Homework #3 Solutions  Chapter 22

22. (a) We use the usual notation for the linear charge density: \( \lambda = q/L \). The arc length is \( L = r\theta \) with \( \theta \) is expressed in radians. Thus,
\[
L = (0.0400 \text{ m})(0.698 \text{ rad}) = 0.0279 \text{ m}.
\]
With \( q = -300(1.602 \times 10^{-19} \text{ C}) \), we obtain \( \lambda = -1.72 \times 10^{-15} \text{ C/m} \).

(b) We consider the same charge distributed over an area \( A = \pi r^2 = \pi (0.0200 \text{ m})^2 \) and obtain \( \sigma = q/A = -3.82 \times 10^{-14} \text{ C/m}^2 \).

(c) Now the area is four times larger than in the previous part \( (A_{\text{sphere}} = 4\pi r^2) \) and thus obtain an answer that is one-fourth as big:
\[
\sigma = q/A_{\text{sphere}} = -9.56 \times 10^{-15} \text{ C/m}^2.
\]
(d) Finally, we consider that same charge spread throughout a volume of \( V = 4\pi r^3/3 \) and obtain the charge density \( \rho = q/V = -1.43 \times 10^{-12} \text{ C/m}^3 \).

28. We use Eq. 22-16, with “q” denoting the charge on the larger ring:
\[
\frac{qz}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}} - \frac{qz}{4\pi\varepsilon_0(z^2 + (3R)^2)^{3/2}} = 0 \quad \Rightarrow \quad q = -Q\left(\frac{13}{5}\right)^{3/2} = -4.19Q.
\]
Note: we set \( z = 2R \) in the above calculation.

29. The smallest arc is of length \( L_1 = \pi r_1/2 = \pi R/2 \); the middle-sized arc has length \( L_2 = \pi r_2/2 = \pi (2R)/2 = \pi R \); and, the largest arc has \( L_3 = \pi (3R)/2 \). The charge per unit length for each arc is \( \lambda = q/L \) where each charge \( q \) is specified in the figure. Following the steps that lead to Eq. 22-21 in Sample Problem 22-3, we find
\[
E_{\text{net}} = \frac{\lambda_1 (2 \sin 45^\circ)}{4\pi\varepsilon_0 r_1} + \frac{\lambda_2 (2 \sin 45^\circ)}{4\pi\varepsilon_0 r_2} + \frac{\lambda_3 (2 \sin 45^\circ)}{4\pi\varepsilon_0 r_3} = \frac{Q}{\sqrt{2\pi^2\varepsilon_0 R^2}}
\]
which yields \( E_{\text{net}} = 1.62 \times 10^6 \text{ N/C} \).

(b) The direction is \( -45^\circ \), measured counterclockwise from the +x axis.

# 32 and #33 at the end
36. We write Eq. 22-26 as

\[
\frac{E}{E_{\text{max}}} = 1 - \frac{z}{(z^2 + R^2)^{1/2}}
\]

and note that this ratio is \(\frac{1}{3}\) (according to the graph shown in the figure) when \(z = 4.0\) cm. Solving this for \(R\) we obtain \(R = z \sqrt{3} = 6.9\) cm.

37. We use Eq. 22-26, noting that the disk in figure (b) is effectively equivalent to the disk in figure (a) plus a concentric smaller disk (of radius \(R/2\)) with the opposite value of \(\sigma\). That is,

\[
E_{(b)} = E_{(a)} - \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{2R}{\sqrt{(2R)^2 + (R/2)^2}} \right)
\]

where

\[
E_{(a)} = \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{2R}{\sqrt{(2R)^2 + R^2}} \right).
\]

We find the relative difference and simplify:

\[
\frac{E_{(a)} - E_{(b)}}{E_{(a)}} = \frac{1 - 2/\sqrt{4 + 1/4}}{1 - 2/\sqrt{4 + 1}} = \frac{1 - 2/\sqrt{17/4}}{1 - 2/\sqrt{5}} = 0.0299 \quad \text{or} \quad 0.283
\]

or approximately 28%.

