{5} \hat{j}\right)
\]

\[
= \frac{36}{5} + \frac{63}{5} = \frac{99}{5}
\]
9. Suppose that \( \vec{E}(x, y, z) = (2xy)i + (x^2 + z^2)j + (2yz)k \) is the electric field due to some configuration of electric charge. It is measured in Newton per coulomb and the coordinates \( x, y, \) and \( z \) are measured in meters.

a. Show that \( \vec{E} \) is conservative in two different ways:

**Method 1:**

Find \( \phi \) so that \( \nabla \phi = \vec{E} \)

\[
\frac{\partial \phi}{\partial x} = 2xy, \quad \text{so} \quad \phi(x, y, z) = x^2y + f(y, z)
\]

\[
\frac{\partial \phi}{\partial y} = x^2 + z^2, \quad \text{so} \quad \phi(x, y, z) = x^2y + yz^2 + g(x, z)
\]

\[
\frac{\partial \phi}{\partial z} = 2yz, \quad \text{so} \quad \phi(x, y, z) = yz^2 + h(x, y)
\]

\( \phi = x^2y + yz^2 \)

**Method 2:**

\[
\nabla \times \vec{E} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xy & x^2 + z^2 & 2yz \\
2y & 2x & 2z
\end{vmatrix} = (2z - 2z)i - 0j + (2x - 2x)k = 0
\]

b. Find \( \nabla \phi \), the electrical potential function. (Hint: Your previous work in this problem may be helpful.)

\[
\nabla \phi(x, y, z) = -\phi(x, y, z) = -x^2y - yz^2
\]
c. Compute the work done by \( \vec{E} \) on an electric charge \( q_0 \) as \( q_0 \) moves from the point \((0,1,2)\) to the point \((1,2,0)\).

\[
W = \sum_{(0,1,2)}^{(1,2,0)} q_0 \vec{E} \cdot \vec{r} = q_0 [\vec{E}(1,2,0) - \vec{E}(0,1,2)]
\]

\[
= q_0 [-2 + 4] = 2q_0
\]

---

d. Let \( G \) be the solid bounded by the xy plane, the \( z=5 \) plane, the \( y=0 \) plane, and that portion of the cylinder \( x^2+y^2 = 9 \) for which \( y \geq 0 \), and let \( S \) be the surface of this solid. Use the divergence theorem to compute the flux of \( \vec{E} \) across \( S \).

\[
\nabla \cdot \vec{E} = 2y + 2y = 4y
\]

\[
\Phi = \iint_S (\vec{E} \cdot \vec{n}) \, dS = \iiint_G (\nabla \cdot \vec{E}) \, dV
\]

\[
= \iiint_G 4y \, dV = \int_0^{2\pi} \int_0^3 \int_0^5 4r^2 \sin \theta \, dr \, d\theta \, dz
\]

\[
= \int_0^{2\pi} \int_0^3 4r^2 \sin \theta \, dr \, d\theta \int_0^5 dz
\]

\[
= \int_0^{2\pi} \left[ \frac{20r^3}{3} \sin \theta \right]_0^3 \, d\theta \int_0^5 4 \, dz
\]

\[
= \int_0^{2\pi} \frac{20 \cdot 3^3}{3} \sin \theta \, d\theta \int_0^5 4 \, dz
\]

\[
= \int_0^{2\pi} \frac{180}{3} \sin \theta \, d\theta \int_0^5 4 \, dz
\]

\[
= \int_0^{2\pi} -180 \cos \theta \bigg|_0^{2\pi} \int_0^5 4 \, dz
\]

\[
= \int_0^{2\pi} 180 \sin \theta \, d\theta \int_0^5 4 \, dz
\]

\[
= 180 + 180 = 360
\]
e. Use your answer to part d to determine the net charge enclosed by $S$ that caused $\vec{E}$. (Note: this is different charge from the charge in part c)

\[
\text{Charge enclosed} = \text{Flux} \cdot \varepsilon_0 \\
= 360 \varepsilon_0 \\
\approx 3.187 \times 10^{-9}
\]

10. An infinitely long current carrying wire, $I = 5\text{A}$, is placed on the $y$-axis. The current flows in the positive $y$-direction.

a. Find the magnetic field $\vec{B}$ (both magnitude and direction) at a point $p = (2\text{mm}, 0, 0)$

\[
\vec{B} = \frac{\mu_0 I}{2\pi r} (-\hat{k})
\]

\[
\vec{B} = \frac{2\times 10^{-7} \text{T} \cdot 5\text{A}}{r} \hat{k} = 5 \times 10^{-4} \text{T} (-\hat{k})
\]

b. Evaluate the integral $\oint_C \vec{B} \cdot d\vec{s}$, where $C$ is a circle of radius 2mm on the $x$-$z$ plane oriented counterclockwise as seen from the positive $y$-axis.

