Voting with Pulleys and Rubber Bands

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Multicandidate voting

3 or more candidates run for office

Multicandidate voting: Set-up

A group must select one option from among several** alternatives:

- Candidates for president:
  - **J**ohn McCain
  - **B**arack Obama
  - **R**on Paul
- What to order for lunch: **P**astrami, **Q**abbage, **R**abbit, **S**alami

** <u>"several" means ≥ 3</u>

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- ♦ What to order for lunch: **P**astrami, **Q**abbage, **R**abbit, **S**alami

** “several” means ≥ 3

General Assumptions:

- ☛ Voters are treated equally
- ☛ More than 2 possible outcomes
- ☛ All possible outcomes are treated equally (no built-in bias favors one candidate)

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We will consider ballots that reveal each voter's full ***preference ranking***.
. . . used in some other countries.

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♦ Candidates for president: **J**ohn McCain, **B**arack Obama, **R**on Paul

Mei-Ling
R
B
J

Multicandidate voting: Examples

1) <u>**Borda Count**</u> Jean Charles de Borda (French Revolution)

- Each voter awards points to the candidates: <u>Ahmed</u>

Q	3 points
P	2 points
S	1 point
R	0 points

- For each alternative, sum the points awarded by all voters

- The winner is the alternative with the most points

Multicandidate voting: Examples

1) <u>**Borda Count**</u> Jean Charles de Borda (French Revolution)

Sample Profile:

3	1	1	2
p	q	r	s
q	s	s	q
r	r	q	r
s	p	p	p

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Sample Profile:

3	1	1	2
p	q	r	s
q	s	s	q
r	r	q	r
s	***p***	***p***	***p***

p's total points:

__ × 3 = __

__ × 2 = __

__ × 1 = __

__ × 0 = __

SUM = —

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p	q	r	s
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r	r	q	r
s	***p***	***p***	***p***

p's total points:

$\underline{3} \times 3 = \underline{9}$

$\underline{0} \times 2 = \underline{0}$

$\underline{0} \times 1 = \underline{0}$

$\underline{4} \times 0 = \underline{0}$

SUM = 9

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1) <u>**Borda Count**</u> Jean Charles de Borda (French Revolution)

Sample Profile:

<u>3</u>	<u>1</u>	<u>1</u>	<u>2</u>
p	*q*	*r*	*s*
q	*s*	*s*	*q*
r	*r*	*q*	*r*
s	*p*	*p*	*p*

q's points:	*r*'s points:	*s*'s points:
<u>1</u> × 3 = <u>3</u>	<u>1</u> × 3 = <u>3</u>	<u>2</u> × 3 = <u>6</u>
<u>5</u> × 2 = <u>10</u>	<u>0</u> × 2 = <u>0</u>	<u>2</u> × 2 = <u>4</u>
<u>1</u> × 1 = <u>1</u>	<u>6</u> × 1 = <u>6</u>	<u>0</u> × 1 = <u>0</u>
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SUM = 14	SUM = 9	SUM = 10
(p had 9 total)		

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r	*r*	*q*	*r*
s	*p*	*p*	*p*

q's points: $\underline{1} \times 3 = \underline{3}$
$\underline{5} \times 2 = \underline{10}$
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SUM = 14
(p had 9 total)

r's points: $\underline{1} \times 3 = \underline{3}$
$\underline{0} \times 2 = \underline{0}$
$\underline{6} \times 1 = \underline{6}$
$\underline{0} \times 0 = \underline{0}$

SUM = 9

s's points: $\underline{2} \times 3 = \underline{6}$
$\underline{2} \times 2 = \underline{4}$
$\underline{0} \times 1 = \underline{0}$
$\underline{3} \times 0 = \underline{0}$

SUM = 10

Borda winner is *q*

Multicandidate voting: Examples

2) <u>Hare</u> <u>Step 1</u> Is some alternative the 1ST choice of a majority of voters?

If so, they win. If not go to step 2.

<u>Step 2</u> Eliminate the alternative(s) having the fewest 1ST choice votes.

<u>Step 3</u> "Squeeze up" to close the gaps left by the eliminations. Then, go to step 1.

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p has a ***plurality*** of 1ST choice votes: 3 of 7. But no alternative has a ***majority***.
Proceed to step 2.

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Alternative s gets 4 of the 1^{ST} place votes – a majority of the 7 votes cast.

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Hare winner is s

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Plurality winner is p

Same election: 3 different voting rules ⇒ 3 different winners

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Does the choice of voting rule really make a difference?

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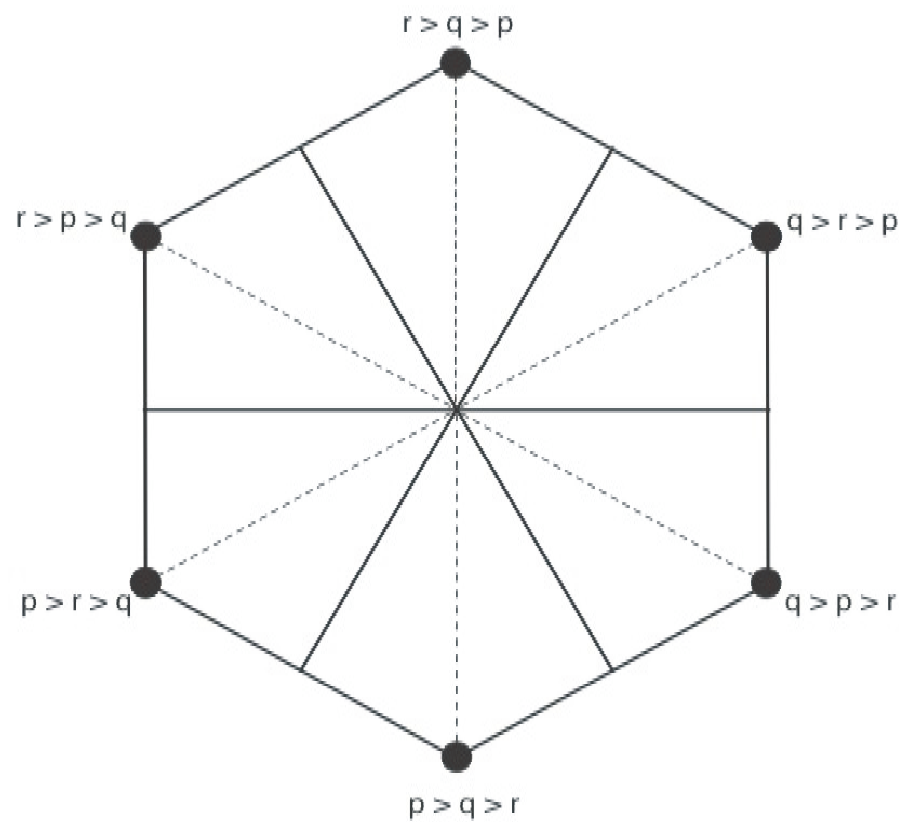
Using Hare? Almost certainly, **Gore.**

Hex-Mean voting rule

- Three alternatives: p, q, r
- 6 possible rankings:
  - p > q > r
  - p > r > q
  - q > p > r
  - q > r > p
  - r > p > q
  - r > q > p
- Label each hex vertex with a ranking, as in the sketch
- What is the labeling pattern?



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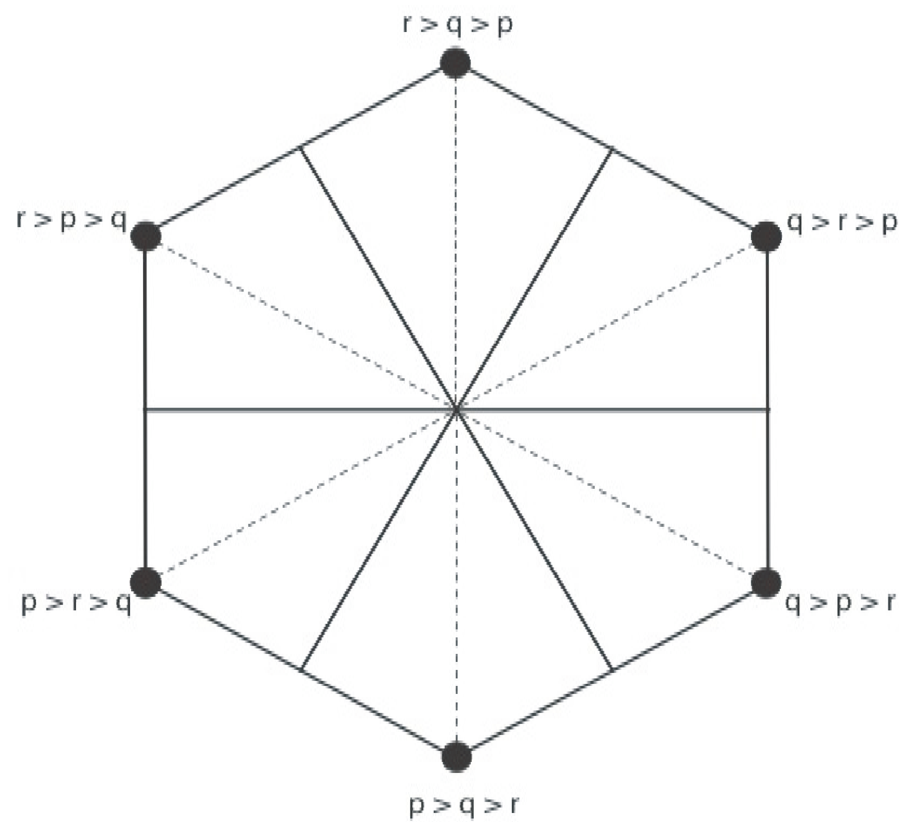
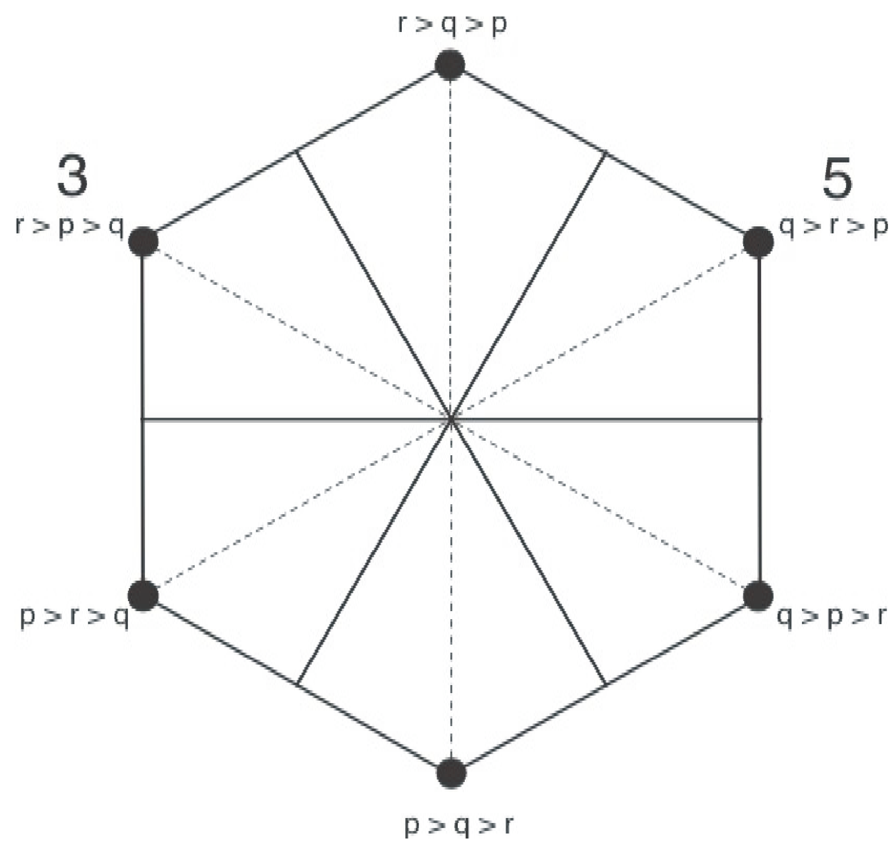
- What is the labeling pattern?
- Adjacent rankings differ by one pairwise reversal



