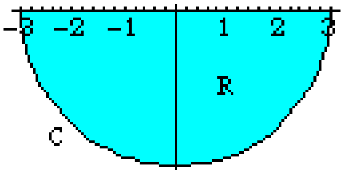
Math 17
Exam III
Solutions

1. Show that the vector field $\mathbf{F}(x,y) = xy^2\mathbf{i} + y\sin x\mathbf{j}$ is not conservative.

$\frac{\partial(y\sin x)}{\partial x} = y\cos x$ and $\frac{\partial(xy^2)}{\partial y} = 2xy$. Since $\frac{\partial(y\sin x)}{\partial x} \neq \frac{\partial(xy^2)}{\partial y}$, it follows that **F** is not conservative.

2. <u>Use</u> <u>Green's</u> <u>theorem</u> to evaluate $\int_C xy^2dx + (2x - \tan y)\,dy$ where C is the bottom half of the circle $x^2+y^2=9$ together with the straight-line segment from (3,0) to (-3,0), traversed in the counter-clockwise direction.

Let R be the region enclosed by C, as shown. Then,



$$\int_C xy^2\,dx + (2x - \tan y)dy =$$
$$\int\int_R \left(\frac{\partial(2x-\tan y)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right)dA =$$
$$\int\int_R (2-2xy)\,dA =$$
$$\int_{\theta=\pi}^{\theta=2\pi}\int_{r=0}^{r=3}(2r - 2r^3\cos\theta\sin\theta)\,dr\,d\theta =$$
$$\int_{\theta=\pi}^{\theta=2\pi}\left(r^2 - \frac{r^4\cos\theta\sin\theta}{2}\right)\Bigg|_{r=0}^{r=3} d\theta =$$
$$\int_{\theta=\pi}^{\theta=2\pi}\left(9 - \frac{81\cos\theta\sin\theta}{2}\right)d\theta =$$
$$\left(9\theta - \frac{81\sin^2\theta}{4}\right)\Bigg|_{\theta=\pi}^{\theta=2\pi} =$$
$$18\pi - 9\pi =$$
$$9\pi$$

3. Consider the portion of the sphere $x^2+y^2+z^2=16$ for which $y\geq 0$ and $z\geq 0$.

   a. Parameterize this surface using ϕ and θ (where these parameters are defined as in the context of spherical coordinates). Be sure to give the range of values of the parameters.

      $x = 4 \sin\phi \cos\theta$
      $y = 4 \sin\phi \sin\theta$
      $z = 4 \cos\phi$

      $0\leq\phi\leq\frac{\pi}{2}$
      $0\leq\theta\leq\pi$

   b. Parameterize this surface using r and θ (where these parameters are defined as in the context of polar coordinates). Be sure to give the range of values of the parameters.

      $x = r \cos\theta$
      $y = r \sin\theta$
      $z = \sqrt{16-r^2}$

      $0\leq r\leq 4$
      $0\leq\theta\leq\pi$

4. Consider the vector field $\mathbf{F}(x,y,z) = <2xy+\cos x, x^2-4y^3>$.

   a. Show that **F** is conservative by finding a potential function φ.

      We wish to find φ so that the following two equalities hold:

$$\frac{\partial\varphi}{\partial x} = 2xy+\cos x \qquad \frac{\partial\varphi}{\partial y} = x^2-4y^3$$

      The first of these statements implies that $\varphi = x^2y + \sin x + u(y)$.
      The second of these statements implies that $\varphi = x^2y - y^4 + v(x)$.

      We observe that $\varphi = x^2y + \sin x - y^4$ satisfies each of these requirements. Thus, φ is a potential function for **F**.

b. Compute $\int_C \mathbf{F} \bullet \mathbf{dr}$ where C is the curve from (π/2,1) to (π,2) shown on the board.