\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I \quad (\text{Since } \vec{B} \text{ and } d\vec{s} \text{ are parallel})
\]

\[
\oint_C \vec{B} \cdot d\vec{s} = (4\pi \times 10^{-7} \text{T} \cdot \text{m}) (5\text{A})
\]

\[
\oint_C \vec{B} \cdot d\vec{s} = 20\pi \times 10^{-7} \text{T} \cdot \text{m}
\]
11. Let $S$ be that portion of the paraboloid $z = 4x^2 + 4y^2$ with $z \leq 100$. Set up, but do not evaluate, integrals to compute each of the following. In each case, your answer should include the integrand and all limits of integration. (In other words, your answer should be a type of integral that you could have computed last term.)

(a) The surface area of $S$.

$$
\begin{align*}
\frac{\partial z}{\partial u} &= 8u, \\
\frac{\partial z}{\partial v} &= 8v, \\
\frac{\partial^2 z}{\partial u \partial v} &= 0.
\end{align*}
$$

Surface Area = \mathcal{S}_S \int \left( \sqrt{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial^2 z}{\partial u \partial v}\right)^2} \right) \, dA
$$

$$
= \mathcal{S}_S \int_0^1 \int_0^{2\pi} \sqrt{64u^4 \cos^2 v + 64u^4 \sin^2 v + u^2} \, du \, dv
$$

$$
= \mathcal{S}_S \int_0^{2\pi} \int_0^5 \sqrt{64u^4 + u^2} \, du \, dv.
$$

(b) The flux of the vector field $\mathbf{F} = 3x\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$ across $S$.

$$
\mathbf{F} = 3u \cos v \mathbf{i} + 8v^2 \mathbf{j} + u \sin v \mathbf{k}
$$

$$
\Phi = \mathcal{S}_S \mathbf{F} \cdot \mathbf{n} \, dS
$$

$$
= \mathcal{S}_S \int \left[ \mathbf{F} \cdot \left( \frac{\partial z}{\partial u} \times \frac{\partial z}{\partial v} \right) \right] \, dS
$$

$$
= \mathcal{S}_S \int_0^{2\pi} \int_0^5 (-24u^3 \cos^2 v - 64u^4 \sin v + u^2 \sin v) \, du \, dv.
$$
12. A charged particle of \( q = 3.2 \times 10^{-16} \text{C} \) enters a region with uniform magnetic field at point A.

The magnetic field is \( \vec{B}(x, y, z) = (0.5T)\hat{k} \) for \( x>0 \) and zero for \( x<0 \). The particle sweeps a semicircle trajectory before it leaves the region at point C. The velocity of the particle at point A is \( \vec{V} = (2 \times 10^6 \text{ m/s})\hat{i} \). Note that the distance from the origin to C is the same as the distance from the origin to A and it is equal to \( r \), the radius of the semicircle.

a. What is the value of the charge \( q \)? (is it positive or negative \( 3.2 \times 10^{-16} \text{C} \)?) (state your reason)

\[ q = -3.2 \times 10^{-16} \text{C} \]

Since the charge experiences a force \( F \) opposite to \( |q| \vec{V} \times \vec{B} \).

b. What is the velocity (magnitude and direction) of the charged particle at point B whose coordinates are \((r,0,0)\)? (state your reason)

\[ \vec{V} = 2 \times 10^6 \text{ m/s} \hat{j} \]

The magnetic field does not increase or decrease the magnitude of \( \vec{V} \), it only changes its direction.

c. Find the force (magnitude and direction) on the charged particle at point A.

\[ \vec{F} = \frac{q\vec{V} \times \vec{B}}{|\vec{B}|} = (3.2 \times 10^{-16} \text{C}) \cdot \frac{2 \times 10^6 \text{ m/s}}{0.5} \hat{j} = (3.2 \times 10^{-16} \text{C}) 4 \times 10^5 \text{J/kg} \hat{j} \]

\[ \vec{F} = 3.2 \times 10^{-10} \text{N} \hat{j} \]

d. Find the radius of the semicircle?

\[ r = \frac{mV}{|F|} = \frac{mV}{q|\vec{B}|} \]

\[ r = \frac{2 \times 10^{-12} \text{kg}}{(3 \times 10^6 \text{ C})(0.57)} \]

\[ r = 400 \text{ km} \]

e. How long does it take for the particle to travel from point A to Point C. (time).

