Math 124
Problem Set IV
Solutions

1. Fix C and $\theta$ so that $\rho_{C,\theta} \neq i$, and let m be a line.
Prove: $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$ if and only if C is on m.

Proof: First, we suppose that $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$ and we must show that C is on m.

Set C' = $\sigma_m$(C). Then C = $\sigma_m$(C') and hence $\sigma_m \rho_{C,\theta} \sigma_m$ (C') = $\sigma_m \rho_{C,\theta}$ (C) = $\sigma_m$ (C) = C'. Since $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$, we know that $\sigma_m \rho_{C,\theta} \sigma_m$ is a rotation with fixed point C. But we have just shown that C' is fixed by $\sigma_m \rho_{C,\theta} \sigma_m$. This implies that C = C'. Then, since C' = $\sigma_m$(C), we know that C is on m.

Next, we suppose that C is on m and we must show that $\sigma_m \rho_{C,\theta} \sigma_m = \rho_{C,-\theta}$. Since C is on m, we can find a line n containing C so that $\rho_{C,\theta} = \sigma_m \sigma_n$. This implies that $\rho_{C,-\theta} = \sigma_n \sigma_m$. Then, $\sigma_m \rho_{C,\theta} \sigma_m = \sigma_m \sigma_m \sigma_n \sigma_m = \sigma_n \sigma_m = \rho_{C,-\theta}$, as desired. □

2. Suppose $\gamma_1$ and $\gamma_2$ are glide reflections with axes $m_1$ and $m_2$ respectively.
Prove:

a. If $m_1$ and $m_2$ are parallel, then $\gamma_1\gamma_2$ is a translation.
b. If $m_1$ and $m_2$ are not parallel, then $\gamma_1\gamma_2$ is a rotation.

Proof: Assume that $\gamma_1$ and $\gamma_2$ are glide reflections with axes $m_1$ and $m_2$ respectively.

a. Assume that $m_1$ and $m_2$ are parallel. We must show that $\gamma_1\gamma_2$ is a translation. Suppose that $\gamma_1 = \sigma_{m_1}\sigma_b\sigma_a$ and $\gamma_2 = \sigma_{m_2}\sigma_q\sigma_p$, where lines a and b are perpendicular to $m_1$, and p and q are perpendicular to $m_2$. Then, $\gamma_1\gamma_2 = \sigma_{m_1}\sigma_b\sigma_a\sigma_{m_2}\sigma_q\sigma_p$. Since a and b are both perpendicular to $m_1$, we know that $\sigma_{m_1}$ commutes with both $\sigma_a$ and $\sigma_b$. Hence, $\gamma_1\gamma_2 = \sigma_{m_1}\sigma_b\sigma_a\sigma_{m_2}\sigma_q\sigma_p = \sigma_b\sigma_a\sigma_{m_1}\sigma_{m_2}\sigma_q\sigma_p$.

Next, we note that since p and q are parallel, $\sigma_q\sigma_p$ is a translation. Similarly, since $m_1$ and $m_2$ are parallel, $\sigma_{m_1}\sigma_{m_2}$ is a translation and, since a and b are parallel, $\sigma_b\sigma_a$ is a translation. Then, $\gamma_1\gamma_2 = (\sigma_b\sigma_a)(\sigma_{m_1}\sigma_{m_2})(\sigma_q\sigma_p)$ is a product of three translations. We know that the product of translations is a translation. Hence, $\gamma_1\gamma_2$ is a translation.

b. Assume that $m_1$ and $m_2$ are not parallel and let C be their point of intersection. We must show that $\gamma_1\gamma_2$ is a rotation. Suppose that $\gamma_1 = \sigma_{m_1}\sigma_b\sigma_a$ and $\gamma_2 = \sigma_{m_2}\sigma_q\sigma_p$, where lines a and b are perpendicular to $m_1$, and p and q are perpendicular to $m_2$.

We can choose lines a' and b' that are perpendicular to $m_1$ so that $\sigma_b\sigma_a = \sigma_{b'}\sigma_{a'}$ and a' contains C. Then, $\gamma_1 = \sigma_{m_1}\sigma_b\sigma_a = \sigma_{m_1}\sigma_{b'}\sigma_{a'}$. Since $m_1$ and b' are perpendicular, $\sigma_{m_1}\sigma_{b'} = \sigma_{b'}\sigma_{m_1}$, and hence $\gamma_1 = \sigma_{b'}\sigma_{m_1}\sigma_{a'}$. And, since $m_1$ and a' are perpendicular and intersect at C, $\sigma_{m_1}\sigma_{a'} = \sigma_C$. Hence, $\gamma_1 = \sigma_{b'}\sigma_C$.

Next, we choose lines p' and q' which are perpendicular to $m_2$ so that $\sigma_q\sigma_p = \sigma_{q'}\sigma_{p'}$ and q' contains C. Then, $\gamma_2 = \sigma_{m_2}\sigma_q\sigma_p = \sigma_{m_2}\sigma_{q'}\sigma_{p'}$ and, since $m_2$ and q' are perpendicular and intersect at C, $\sigma_{m_2}\sigma_{q'} = \sigma_C$. Hence, $\gamma_2 = \sigma_C\sigma_{p'}$.