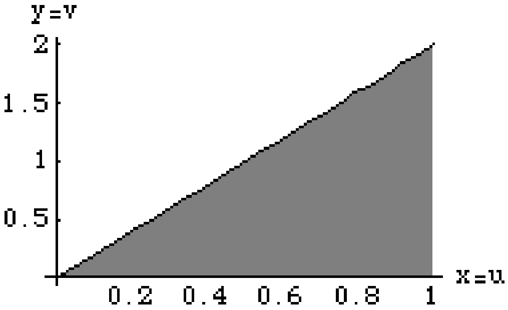
Since **F** is conservative, the given line integral is path independent and the line integral over any path from(π/2,1) to (π,2) is given by

$$\int_C \mathbf{F} \bullet \mathbf{dr} =$$
$$\varphi(\pi,2) - \varphi(\pi/2,1) =$$
$$\left(2\pi^2 - 16\right) - \left(\tfrac{\pi^2}{4} + 1 - 1\right) =$$
$$\tfrac{7\pi^2}{4} - 16$$

5. The surface σ is that portion of the plane 3x+4y+z = 20 that lies above the triangle in the xy plane with vertices (0,0), (1,0), and (1,2). Compute the mass of this surface if its density function is δ(x,y,z) = y+4z.

We parameterize the surface 3x+4y+z = 20 as follows:

x = u
y = v
z = 20 - 3u - 4v



The parameters u and v range over the region shown. Hence, their ranges are given by

$0 \le v \le 2u$
$0 \le u \le 1$

Then, **r**(u,v) = <u,v,20-3u-4v> and so

$$\frac{\partial \mathbf{r}}{\partial u} \text{x} \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{vmatrix} = \langle 3,4,1 \rangle \text{ and}$$

$$\left| \frac{\partial \mathbf{r}}{\partial u} \text{x} \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{9+16+1} = \sqrt{26}.$$

The density function is given by
δ = y + 4z = v + 80 - 12u - 16v = 80 - 12u - 15v.

Then,

Mass =

$$\int_{u=0}^{u=1}\int_{v=0}^{v=2u}\sqrt{26}(80-12u-15v)\,dv\,du =$$

$$\sqrt{26}\int_{u=0}^{u=1}\left(80v-12uv-\frac{15v^2}{2}\right)\Bigg|_{v=0}^{v=2u} du =$$

$$\sqrt{26}\int_{u=0}^{u=1}\left(160u-24u^2-30u^2\right)du =$$

$$\sqrt{26}\int_{u=0}^{u=1}\left(160u-54u^2\right)du =$$

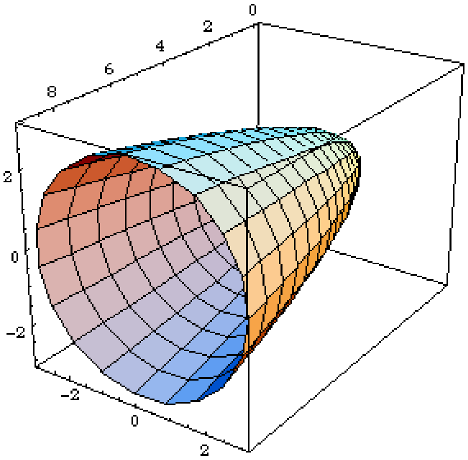
$$\sqrt{26}\left(80u^2-18u^3\right)\Big|_{u=0}^{u=1} =$$

$$\sqrt{26}(80-18)=$$

$$62\sqrt{26}$$

6. Find the flux of the vector field **F**(x,y,z) = <2,-z,0> across the portion of the paraboloid $x = y^2 + z^2$ between x=0 and x=9.

We parameterize the surface as follows:



$x = u^2$
$y = u \cos v$
$z = u \sin v$

$0 \le u \le 3$
$0 \le v \le 2\pi$

Then, $\mathbf{r}(u,v) = \langle u^2, u\cos v, u\sin v\rangle$ and so

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix} = \left\langle u, -2u^2\cos v, -2u^2\sin v \right\rangle \text{ and}$$

F = <2, -usinv, 0>.

Then,

$$\Phi =$$

$$\int \int_S (\mathbf{F} \bullet \mathbf{n})\, dS =$$

$$\int \int_R \left(\mathbf{F} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \, x \, \frac{\partial \mathbf{r}}{\partial v} \right) \right) dA =$$

$$\int_{u=0}^{u=3} \int_{v=0}^{v=2\pi} \left(2u + 2u^3 \sin v \cos v \right) dv\, du =$$

$$\int_{u=0}^{u=3} \left(2uv + u^3 \sin^2 v \right) \Big|_{v=0}^{v=2\pi} du =$$

$$\int_{u=0}^{u=3} (4\pi u)\, du =$$

$$2\pi u^2 \Big|_{u=0}^{u=3} =$$

$$18\pi$$