\[ \text{time} = \frac{\text{distance}}{\text{speed}} = \frac{\pi r}{V} = \frac{\pi \cdot 400}{2 \times 10^6} \]

\[ \text{time} = \frac{(\pi)(2.2 \times 10^{-12} \text{ kg})}{(9.2 \times 10^6 \text{ C})(0.57)} \]

\[ \text{time} = 2 \times 10^{-17} \text{ s} \]
13. Suppose that the vector field \( \vec{F}(x, y, z) = yzi + \frac{z^2}{2} + 4y\hat{k} \) gives the velocity of fluid flow.

a. Compute \( \nabla \times \vec{F}(x, y, z) = \)

\[
\nabla \times \vec{F} = \begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
yz & z^2 & x \\
2z & 4y & 0
\end{vmatrix} = (4z-2z)\hat{i} + y\hat{j} - 2\hat{k}
\]

b. Compute \( \nabla \times \vec{F}(1, 2, 3) \) and \( \nabla \times \vec{F}(1, 3, 1) \)

\[
\nabla \times \vec{F}(1, 2, 3) = -2\hat{i} + 2\hat{j} - 3\hat{k}
\]

\[
\nabla \times \vec{F}(1, 3, 1) = 2\hat{i} + 3\hat{j} - \hat{k}
\]

c. Briefly contrast your two answers to part b. What do they tell us about the flow of fluid near the point (1,2,3) and the flow of fluid near the point (1,3,1)? Be sure to comment on the significance of both the direction and the magnitude of your answers to part b.

Note that \( |\nabla \times \vec{F}(1, 2, 3)| = \sqrt{9+4+9} = \sqrt{17} \) and

\( |\nabla \times \vec{F}(1, 3, 1)| = \sqrt{4+9+1} = \sqrt{14} \).

The maximal rotational tendencies at (1,2,3) occur in a plane perpendicular to \(-2\hat{i} + 2\hat{j} - 3\hat{k}\). The maximal rotational tendencies at (1,3,1) occur in a plane perpendicular to \(2\hat{i} + 3\hat{j} - \hat{k}\). Since \( |\nabla \times \vec{F}(1, 2, 3)| > |\nabla \times \vec{F}(1, 3, 1)| \), these rotational tendencies are stronger at (1,2,3).
14. Refer to the figure below and do not change the direction of flow of currents.

a. Write a system of equations (three) to solve the current in the circuit.

\[ \begin{align*}
I_1 + I_2 &= I_3 \Rightarrow I_1 + I_2 - I_3 &= 0 \quad (1) \\
10I_2 + 14 - 6I_1 + 10 &= 0 \\
6I_1 - 4I_2 &= 24 \quad (2) \\
-10 + 6I_1 + 0 + 2I_3 &= 0 \\
6I_1 + 2I_3 &= 10 \quad (3)
\end{align*} \]

b. Write the system of equations in matrix form

\[ \begin{pmatrix} 1 & 1 & -1 \\ 6 & -4 & 0 \\ 6 & 0 & 2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ 10 \end{pmatrix} \]

\[ A \mathbf{I} = \mathbf{B} \]

c. Find the determinant of the coefficient matrix.

\[ \det A = \begin{vmatrix} 1 & 1 & -1 \\ 6 & -4 & 0 \\ 6 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 6 & -4 \end{vmatrix} + 2 \begin{vmatrix} 6 & 0 \\ 6 & 2 \end{vmatrix} = -24 + 20 = -4 \]

\[ \det A = (1 - (4)) + 2(6 - 0) = -24 + 20 = -4 \]

d. Using Cramer's rule solve for \( I_2 \).

\[ I_2 = \frac{\begin{vmatrix} 0 & -1 \\ 6 & 2 \end{vmatrix}}{\det A} = \frac{1 \begin{vmatrix} 6 & 2 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \end{vmatrix}}{\det A} = \frac{1 \cdot 14 - 1 \cdot (60 - 94)}{-4} \]

\[ I_2 = \frac{132 - 94}{-4} = -3 \]
15. Use Green's theorem to compute the work done by the force field \( \mathbf{F}(x, y) = 3y^2 \hat{i} + 2xy \hat{j} \) on an object as that object moves from the point (2,0), around the top half of the circle with center at the origin and radius 2, to the point (-2,0), and then back to (2,0) along the x axis.

