1. Fix $C$ and $\theta$ so that $\rho_{C,\theta} \neq \text{id}$, and let $m$ be a line.

**Prove:** $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$ if and only if $C$ is on $m$.

**Proof:** First, we suppose that $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$ and we must show that $C$ is on $m$.

Set $C' = \sigma_m(C)$. Then $C = \sigma_m(C')$ and hence $\sigma_m \rho_{C,\theta} \sigma_m (C') = \sigma_m \rho_{C,\theta} (C) = \sigma_m (C) = C'$. Since $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$, we know that $\sigma_m \rho_{C,\theta} \sigma_m$ is a rotation with fixed point $C$. But we have just shown that $C'$ is fixed by $\sigma_m \rho_{C,\theta} \sigma_m$. This implies that $C = C'$. Then, since $C' = \sigma_m(C)$, we know that $C$ is on $m$.

Next, we suppose that $C$ is on $m$ and we must show that $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$. Since $C$ is on $m$, we can find a line $n$ containing $C$ so that $\rho_{C,\theta} = \sigma_m \sigma_n$. This implies that $\rho_{C,-\theta} = \sigma_n \sigma_m$. Then, $\sigma_m \rho_{C,\theta} \sigma_m = \sigma_m \sigma_n \sigma_m \sigma_n = \sigma_n \sigma_m = \rho_{C,-\theta}$, as desired. \(\square\)

2. Suppose $\gamma_1$ and $\gamma_2$ are glide reflections with axes $m_1$ and $m_2$ respectively.

**Prove:**

a. If $m_1$ and $m_2$ are parallel, then $\gamma_1 \gamma_2$ is a translation.

b. If $m_1$ and $m_2$ are not parallel, then $\gamma_1 \gamma_2$ is a rotation.

**Proof:** Assume that $\gamma_1$ and $\gamma_2$ are glide reflections with axes $m_1$ and $m_2$ respectively.

a. Assume that $m_1$ and $m_2$ are parallel. We must show that $\gamma_1 \gamma_2$ is a translation. Suppose that $\gamma_1 = \sigma_{m_1} \sigma_b \sigma_a$ and $\gamma_2 = \sigma_{m_2} \sigma_q \sigma_p$, where lines $a$ and $b$ are perpendicular to $m_1$ and $p$ and $q$ are perpendicular to $m_2$. Then, $\gamma_1 \gamma_2 = \sigma_{m_1} \sigma_b \sigma_a \sigma_{m_2} \sigma_q \sigma_p$. Since $a$ and $b$ are both perpendicular to $m_1$, we know that $\sigma_{m_1}$ commutes with both $\sigma_a$ and $\sigma_b$. Hence, $\gamma_1 \gamma_2 = \sigma_{m_1} \sigma_b \sigma_a \sigma_{m_2} \sigma_q \sigma_p = \sigma_b \sigma_a \sigma_{m_1} \sigma_{m_2} \sigma_q \sigma_p$.

Next, we note that since $p$ and $q$ are parallel, $\sigma_q \sigma_p$ is a translation. Similarly, since $m_1$ and $m_2$ are parallel, $\sigma_{m_1} \sigma_{m_2}$ is a translation and, since $a$ and $b$ are parallel, $\sigma_a \sigma_b$ is a translation. Then, $\gamma_1 \gamma_2 = (\sigma_b \sigma_a)(\sigma_{m_1} \sigma_{m_2})(\sigma_q \sigma_p)$ is a product of three translations. We know that the product of translations is a translation. Hence, $\gamma_1 \gamma_2$ is a translation.

b. Assume that $m_1$ and $m_2$ are not parallel and let $C$ be their point of intersection. We must show that $\gamma_1 \gamma_2$ is a rotation. Suppose that $\gamma_1 = \sigma_{m_1} \sigma_b \sigma_a$ and $\gamma_2 = \sigma_{m_2} \sigma_q \sigma_p$, where lines $a$ and $b$ are perpendicular to $m_1$, and $p$ and $q$ are perpendicular to $m_2$. 