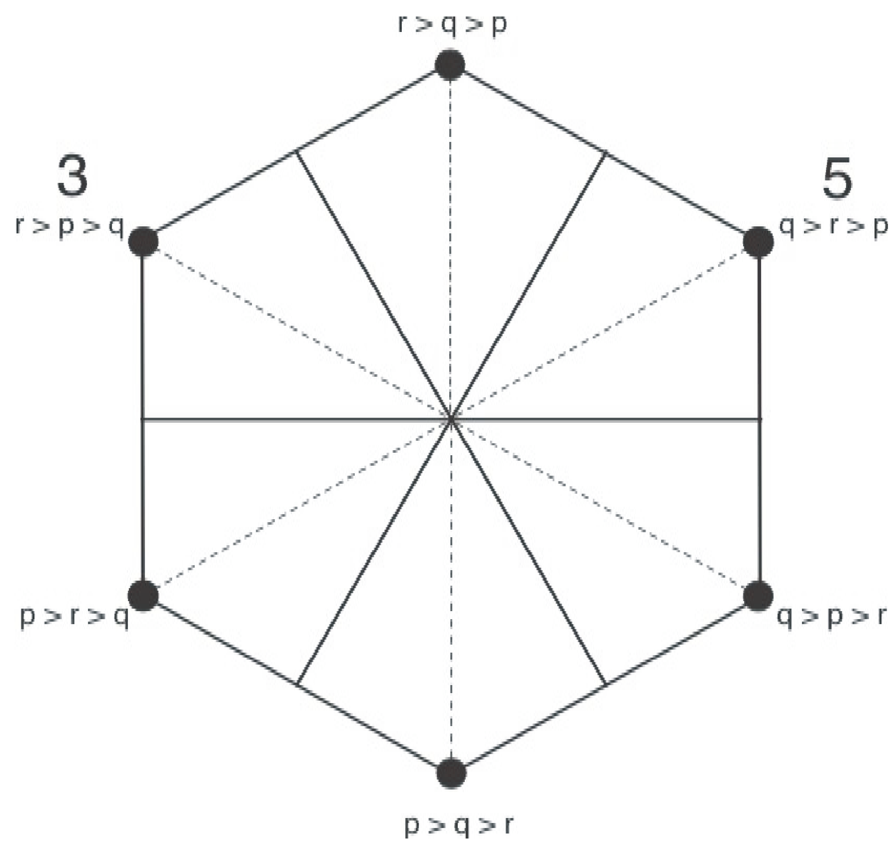
Hex-Mean voting rule

- Each voter chooses a vertex



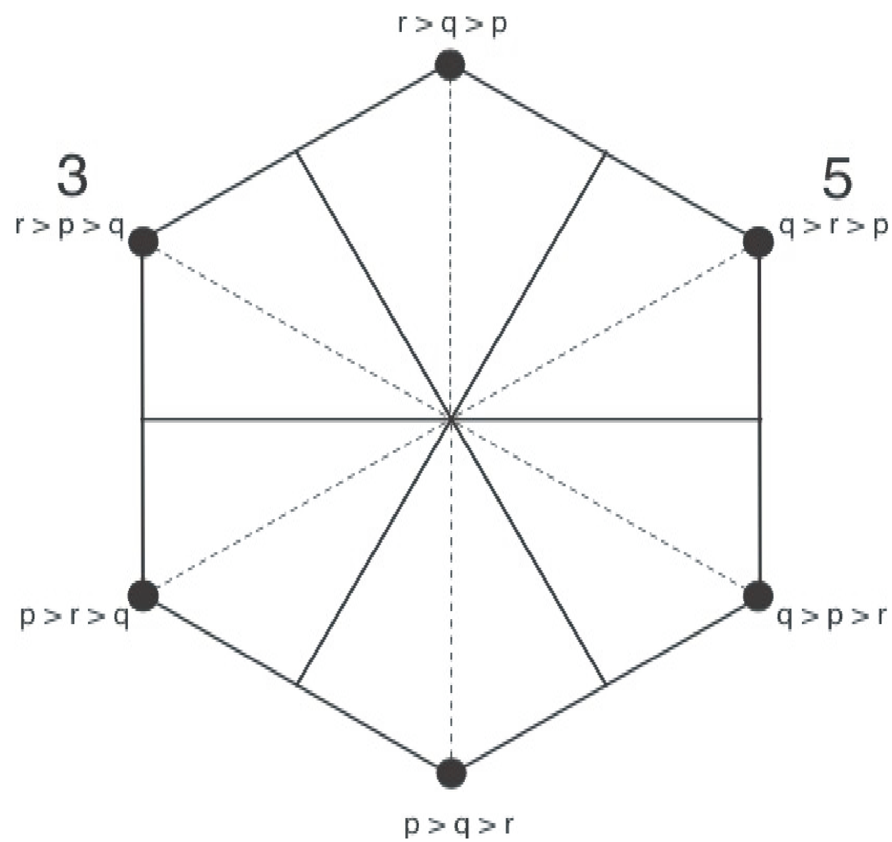
Hex-Mean voting rule

- Each voter chooses a vertex
- **○** = mean location of all votes



Hex-Mean voting rule

- Each voter chooses a vertex
- **O** = mean location of all votes
- How do we find the “mean” of points in the plane? We’ll come back to that.
- Where is **O** ?



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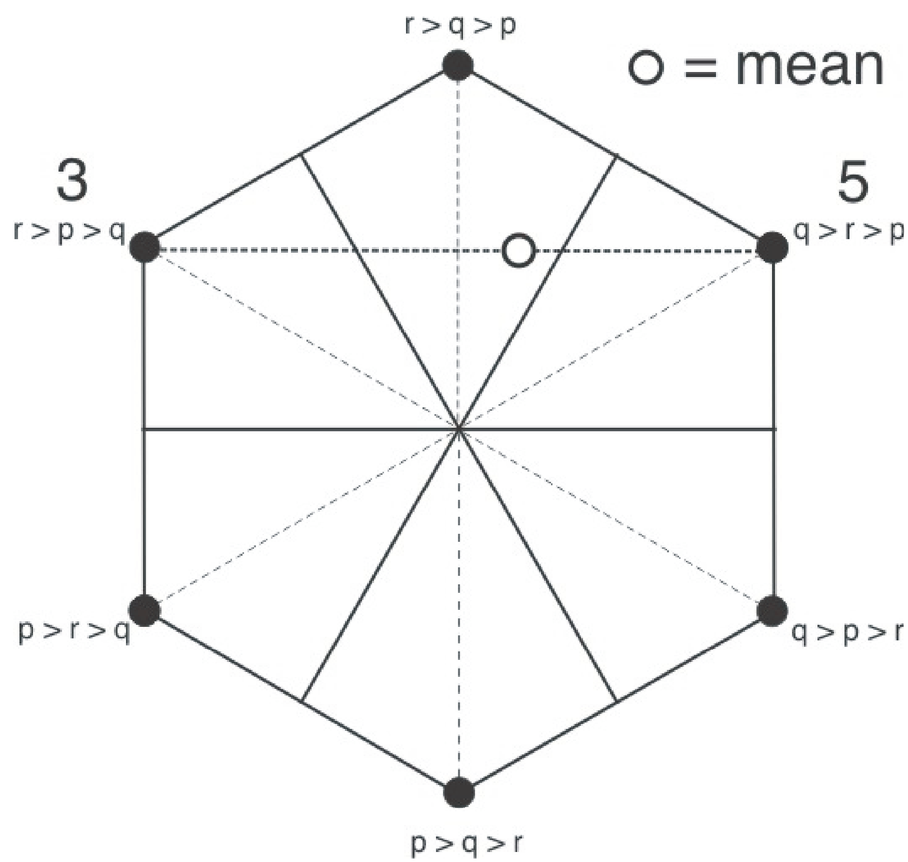
- The winning ranking is that of the vertex <u>closest</u> to the mean