\[
\begin{align*}
W &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_R \left( \frac{\partial}{\partial x} (2xy) - \frac{\partial}{\partial y}(3y^2) \right) \, dA \\
&= \iint_R (2y - 6y) \, dA = \iint_R -4y \, dA \\
&= \int_0^{\pi} \int_0^2 -4r^2 \sin \theta \, dr \, d\theta \\
&= -\int_0^{\pi} \left[ \frac{r^3 \sin \theta}{3} \right]_0^2 \, d\theta \\
&= -\int_0^{\pi} -\frac{32 \sin \theta}{3} \, d\theta \\
&= -\frac{32}{3} \cos \theta \bigg|_0^{\pi} = -\frac{32}{3} - \frac{32}{3} = -\frac{64}{3}
\end{align*}
\]

b. What does your answer to part a tell us about whether \( \mathbf{F} \) is conservative or not? Give a reason for your answer.

\textit{Not conservative. If it were conservative then since } \( C \) \textit{is a closed path the integral would be a}
16. Use Stokes' Theorem to compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$ and $C$ is the triangle with vertices (10, 0, 0), (0, 2, 0), and (0, 0, 5). (Hint: This triangle lies in the plane $x + 5y + 2z = 10$.)

\[ \nabla \times \mathbf{F} = \begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \]

$x = u$

$y = v$

$z = \frac{1}{2} (10 - u - 5v)$

$0 \to \frac{1}{2}(10 - u)$

$0 \to u \to 10$

\[ \dd S_c \dd x = \dd S_v (\nabla \times \mathbf{F}) \cdot \mathbf{n} \dd S_v \]

\[ = \iint_R \left[ (\nabla \times \mathbf{F}) \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \right] \dd A \]

\[ = \iint_R \left( \frac{1}{2} + \frac{5}{2} + 1 \right) \dd A \]

\[ = \iint_R 4 \dd A \]

\[ = 4 \text{ (Area of right triangle with base 10 and height 2)} \]

\[ = (4)(\frac{1}{2})(10)(2) = 40 \]
17. Write the augmented matrix for each of the following systems of linear equations and then solve:

\[
\begin{align*}
  x - 3z &= 1 \\
  4x - y + z &= 3 \\
  x - y + 10z &= 2
\end{align*}
\]

a. \[
\begin{bmatrix}
  1 & 0 & -3 & 1 \\
  4 & -1 & 1 & 3 \\
  1 & -1 & 10 & 2
\end{bmatrix}
\]

b. \[
\begin{align*}
  w + x - y + z &= 4 \\
  -w - 2x + y - 3z &= -2
\end{align*}
\]

\[
\begin{bmatrix}
  -1 & 1 & -1 & 1 & 4 \\
  0 & -1 & 1 & 2 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 1 & -1 & 4 \\
  0 & 1 & 2 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 & -1 & 1 & 6 \\
  0 & 1 & 0 & 2 & 2
\end{bmatrix}
\]
18. In the figure below, evaluate the integral \( \oint_C \vec{B} \cdot d\vec{s} \) clockwise

a. if \( C \) is the closed curve labeled a
\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \left(-1A - 5A + 2A\right) = -\mu_0 (4A)
\]

b. if \( C \) is the closed curve labeled b
\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \left(-1A - 2A\right) = \mu_0 (1A)
\]

c. if \( C \) is the closed curve labeled c
\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \left(-1A - 5A\right) = -\mu_0 (6A)
\]

d. if \( C \) is the closed curve labeled d
\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 (-5A + 2A) = -\mu_0 (3A)
\]

19. Using the figure below,

a. Evaluate the electrical flux through the closed surfaces \( S_1 \) if \( q = 6 \mu C \).
\[
\Phi_{\text{F}} = \oint_{S_1} \vec{E} \cdot dA = \frac{q}{\varepsilon_0} = \frac{6 \mu C}{\varepsilon_0}
\]

b. Evaluate the electrical flux through the closed surfaces \( S_1 \) if \( q = 12 \mu C \).
\[
\Phi_{\text{F}} = \oint_{S_1} \vec{E} \cdot dA = \frac{q}{\varepsilon_0} = \frac{12 \mu C}{\varepsilon_0}
\]

c. Evaluate the electrical flux through the closed surfaces \( S_2 \) if \( q = 6 \mu C \).
\[
\Phi_{\text{F}} = \oint_{S_2} \vec{E} \cdot dA = \frac{q}{\varepsilon_0} = \frac{6 \mu C}{\varepsilon_0}
\]

d. Evaluate the electrical flux through the closed surfaces \( S_1 \) if \( q = 0 \).
\[
\Phi_{\text{F}} = \oint_{S_1} \vec{E} \cdot dA = 0
\]