We can choose lines $a'$ and $b'$ that are perpendicular to $m_1$ so that $\sigma_{b'}\sigma_a = \sigma_{b}\sigma_{a'}$, and $a'$ contains $C$. Then, $\gamma_1 = \sigma_{m_1}\sigma_{b'}\sigma_a = \sigma_{m_1}\sigma_{b}\sigma_{a'}$. Since $m_1$ and $b'$ are perpendicular, $\sigma_{m_1}\sigma_{b'} = \sigma_{b}\sigma_{m_1}$, and hence $\gamma_1 = \sigma_{b}\sigma_{m_1}\sigma_{a'}$. And, since $m_1$ and $a'$ are perpendicular and intersect at $C$, $\sigma_{m_1}\sigma_{a'} = \sigma_{C}$. Hence, $\gamma_1 = \sigma_{b}\sigma_{C}$.

Next, we choose lines $p'$ and $q'$ which are perpendicular to $m_2$ so that $\sigma_{q}\sigma_{p} = \sigma_{q}\sigma_{p'}$ and $q'$ contains $C$. Then, $\gamma_2 = \sigma_{m_2}\sigma_{q}\sigma_{p} = \sigma_{m_2}\sigma_{q}\sigma_{p'}$ and, since $m_2$ and $q'$ are perpendicular and intersect at $C$, $\sigma_{m_2}\sigma_{q'} = \sigma_{C}$. Hence, $\gamma_2 = \sigma_{C}\sigma_{p'}$.

Then, $\gamma_1\gamma_2 = \sigma_{b}\sigma_{C}\sigma_{C}\sigma_{p'} = \sigma_{b}\sigma_{p'}$. Recall now that $b'$ and $p'$ are perpendicular to $m_1$ and $m_2$ respectively, and $m_1$ and $m_2$ are not parallel. Hence, $b'$ and $p'$ are not parallel. It follows that $\gamma_1\gamma_2 = \sigma_{b}\sigma_{p}$ is a rotation. \hfill \Box

3. **Prove:** If $\gamma$ is a glide reflection, then $\gamma = \sigma_{p}\sigma_{n}\sigma_{m}$, where any one of lines $m$, $n$, and $p$ can be any arbitrarily chosen line not parallel to the axis of $\gamma$.

Suppose $\gamma$ is a glide reflection. Then, for some lines $a$, $b$, and $c$, where $a$ and $b$ are both perpendicular to $c$, $\gamma = \sigma_{c}\sigma_{b}\sigma_{a}$. Also, $c$ is the axis of $\gamma$. We must show that $\gamma = \sigma_{p}\sigma_{n}\sigma_{m}$ where any one of $m$, $n$, $p$ can be any arbitrarily chosen line not parallel to $c$.

First, let $m$ be any line not parallel to $c$. We must find $n$ and $p$ such that $\gamma = \sigma_{p}\sigma_{n}\sigma_{m}$. Let $P$ be the point of intersection of $m$ and $c$. Let $a'$ be the line perpendicular to $c$ that contains $P$, and let $b'$ be such that $\sigma_{b'}\sigma_{a} = \sigma_{b}\sigma_{a'}$. Then we have $\gamma = \sigma_{c}\sigma_{b}\sigma_{a'}$. Since $m$, $c$, and $a'$ are concurrent at $P$, we can find a line $n$ containing $P$ such that $\sigma_{n}\sigma_{m} = \sigma_{c}\sigma_{a'}$. We next note that since $b'$ and $c$ are perpendicular, $\sigma_{c}\sigma_{b'} = \sigma_{b'}\sigma_{c}$. Then, setting $p=b'$ we have $\gamma = \sigma_{c}\sigma_{b}\sigma_{a} = \sigma_{c}\sigma_{b}\sigma_{a'} = \sigma_{b'}\sigma_{c}\sigma_{a'} = \sigma_{p}\sigma_{n}\sigma_{m}$, as desired.