Hex-Mean voting rule

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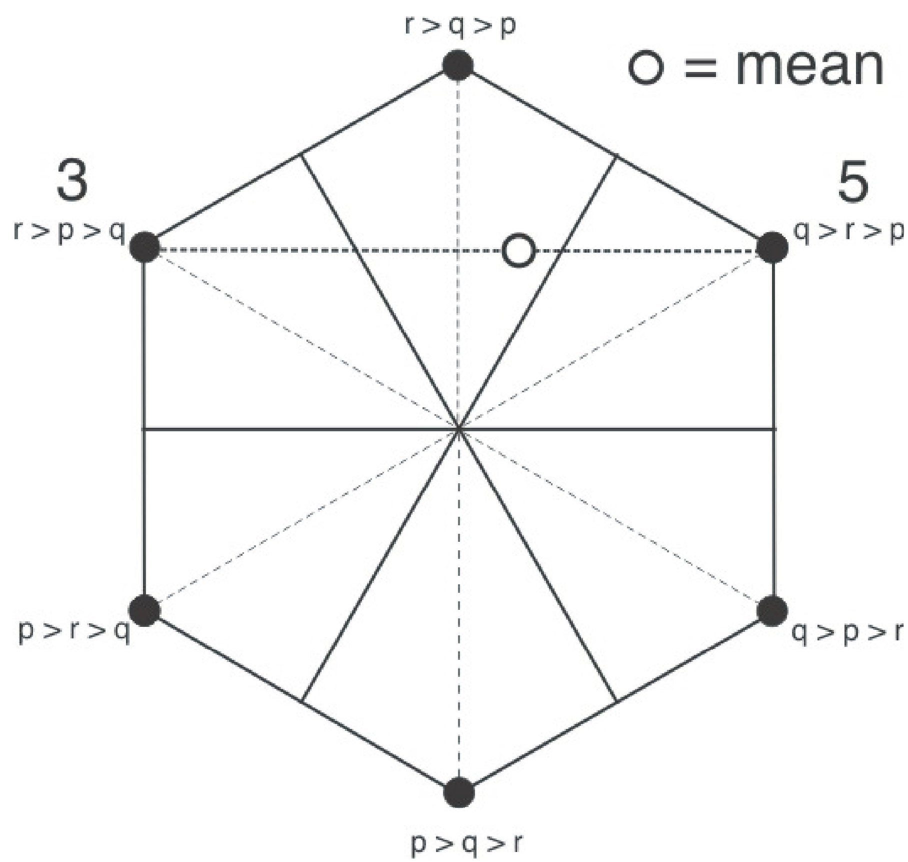
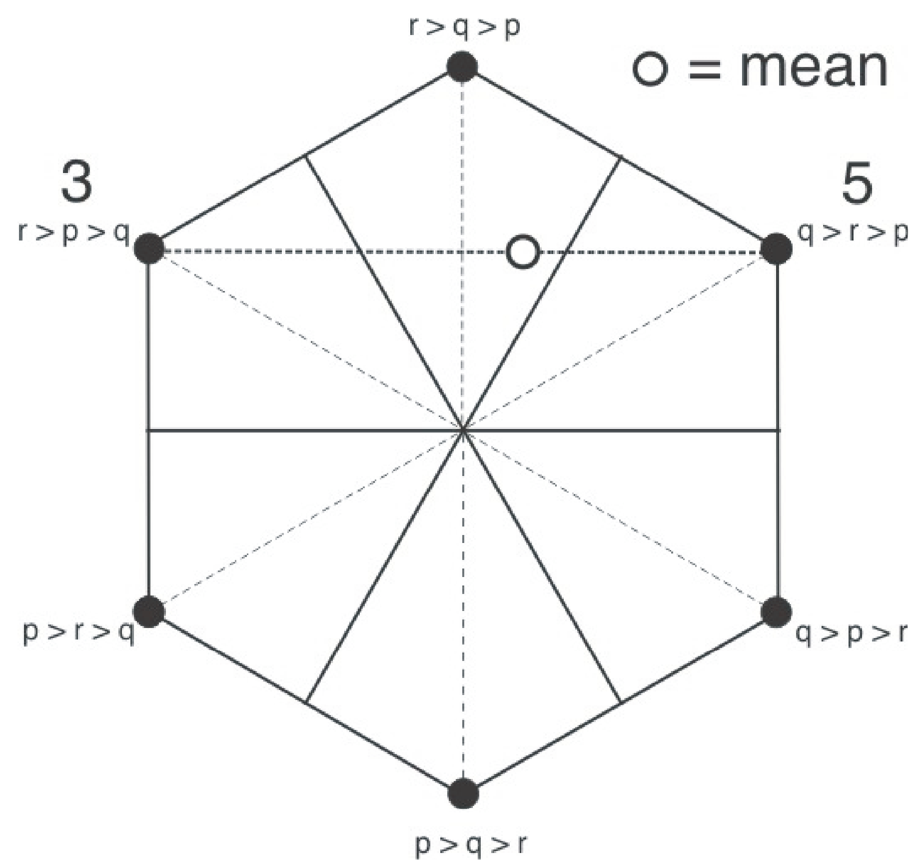
- **O** = mean location of all votes
- How do we find the “mean” of points in the plane? We’ll come back to that.
- Where is **O** ?
- The winning ranking is that of the vertex <u>closest</u> to the mean:

  r > q > p
- The Hex-Mean winner is **r**
- Who cares?



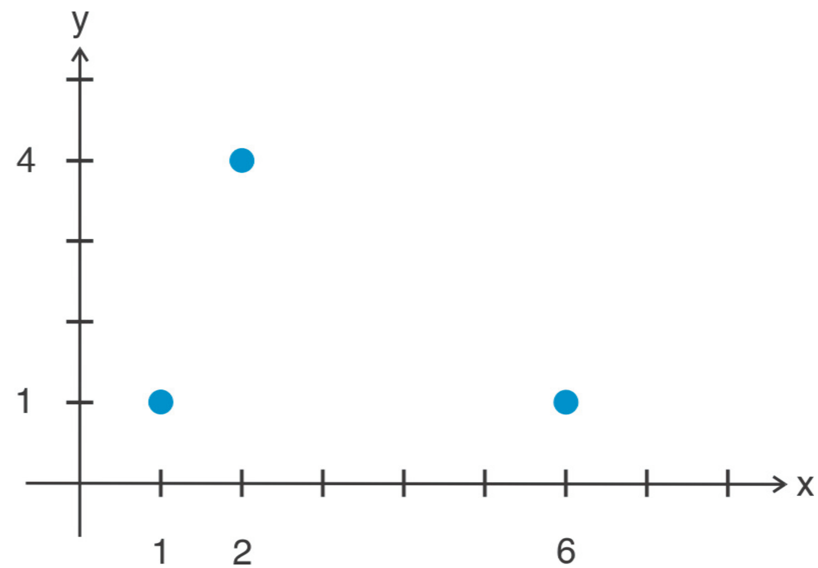
Hex-Mean voting rule

- **Theorem** The Hex-Mean rule is the same as the Borda Count



The Mean
2 equivalent definitions

- Given three (blue) points in the plane (or on a number line, or in space)



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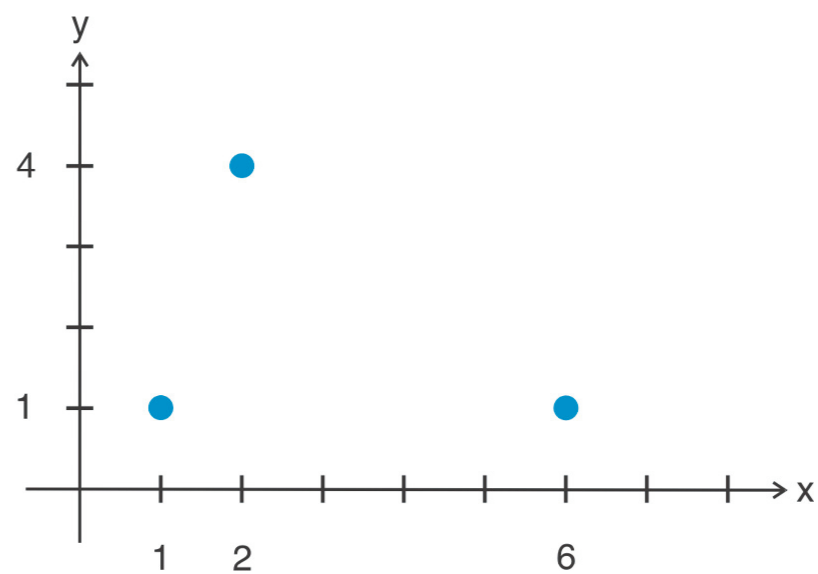
   1. <u>Average Coordinate Method</u>



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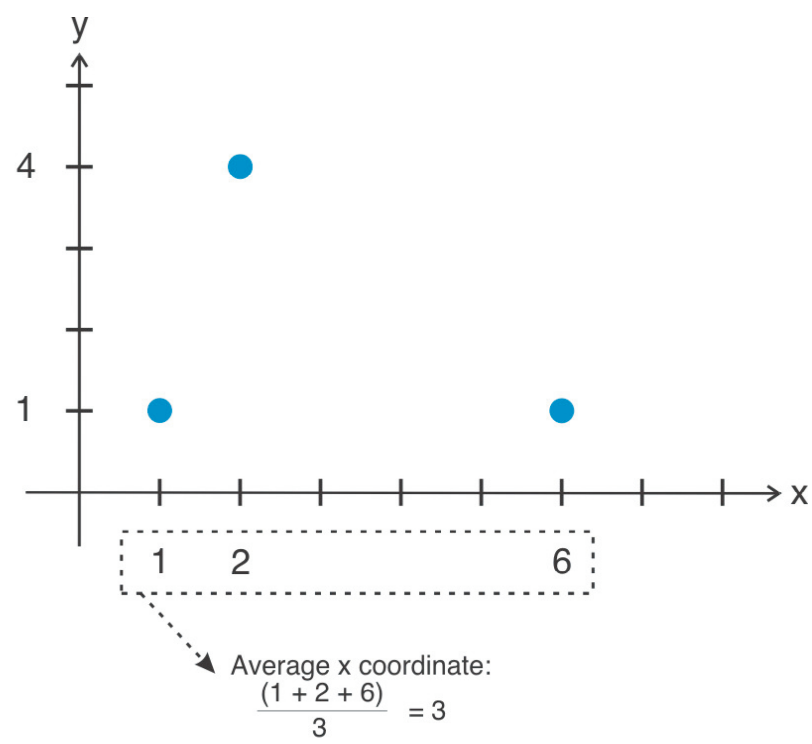
- Find the average x coordinate