32. We assume \(q > 0\). Using the notation \(\lambda = q/L\) we note that the (infinitesimal) charge on an element \(dx\) of the rod contains charge \(dq = \lambda \, dx\). By symmetry, we conclude that all horizontal field components (due to the \(dq\)'s) cancel and we need only “sum” (integrate) the vertical components. Symmetry also allows us to integrate these contributions over only half the rod \((0 \leq x \leq L/2)\) and then simply double the result. In that regard we note that \(\sin \theta = R/r\) where \(r = \sqrt{x^2 + R^2}\).

(a) Using Eq. 22-3 (with the 2 and \(\sin \theta\) factors just discussed) the magnitude is

\[
|\vec{E}| = 2 \int_0^{L/2} \left( \frac{dq}{4\pi\varepsilon_0 r^2} \right) \sin \theta = \frac{\lambda}{4\pi\varepsilon_0} \int_0^{L/2} \left( \frac{\lambda \, dx}{x^2 + R^2} \right) \left( \frac{y}{\sqrt{x^2 + R^2}} \right)
\]

\[
= \frac{\lambda R}{2\pi\varepsilon_0} \int_0^{L/2} \frac{dx}{(x^2 + R^2)^{3/2}} = \left( \frac{q/L}{2\pi\varepsilon_0} \right) \left( \frac{R}{R^2 \sqrt{x^2 + R^2}} \right)_{x=0}^{x=L/2}
\]

\[
= \frac{q}{2\pi\varepsilon_0 LR} \frac{L/2}{\sqrt{(L/2)^2 + R^2}} = \frac{q}{2\pi\varepsilon_0 R \sqrt{L^2 + 4R^2}}
\]

where the integral may be evaluated by elementary means or looked up in Appendix E (item #19 in the list of integrals). With \(q = 7.81 \times 10^{-12}\) C, \(L = 0.145\) m and \(R = 0.0600\) m, we have \(|\vec{E}| = 12.4\) N/C.

(b) As noted above, the electric field \(\vec{E}\) points in the +y direction, or +90° counterclockwise from the +x axis.
33. Consider an infinitesimal section of the rod of length \( dx \), a distance \( x \) from the left end, as shown in the following diagram. It contains charge \( dq = \lambda \, dx \) and is a distance \( r \) from \( P \). The magnitude of the field it produces at \( P \) is given by

\[
dE = \frac{1}{4\pi \varepsilon_0} \frac{\lambda \, dx}{r^2}.
\]

The \( x \) and the \( y \) components are

\[
dE_x = -\frac{1}{4\pi \varepsilon_0} \frac{\lambda \, dx}{r^2} \sin \theta
\]

and

\[
dE_y = -\frac{1}{4\pi \varepsilon_0} \frac{\lambda \, dx}{r^2} \cos \theta,
\]

respectively. We use \( \theta \) as the variable of integration and substitute \( r = R/\cos \theta \), \( x = R \tan \theta \) and \( dx = (R/\cos^2 \theta) \, d\theta \). The limits of integration are 0 and \( \pi/2 \) rad. Thus,

\[
E_x = -\frac{\lambda}{4\pi \varepsilon_0 R} \int_0^{\pi/2} \sin \theta \, d\theta = \frac{\lambda}{4\pi \varepsilon_0 R} \cos \theta \bigg|_0^{\pi/2} = -\frac{\lambda}{4\pi \varepsilon_0 R}
\]

and

\[
E_y = -\frac{\lambda}{4\pi \varepsilon_0 R} \int_0^{\pi/2} \cos \theta \, d\theta = -\frac{\lambda}{4\pi \varepsilon_0 R} \sin \theta \bigg|_0^{\pi/2} = -\frac{\lambda}{4\pi \varepsilon_0 R}.
\]

We notice that \( E_x = E_y \) no matter what the value of \( R \). Thus, \( \vec{E} \) makes an angle of \( 45^\circ \) with the rod for all values of \( R \).