Next, let $n$ be any line not parallel to $c$. We must find $m$ and $p$ such that $\gamma = \sigma_{p}\sigma_{n}\sigma_{m}$. Let $P$ be the point of intersection of $n$ and $c$. Let $b'$ be the line perpendicular to $c$ containing $P$, and let $a'$ be such that $\sigma_{b'}\sigma_{a} = \sigma_{b}\sigma_{a'}$. Then we have $\gamma = \sigma_{c}\sigma_{b}\sigma_{a'}$. Since $n$, $c$, and $b'$ are concurrent at $P$, we can find a line $p$ containing $P$ such that $\sigma_{p}\sigma_{n} = \sigma_{c}\sigma_{b'}$. Then, setting $m=a'$, we have $\gamma = \sigma_{c}\sigma_{b}\sigma_{a} = \sigma_{c}\sigma_{b}\sigma_{a'} = \sigma_{p}\sigma_{n}\sigma_{m}$, as desired.

Finally, let $p$ be any line not parallel to $c$. We must find $m$ and $n$ such that $\gamma = \sigma_{p}\sigma_{n}\sigma_{m}$. Let $P$ be the point of intersection of $p$ and $c$ and, as above, let $b'$ be the line perpendicular to $c$ containing $P$. We can find a line $a'$ such that $\sigma_{b'}\sigma_{a} = \sigma_{b}\sigma_{a'}$. Then we have $\gamma = \sigma_{c}\sigma_{b}\sigma_{a'}$. Next, since $p$, $c$, and $b'$ are concurrent at $P$, we can find a line $n$ containing $P$ such that $\sigma_{n}\sigma_{p} = \sigma_{c}\sigma_{b'}$. Then, setting $m=a'$, we have $\gamma = \sigma_{c}\sigma_{b}\sigma_{a} = \sigma_{c}\sigma_{b}\sigma_{a'} = \sigma_{p}\sigma_{n}\sigma_{m}$, as desired. \hfill \Box
Extra Credit:  **Prove:** The above statement also holds if the chosen line is parallel to the axis of $\gamma$.

As in problem 3, we suppose that $\gamma$ is a glide reflection and $\gamma = \sigma_c \sigma_b \sigma_a$, where $a$ and $b$ are both perpendicular to $c$. Then $c$ is the axis of $\gamma$. We must show that $\gamma = \sigma_p \sigma_n \sigma_m$ where any one of $m$, $n$, $p$ can be any arbitrarily chosen line parallel to $c$.

Let $p$ be any line parallel to $c$. We must find $m$ and $n$ such that $\gamma = \sigma_p \sigma_n \sigma_m$. We note that $a$ and $b$ are perpendicular to $p$. Let $P$ be the point of intersection of $a$ and $c$, and let $Q$ be the point of intersection of $b$ and $p$. Next, let $a'$ be the line containing $P$ and $Q$. Then, since $a$, $c$, and $a'$ are concurrent at $P$, there is a line $m$ such that $\sigma_a \sigma_c = \sigma_a \sigma_m$. Also, since $a$ and $b$ are each perpendicular to $c$, it follows that $\sigma_b \sigma_c = \sigma_c \sigma_b$ and $\sigma_a \sigma_c = \sigma_c \sigma_a$. Hence, we have $\gamma = \sigma_c \sigma_b \sigma_a = \sigma_b \sigma_c \sigma_a = \sigma_b \sigma_a \sigma_c = \sigma_b \sigma_a \sigma_m$. Next, since $b$, $a'$, and $p$ are concurrent at $Q$, there is a line $n$ such that $\sigma_b \sigma_a' = \sigma_p \sigma_n$. Hence, we have $\gamma = \sigma_b \sigma_a' \sigma_m = \sigma_p \sigma_n \sigma_m$, as desired.

The proof for $m$ and $n$ is similar. \qed