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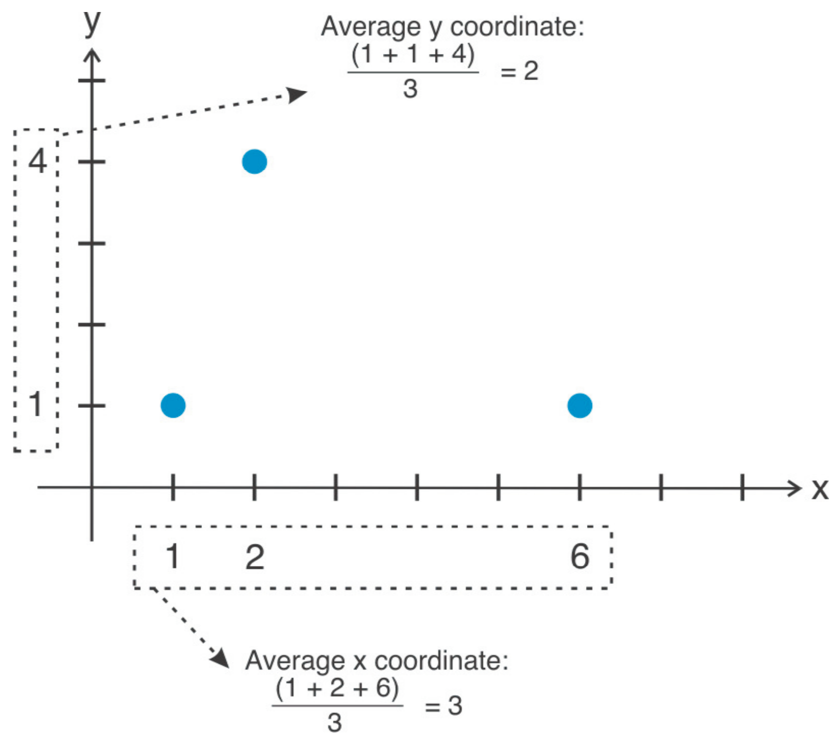

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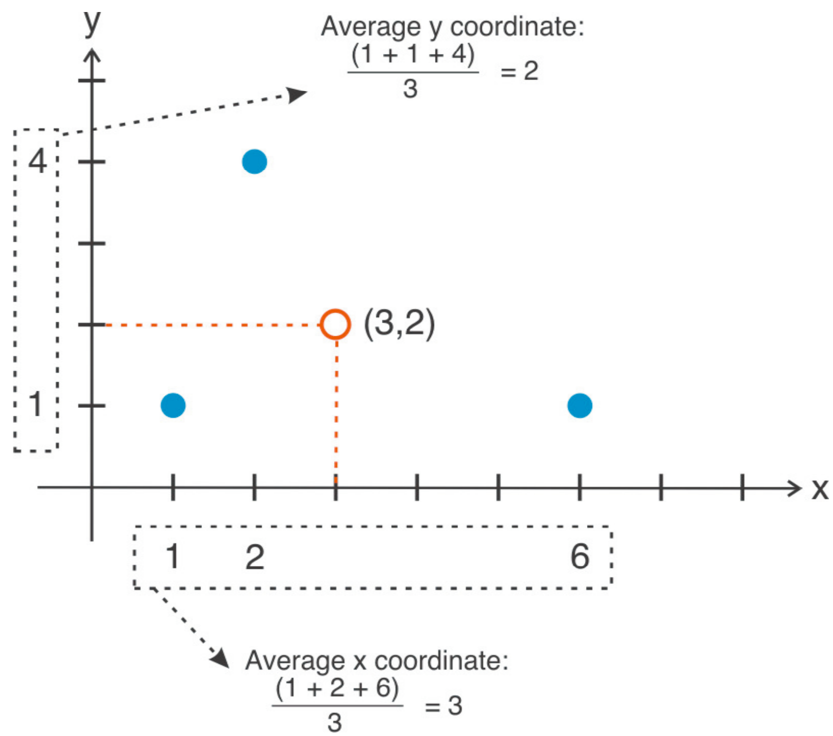
1. <u>Average Coordinate Method</u>

- Find the average x coordinate
- Find the average y coordinate
- Use these as the coordinates of the mean point ○



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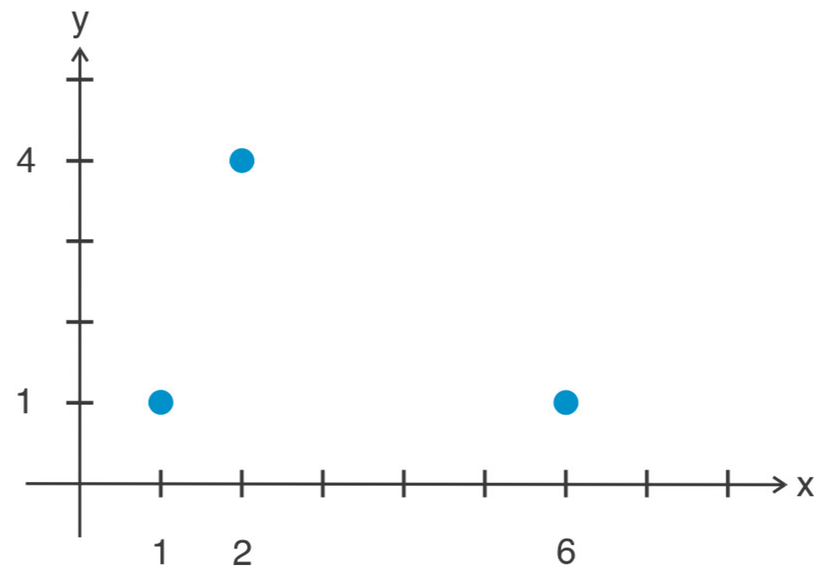

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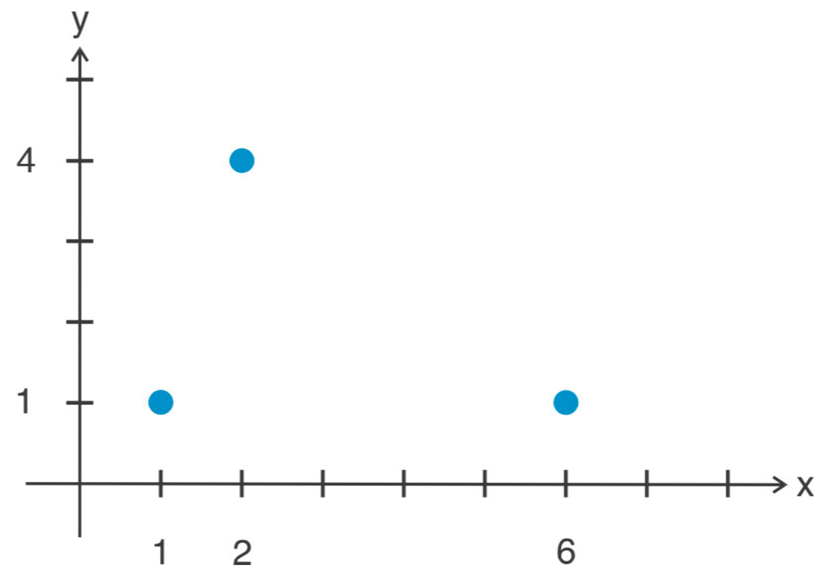
  - will shrink to a point if you let go of both ends
  - Tension is proportional to stretch



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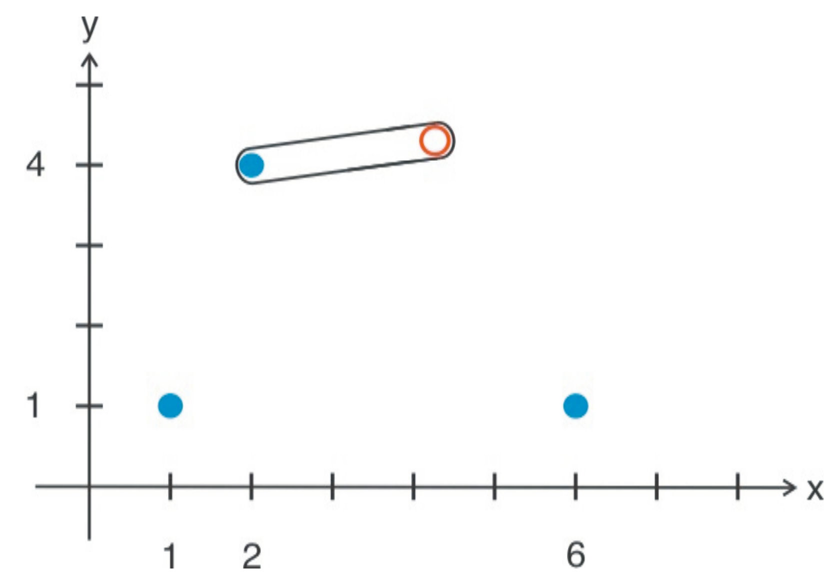
- Loop one end of an i.r.b. around a blue point, and the other end about a movable point **o**



