Math 124
Exam 2
Solutions

1. Complete each of the following:

a. The product of three reflections is a reflection if and only if the lines about which we are reflecting are parallel or concurrent.

b. α is a dilation if and only if it is a stretch about some point C, or else a stretch about some point C followed by a halfturn about C.

c. α is a similarity of ratio 3 if and only if for all points P and Q, P'Q' = 3PQ

2. Give an explanation involving fixed points or fixed lines for why each of the following statements is false:

a. There is an isometry that is both a (non-identity) translation and a glide reflection.

A (non-identity) translation has infinitely many fixed lines (all lines parallel to the direction of the translation), but a glide reflection has only one fixed line (its axis).

b. There is an isometry that is both a (non-identity) rotation and a reflection.

A (non-identity) rotation has exactly one fixed point (its center), but a reflection has infinitely many fixed points (all points on the line of reflection).

3. Which isometries are dilatations? Justify your answer.

By the classification theorem for isometries, we need only consider reflections, translations, rotations, and glide reflections. No reflection is a dilatation. Every translation is a dilatation. A rotation is a dilatation if and only if it is a halfturn. No glide reflection is a dilatation. Thus, an isometry is a dilatation if and only if it is a translation or a halfturn.

4. Each of the following is a description of an isometry. In each case determine, if possible, whether the given isometry is a reflection, a rotation, a translation, or a glide reflection. In some cases, the given information may not be enough to allow you to make a precise determination, and your answer may be of the form "either A or B". You are not being asked to justify your answer.

a. The product of 2 rotations.

Either a rotation or a translation

b. The product of 1001 translations.

Translation

c. A glide reflection followed by a reflection about some line not parallel to the axis of the glide reflection.

Rotation

d. $\alpha^{2001}\beta^{300}$ where α is an odd isometry and β is an even isometry.

Either a reflection or a glide reflection

e. An odd isometry with no fixed points.

Glide reflection

f. An even isometry with no fixed points.

Translation

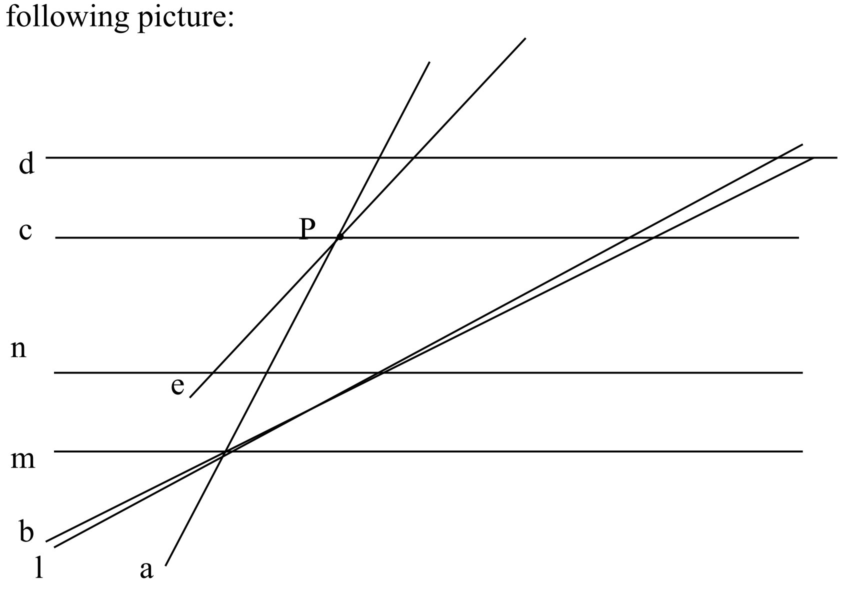
g. A non-identity isometry with a line that is fixed pointwise.

Reflection

h. $\rho_{C,\theta}\sigma_n\gamma\sigma_m\tau_{P,Q}\sigma_R$, where C, P, Q, and R are points and m and n are lines. (Note: γ is a glide reflection.)

Either a reflection or a glide reflection

5. Consider the following picture:



Draw lines a, b, c, d , and e such that a, c, and e each contain point P, $\sigma_m\sigma_l=\sigma_b\sigma_a$, $\sigma_n\sigma_m=\sigma_d\sigma_c$, and $\sigma_e\sigma_c=\rho_{P,90^0}$

6. <u>Prove</u>: If α is an non-identity isometry, then α has either no fixed points, exactly one fixed point, or the set of fixed points of α is a line. (Hint: Use the Classification Theorem)

Proof: We assume that α is a non-identity isometry. We must show that α has either no fixed points, exactly one fixed point, or the set of fixed points of α is a line. By the classification theorem, α is either a reflection, a translation, a rotation, or a glide reflection. We consider each of these cases.

Case 1: α is a reflection. Then, the set of all fixed points of α is a line.

Case 2: α is a translation. Then, α has no fixed points.

Case 3: α is a rotation. Then, α has exactly one fixed point.

Case 4: α is a glide reflection. Then, α has no fixed points.

We have shown that, in each case, α has either no fixed points, exactly one fixed point, or the set of fixed points of α is a line. □

7. Prove: If γ is a glide reflection, then γ^2 is a translation.

Proof: We may assume that $\gamma = \sigma_c\sigma_b\sigma_a$, where lines a and b are parallel and line c is perpendicular to a and to b. Then, $\sigma_c\sigma_b = \sigma_b\sigma_c$ and $\sigma_c\sigma_a = \sigma_a\sigma_c$. Hence, we have $\gamma^2 = \sigma_c\sigma_b\sigma_a\sigma_c\sigma_b\sigma_a = \sigma_b\sigma_c\sigma_a\sigma_c\sigma_b\sigma_a = \sigma_b\sigma_a\sigma_c\sigma_c\sigma_b\sigma_a = \sigma_b\sigma_a\sigma_b\sigma_a$.

Since a and b are parallel, $\sigma_b\sigma_a$ is a translation. Hence, γ^2 is the composition of two translations and is therefore a translation. □

8. Assume: If an isometry fixes at least one point, then it is the product of at most two reflections.

Prove: Every isometry is the product of at most three reflections. Do not use the Classification Theorem.

Proof: Let α be an isometry. We must show that α is the product of at most three reflections. Pick any point P. We consider two cases:

Case 1: P is fixed by α. Then, by our assumption, α is the product of at most two reflections, and so it is obviously the product of at most three reflections.

Case2: P is not fixed by α. Let $P' = \alpha(P)$, and let m be the perpendicular bisector of $\overline{PP'}$. Set $\beta = \sigma_m\alpha$. Then, $\beta(P) = \sigma_m\alpha(P) = \sigma_m(P') = P$. So, P is a fixed point of β. By our assumption, β is the product of at most two reflections. Hence, since $\alpha = \sigma_m\beta$, α is the product of at most three reflections. □