Then, $\gamma_1\gamma_2 = \sigma_{b'}\sigma_C\sigma_C\sigma_{p'} = \sigma_{b'}\sigma_{p'}$. Recall now that b' and p' are perpendicular to $m_1$ and $m_2$ respectively, and $m_1$ and $m_2$ are not parallel. Hence, b' and p' are not parallel. It follows that $\gamma_1\gamma_2 = \sigma_{b'}\sigma_{p'}$ is a rotation. □

3. <u>Prove</u>: If $\gamma$ is a glide reflection, then $\gamma = \sigma_p\sigma_n\sigma_m$, where any one of lines m, n, and p can be any arbitrarily chosen line not parallel to the axis of $\gamma$.

Suppose $\gamma$ is a glide reflection. Then, for some lines a, b, and c, where a and b are both perpendicular to c, $\gamma = \sigma_c\sigma_b\sigma_a$. Also, c is the axis of $\gamma$. We must show that $\gamma = \sigma_p\sigma_n\sigma_m$ where any one of m, n, p can be any arbitrarily chosen line not parallel to c.

First, let m be any line not parallel to c. We must find n and p such that $\gamma = \sigma_p\sigma_n\sigma_m$. Let P be the point of intersection of m and c. Let a' be the line perpendicular to c that contains P, and let b' be such that $\sigma_b\sigma_a = \sigma_{b'}\sigma_{a'}$. Then we have $\gamma = \sigma_c\sigma_{b'}\sigma_{a'}$. Since m, c, and a' are concurrent at P, we can find a line n containing P such that $\sigma_n\sigma_m = \sigma_c\sigma_{a'}$. We next note that since b' and c are perpendicular, $\sigma_c\sigma_{b'} = \sigma_{b'}\sigma_c$. Then, setting p=b' we have $\gamma = \sigma_c\sigma_b\sigma_a = \sigma_c\sigma_{b'}\sigma_{a'} = \sigma_{b'}\sigma_c\sigma_{a'} = \sigma_p\sigma_n\sigma_m$, as desired.

Next, let n be any line not parallel to c. We must find m and p such that $\gamma = \sigma_p\sigma_n\sigma_m$. Let P be the point of intersection of n and c. Let b' be the line perpendicular to c containing P, and let a' be such that $\sigma_b\sigma_a = \sigma_{b'}\sigma_{a'}$. Then we have $\gamma = \sigma_c\sigma_{b'}\sigma_{a'}$. Since n, c, and b' are concurrent at P, we can find a line p containing P such that $\sigma_p\sigma_n = \sigma_c\sigma_{b'}$. Then, setting m=a', we have $\gamma = \sigma_c\sigma_b\sigma_a = \sigma_c\sigma_{b'}\sigma_{a'} = \sigma_p\sigma_n\sigma_m$, as desired.

Finally, let p be any line not parallel to c. We must find m and n such that $\gamma = \sigma_p\sigma_n\sigma_m$. Let P be the point of intersection of p and c and, as above, let b' be the line perpendicular to c containing P. We can find a line a' such that $\sigma_b\sigma_a = \sigma_{b'}\sigma_{a'}$. Then we have $\gamma = \sigma_c\sigma_{b'}\sigma_{a'}$. Next, since p, c, and b' are concurrent at P, we can find a line n containing P such that $\sigma_p\sigma_n = \sigma_c\sigma_{b'}$. Then, setting m=a', we have $\gamma = \sigma_c\sigma_b\sigma_a = \sigma_c\sigma_{b'}\sigma_{a'} = \sigma_p\sigma_n\sigma_m$, as desired. □

Extra Credit: Prove: The above statement also holds if the chosen line is parallel to the axis of $\gamma$.

As in problem 3, we suppose that $\gamma$ is a glide reflection and $\gamma = \sigma_c\sigma_b\sigma_a$, where a and b are both perpendicular to c. Then c is the axis of $\gamma$. We must show that $\gamma = \sigma_p\sigma_n\sigma_m$ where any one of m, n, p can be any arbitrarily chosen line parallel to c.

Let p be any line parallel to c. We must find m and n such that $\gamma = \sigma_p\sigma_n\sigma_m$. We note that a and b are perpendicular to p. Let P be the point of intersection of a and c, and let Q be the point of intersection of b and p. Next, we let a' be the line containing P and Q. Then, since a, c, and a' are concurrent at P, there is a line m such that $\sigma_a\sigma_c = \sigma_{a'}\sigma_m$. Also, since a and b are each perpendicular to c, it follows that $\sigma_b\sigma_c = \sigma_c\sigma_b$ and $\sigma_a\sigma_c = \sigma_c\sigma_a$. Hence, we have $\gamma = \sigma_c\sigma_b\sigma_a = \sigma_b\sigma_c\sigma_a = \sigma_b\sigma_a\sigma_c = \sigma_b\sigma_{a'}\sigma_m$. Next, since b, a', and p are concurrent at Q, there is a line n such that $\sigma_b\sigma_{a'} = \sigma_p\sigma_n$. Hence, we have $\gamma = \sigma_b\sigma_{a'}\sigma_m = \sigma_p\sigma_n\sigma_m$, as desired.

The proof for m and n is similar. □