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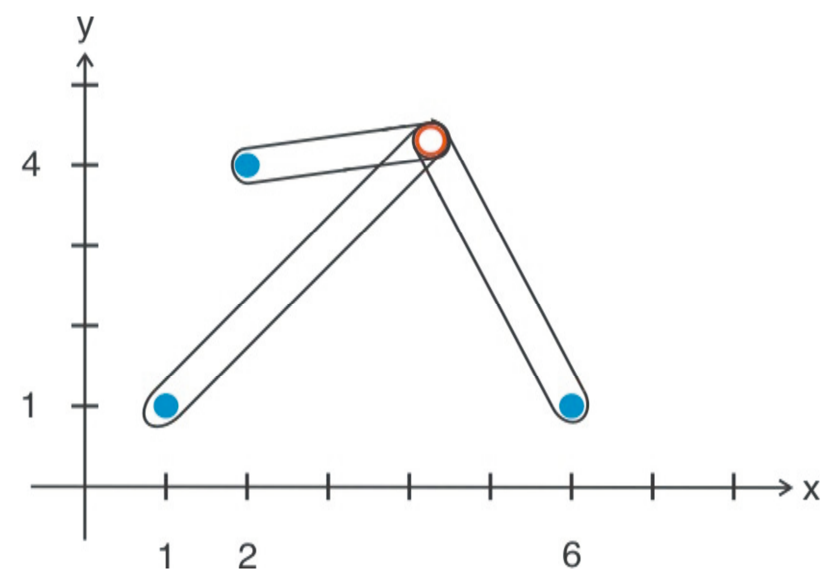
- Repeat with the other blue points



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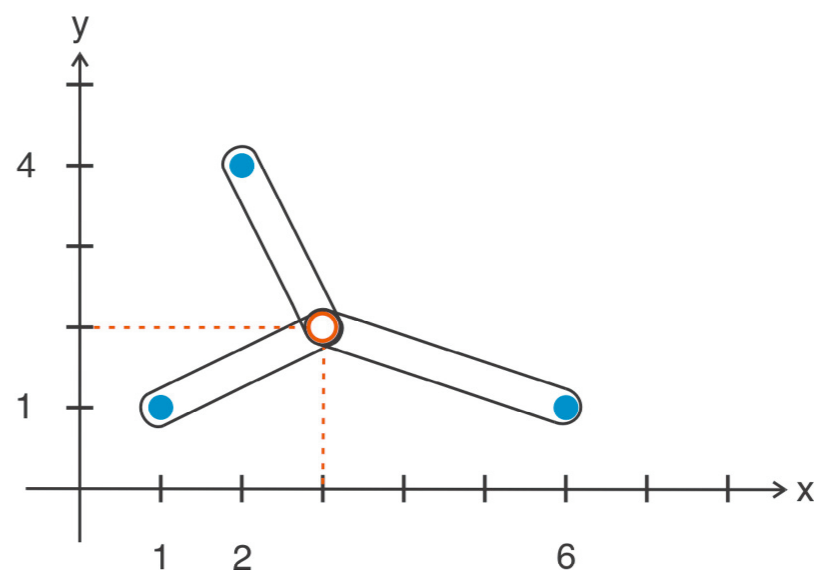
- Release **O** and let it reach equilibrium
  – rubber band forces cancel out exactly



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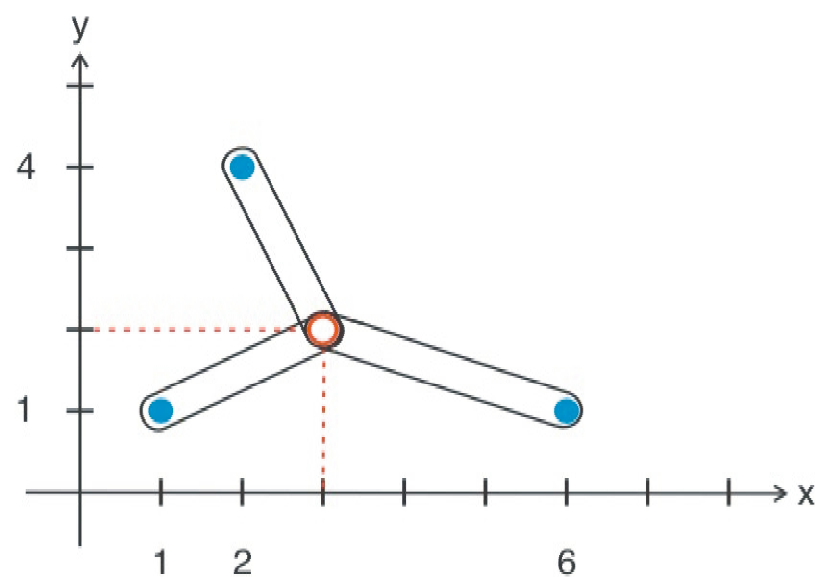


○ is the same as it was with the average method!

- An **i.r.b.**
  ◆ will shrink to a point if you let go of both ends
  ◆ Tension is proportional to stretch
- Loop one end of an i.r.b. around a blue point, and the other end about a movable point **O**
- Repeat with the other blue points
- Release **O** and let it reach equilibrium
  – *rubber band forces cancel out exactly*
- The two methods always agree, producing the same point **O**

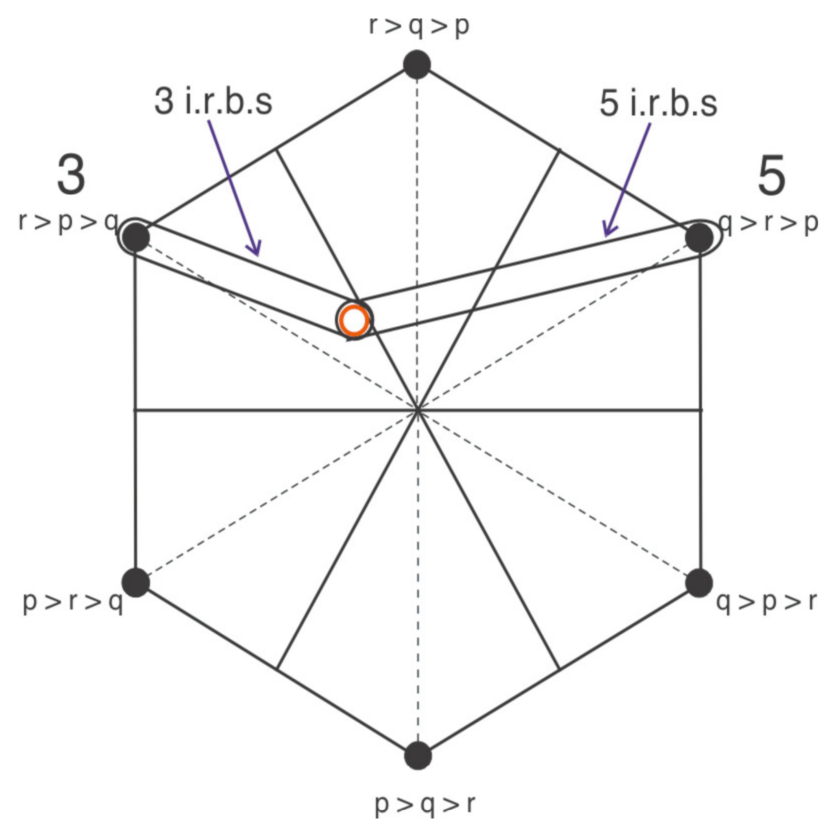
- **Theorem** The Hex-Mean rule is the <u>same</u> as the Borda Count

- And the mean can be found using rubber bands

- Putting these together we get…

Physical model for Borda count

- Tie 3 i.r.b.s around r>p>q and a movable point ○
- Tie 5 i.r.b.s around q>r>p & ○
- Release and let it reach equilibrium – *rubber band forces cancel out exactly*



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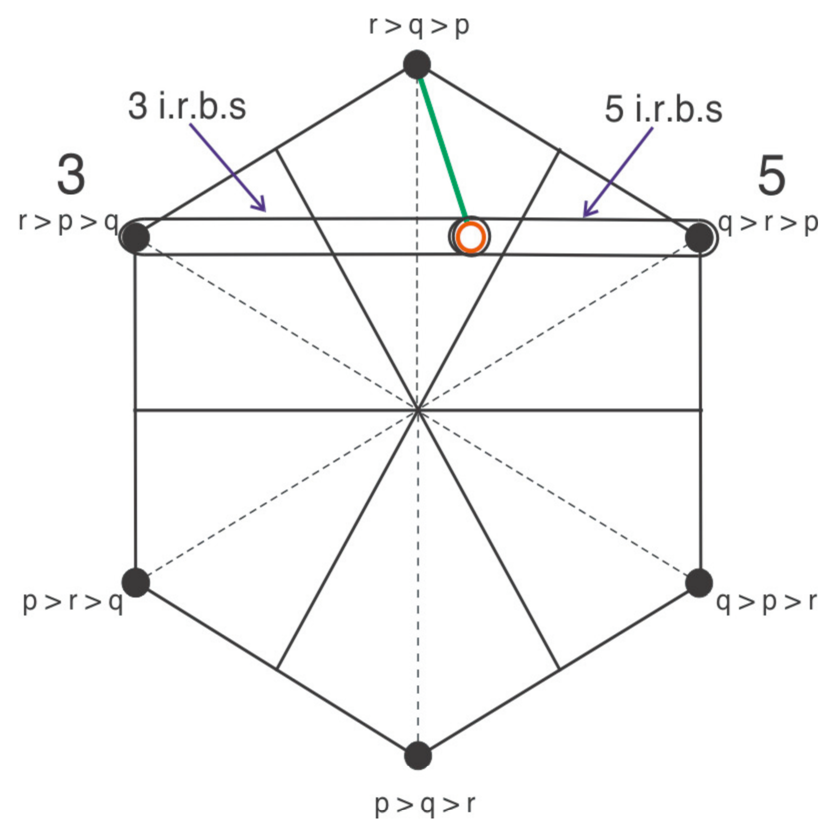
- Release and let it reach equilibrium – *rubber band forces cancel out exactly*
- The vertex closest to **o** (green line) tell us the Borda winner
- **Conclusion** Borda count = voting with rubber bands on the hexagon (3 alternatives)



