1. **Prove**: For any two distinct lines \( m \) and \( n \) and any point \( P \), \( \sigma_m \sigma_n \) fixes \( P \) if and only if \( P \) is on both \( m \) and \( n \).

**Proof**: Fix distinct lines \( m \) and \( n \) and point \( P \). We assume first that \( \sigma_m \sigma_n \) fixes \( P \). In other words, \( \sigma_m \sigma_n (P) = P \). We must show that \( P \) is on both \( m \) and \( n \).

Suppose, by way of contradiction, that \( P \) is not on \( n \). Then, \( P \neq \sigma_n(P) \) and \( n \) is the perpendicular bisector of \( P\sigma_n(P) \). By assumption, \( \sigma_m \sigma_n(P) = P \). This implies that \( m \) is the perpendicular bisector of \( P\sigma_n(P) \). But since \( n \) and \( m \) are both perpendicular bisectors of \( P\sigma_n(P) \), it follows that \( m = n \). This contradicts our assumption that \( m \) and \( n \) are distinct. Hence, \( P \) is on \( n \).

Then, we know that \( \sigma_n(P) = P \) and hence, since \( \sigma_m \sigma_n(P) = P \), we know that \( \sigma_m(P) = P \). This implies that \( P \) is on \( m \). Thus, we have established that \( P \) is on both \( m \) and \( n \).

Next, we assume that \( P \) is on both \( m \) and \( n \). We must show that \( \sigma_m \sigma_n \) fixes \( P \). Since \( P \) is on \( n \), it follows that \( \sigma_n(P) = P \). Since \( P \) is on \( m \), it follows that \( \sigma_m(P) = P \). Hence, we have shown that \( \sigma_m \sigma_n(P) = \sigma_m(P) = P \), as desired.

A slightly different proof of the left-to-right direction:

We assume that \( \sigma_m \sigma_n(P) = P \). This implies that \( \sigma_n(P) = \sigma_m(P) \). Call this point \( P' \). We wish to show that \( P \) is on both \( m \) and \( n \). Clearly, this is equivalent to showing that \( P \) is fixed by both \( \sigma_m \) and \( \sigma_n \). Thus, we must show that \( P = P' \). Suppose, by way of contradiction that \( P \neq P' \). Then, since \( \sigma_n(P) = P' \) and \( \sigma_m(P) = P' \), we know that \( m \) and \( n \) are each perpendicular bisectors of the segment \( PP' \). This implies that \( m = n \) and contradicts our assumption that \( m \) and \( n \) are distinct. We conclude that \( P \) is on both \( m \) and \( n \). 

\( \square \)
2. Suppose that \( A = (-1,0), \ B = (-4,3), \ C = (3,-2), \) and \( D = (5,12). \) Find equations of lines such that the product of reflections in these lines sends ray \( \overrightarrow{CD} \) to ray \( \overrightarrow{AB}. \) Clearly explain your work.

First, we look for a line \( n \) so that \( \sigma_n(C) = A. \) Line \( n \) is the perpendicular bisector of segment \( \overline{AC}. \) The midpoint of this segment is \( \left( \frac{-1+3}{2}, \frac{0-2}{2} \right) = (1,-1). \) The slope of segment \( \overline{AC} \) is \( \frac{-2-0}{3+1} = -\frac{1}{2}. \) Hence, the slope of the perpendicular bisector is 2. Then, we can find an equation for the perpendicular bisector \( n \) as follows:

\[
\begin{align*}
y + 1 &= 2(x - 1) \\
y + 1 &= 2x - 2 \\
2x - y - 3 &= 0
\end{align*}
\]

Then, the equations for \( \sigma_n \) are as follows:

\[
\begin{align*}
x' &= x - \frac{4(2x - y - 3)}{5} \\
y' &= y + \frac{2(2x - y - 3)}{5}
\end{align*}
\]

We wish to determine \( \sigma_n(D). \)

\[
\sigma_n(D) = \\
\sigma_n(5,12) = \\
\left( 5 - \frac{4(10 - 12 - 3)}{5}, 12 + \frac{2(10 - 12 - 3)}{5} \right) = (9,10)
\]

We need to determine a line \( m \) so that \( \sigma_m \sigma_n(\overrightarrow{CD}) = \overrightarrow{AB}. \) We know that \( \sigma_n(C) = A. \) We need only find \( m \) so that \( m \) is the angle bisector of the angle between \( \overrightarrow{AB} \) and \( \sigma_n(\overrightarrow{CD}). \)

Note that

\[
\text{slope of } \overrightarrow{AB} = \frac{3-0}{-4+1} = -1 \text{ and } \\
\text{slope of } \sigma_n(\overrightarrow{CD}) = \\
\text{slope between } \sigma_n(\overrightarrow{C}) \text{ and } \sigma_n(\overrightarrow{D}) = \\
\text{slope between } (-1,0) \text{ and } (9,10) = 1
\]
Then, since $\overrightarrow{AB}$ has slope $-1$ and $\sigma_n(\overrightarrow{CD})$ has slope $1$, it is not hard to see that the angle bisector of the angle between $\overrightarrow{AB}$ and $\sigma_n(\overrightarrow{CD})$ is the vertical line $x=-1$.

Thus, we have shown that if $n$ is the line $2x-y-3=0$ and $m$ is the line $x=-1$, then $\sigma_m\sigma_n(\overrightarrow{CD}) = \overrightarrow{AB}$.

3. **Prove:** If $\sigma_n\sigma_m\sigma_l$ is a reflection, then lines $l$, $m$, and $n$ are concurrent or parallel.

**Proof:** Suppose $\sigma_n\sigma_m\sigma_l = \sigma_p$, for some line $p$. Then, $\sigma_n\sigma_m = \sigma_p\sigma_l$. We consider the following two cases:

**Case I:** $m$ and $n$ are parallel. Then $\sigma_n\sigma_m$ is a translation. It follows that $\sigma_p\sigma_l$ is the same translation. Then $m$, $n$, $l$, and $p$ are all perpendicular to the direction of this translation. Hence $l$, $m$, $n$, and $p$ are parallel.

**Case II:** $m$ and $n$ are not parallel. Suppose $C$ is the point of intersection of $m$ and $n$. Then $\sigma_n\sigma_m$ is a rotation with center $C$, and therefore $\sigma_p\sigma_l$ is a rotation with center $C$. Hence, $l$, $m$, $n$, and $p$ all contain the point $C$.

We have established that $l$, $m$, and $n$ are concurrent or parallel, as desired. $\Box$