Physical model for Borda count

- How about **four** alternatives?
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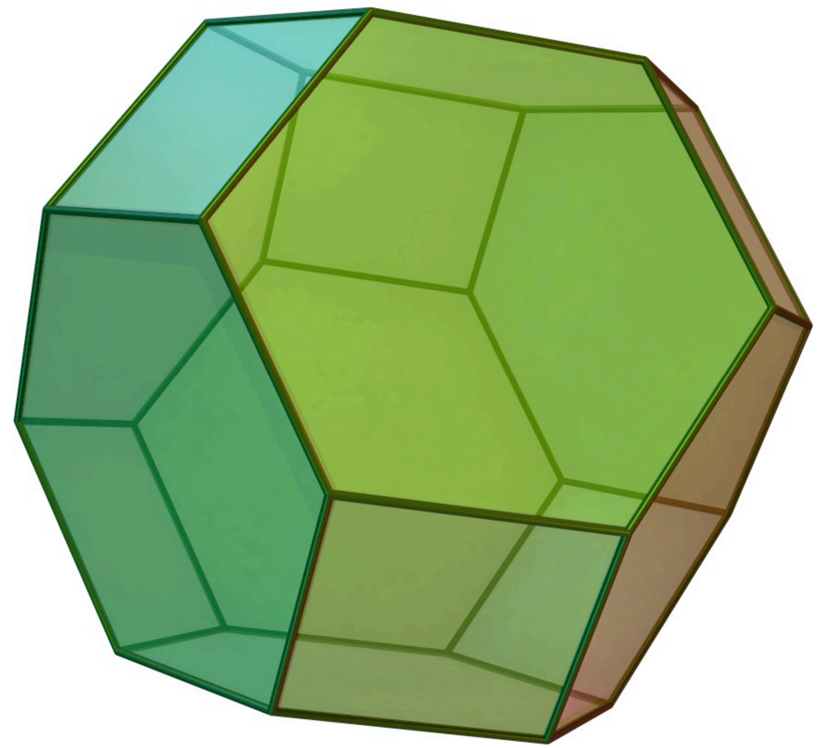

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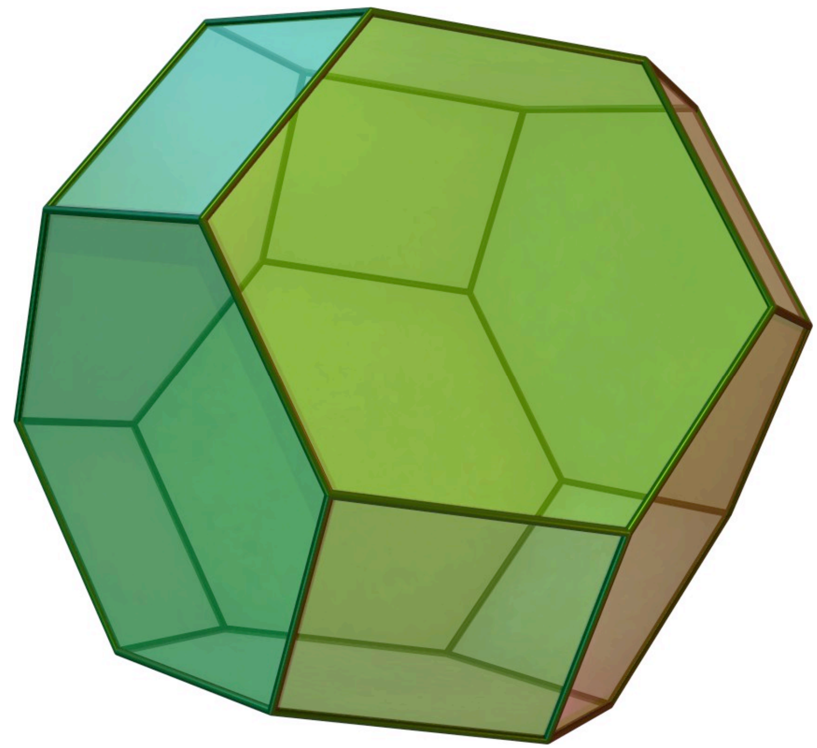

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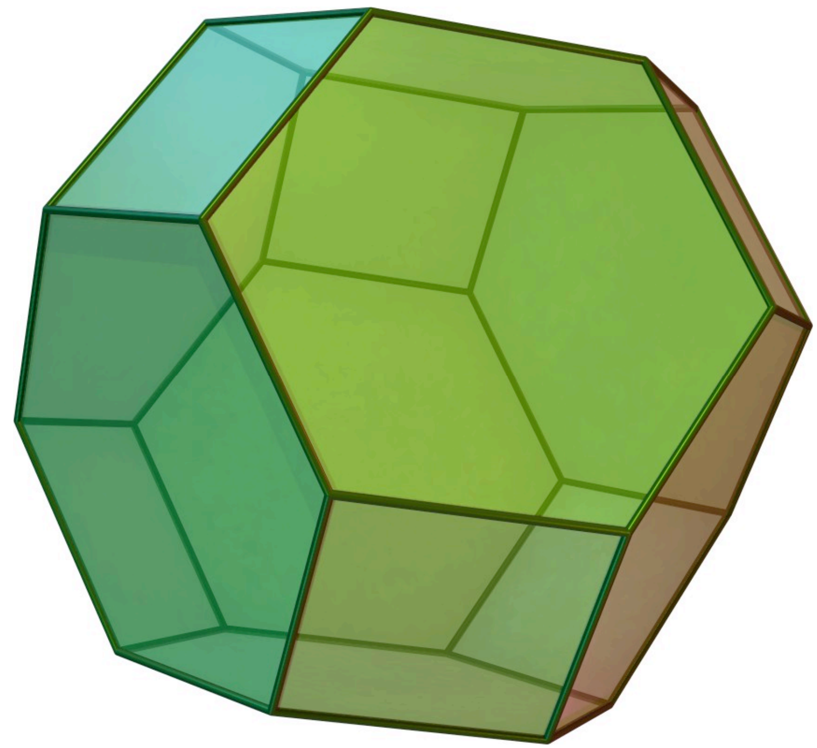
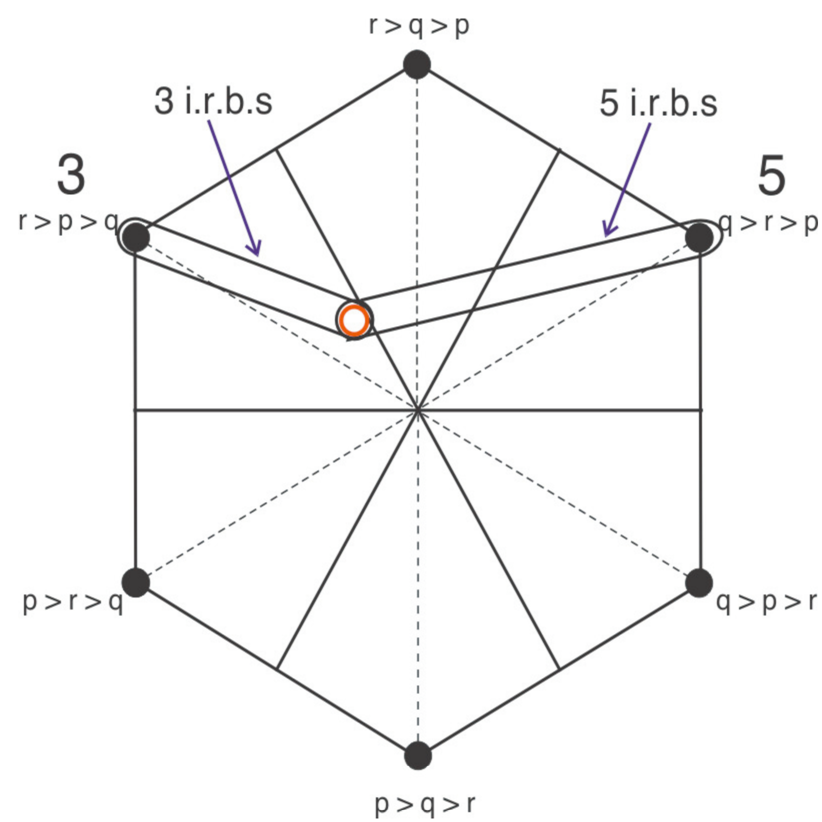
- Then vote with i.r.b.s; choose vertex closest to **O**



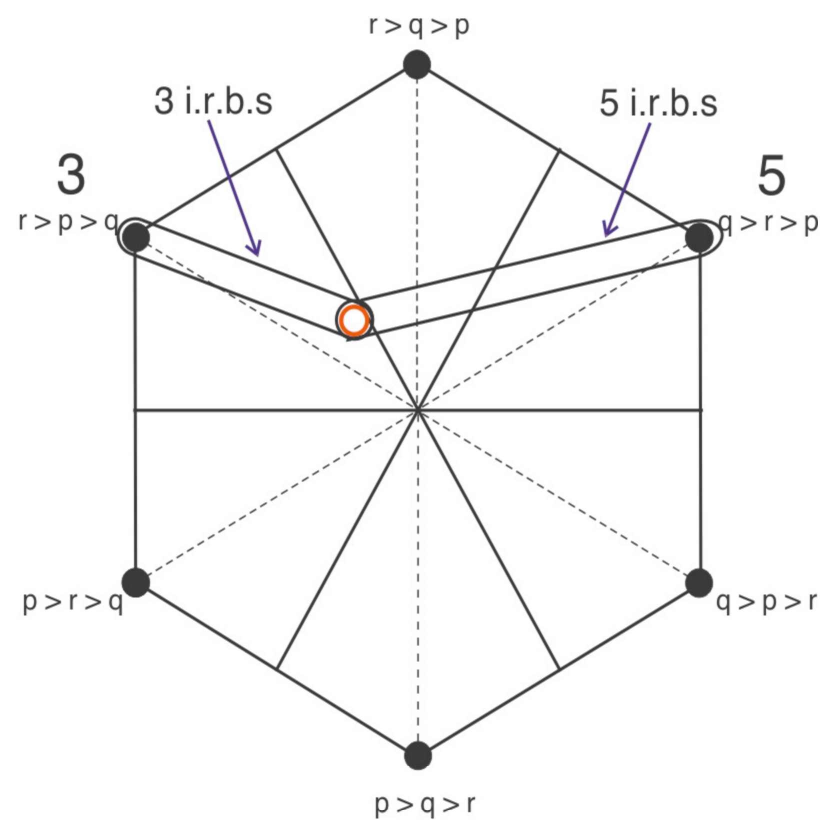
Physical model for Borda count

- **Conclusion** Borda count = voting with rubber bands on the hexagon (3 alternatives)
- With rubber bands, greater distance = harder pull
- Is there an alternative, with greater distance = same pull ?



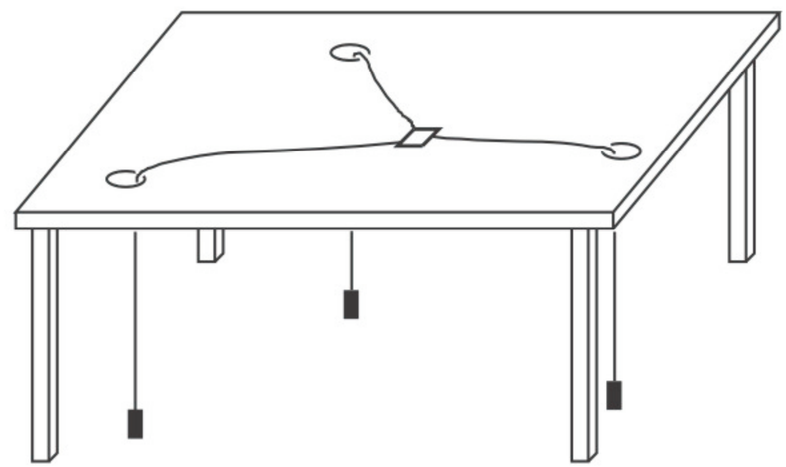
Physical model for Borda count

- **Conclusion** Borda count = voting with rubber bands on the hexagon (3 alternatives)
- With rubber bands, greater distance = harder pull
- Is there an alternative, with greater distance = same pull ?
- **Yes.** Replace rubber bands with weights and strings



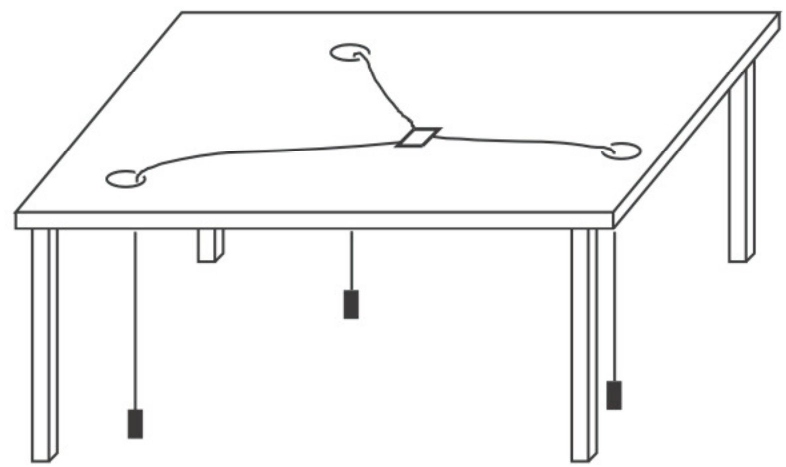
An Alternative to the Mean

- Choose 3 points on the plane



An Alternative to the Mean

- Choose 3 points on the plane
- Drill a hole through at each point, and pass a string through each hole



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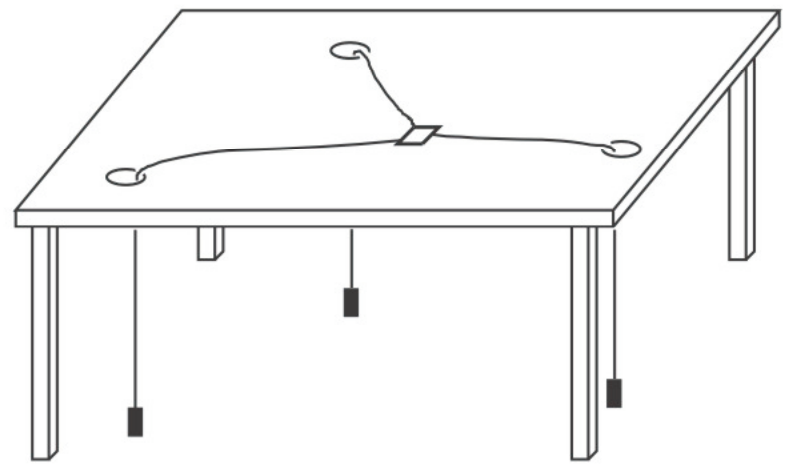
- Choose 3 points on the plane
- Drill a hole through at each point, and pass a string through each hole
- Attach a unit weight ▮ to each end below the table
- Tie all other ends to one movable point ◘



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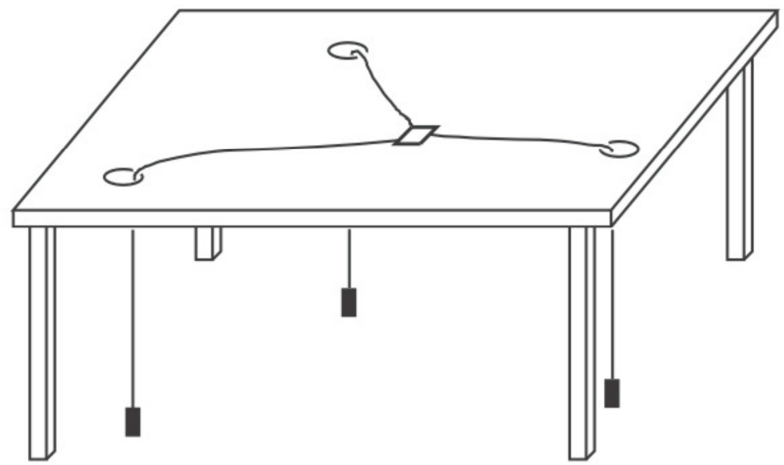

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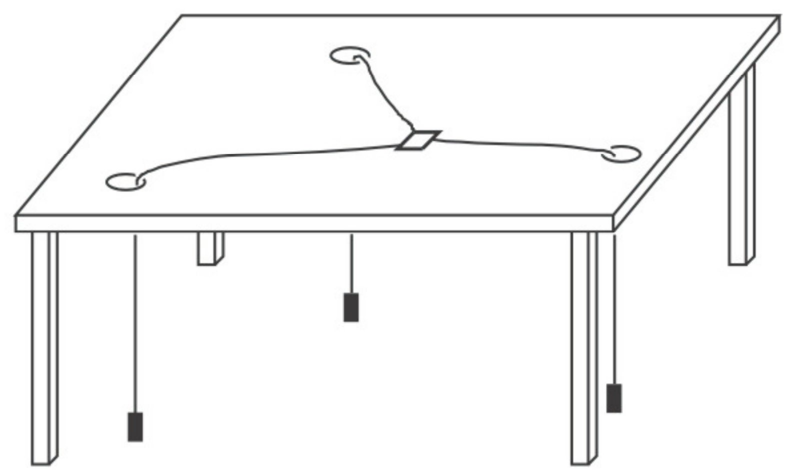
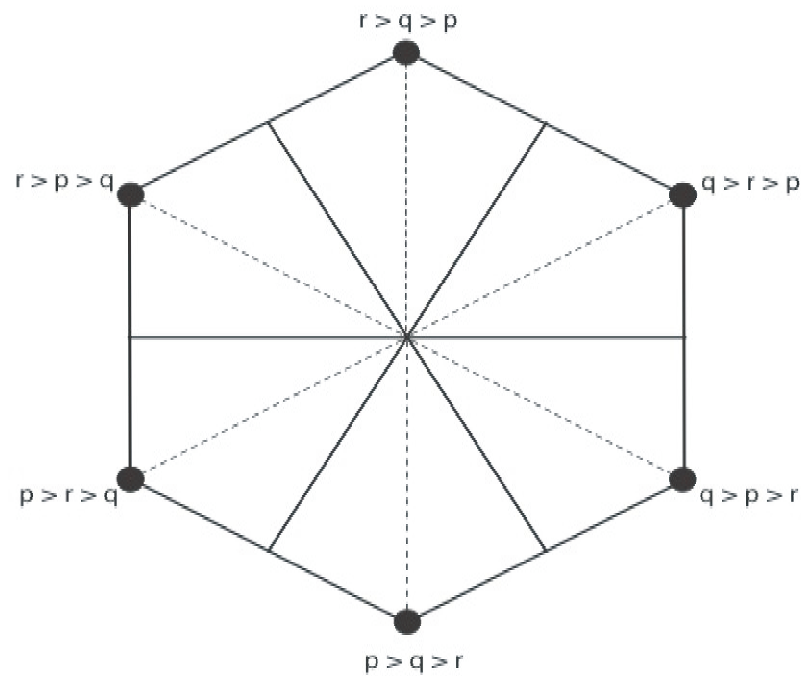
- Release, allow ◘ to reach equilibrium
- This point is called the ***mediancentre*** . . .
- . . . and it is different from the mean



A New Voting Rule

- Each voter chooses a vertex



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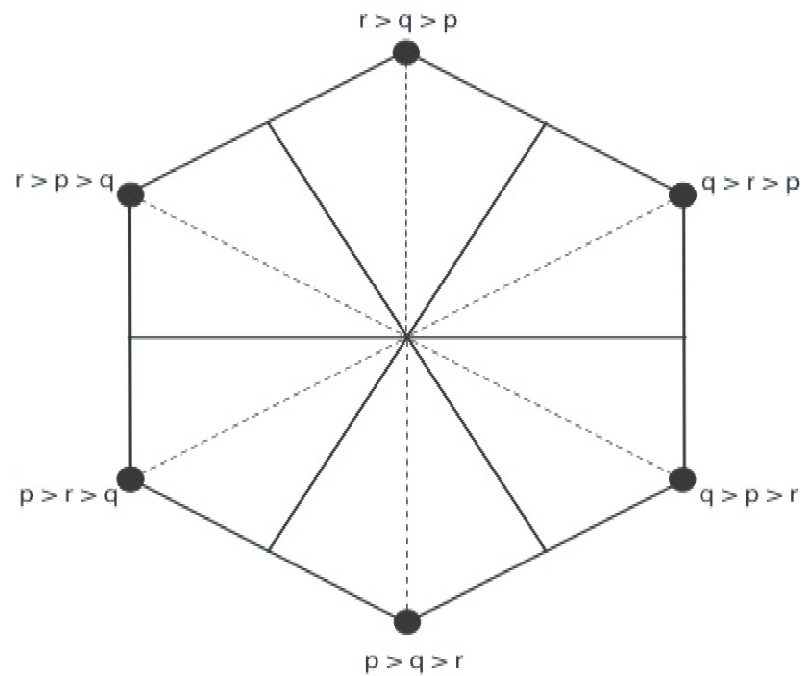

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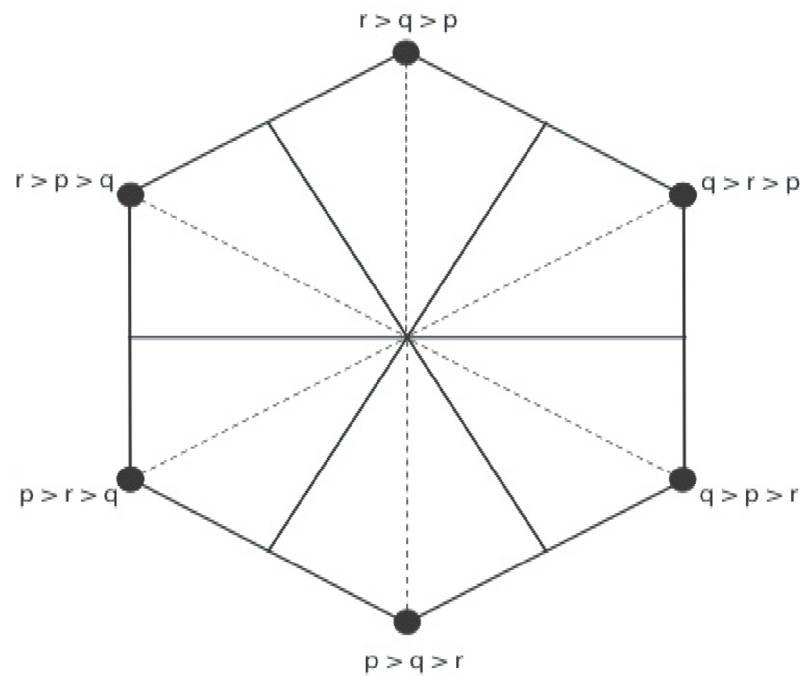

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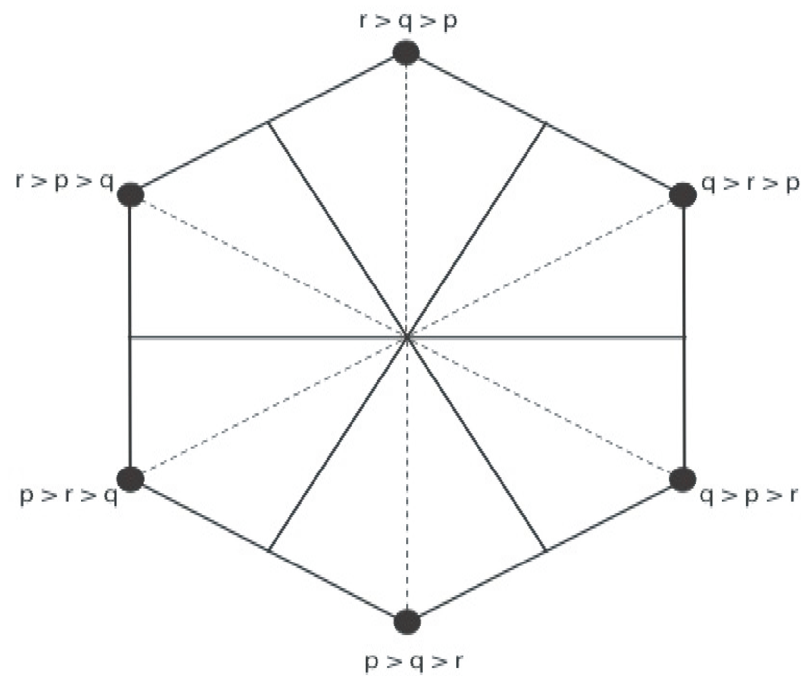

A New Voting Rule

- Each voter chooses a vertex
- ☐ = mediancentre of all votes
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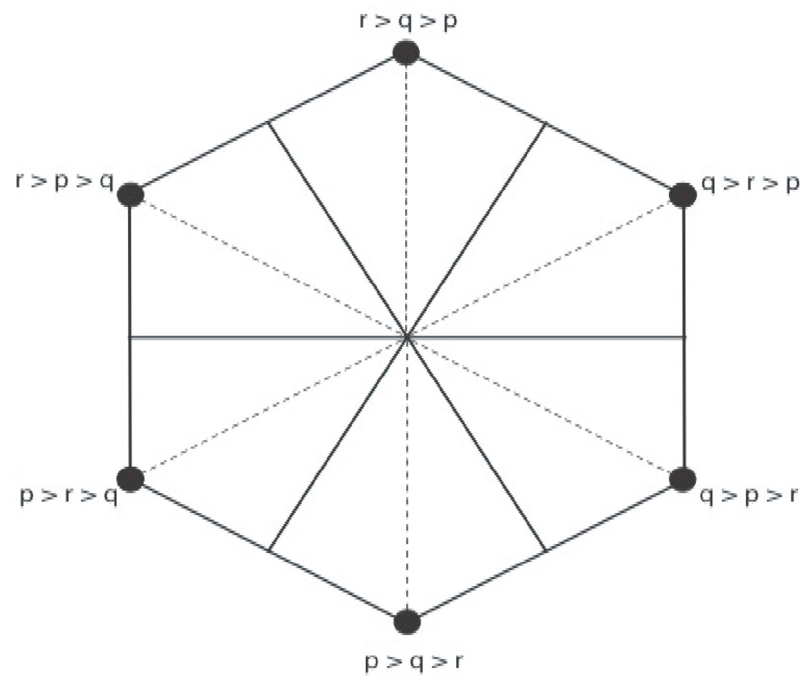
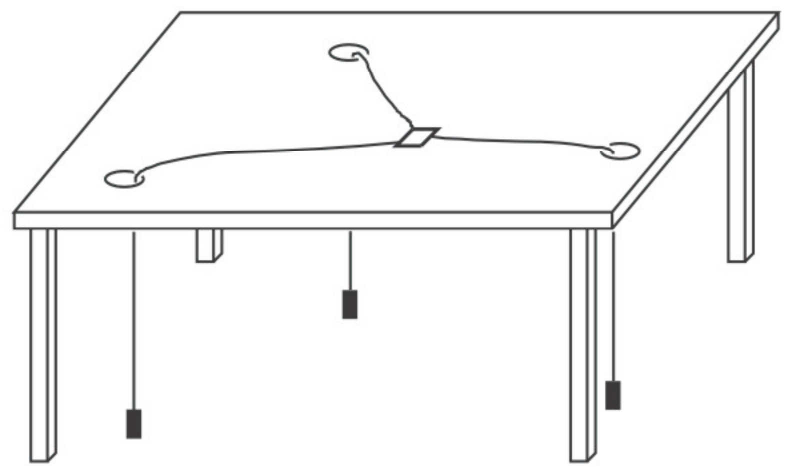
- We call this new voting rule the M^CBorda rule
- It is so new that we are still learning about its basic properties

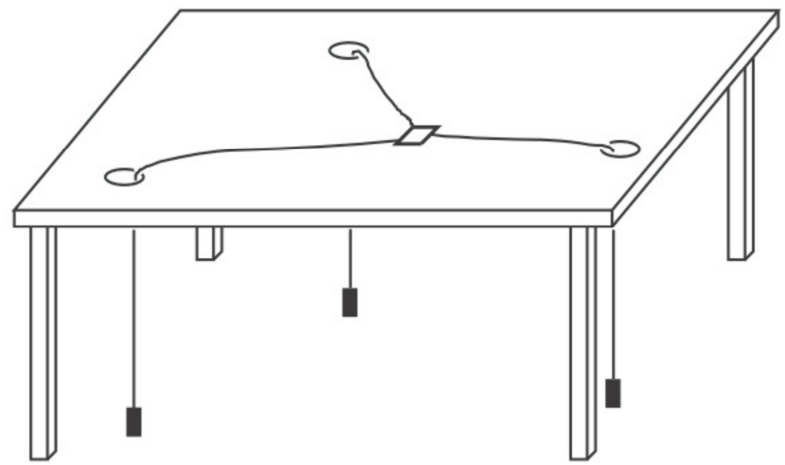


3 **BIG** Questions



3 **BIG** Questions

1) How does the mediancentre differ from the mean?



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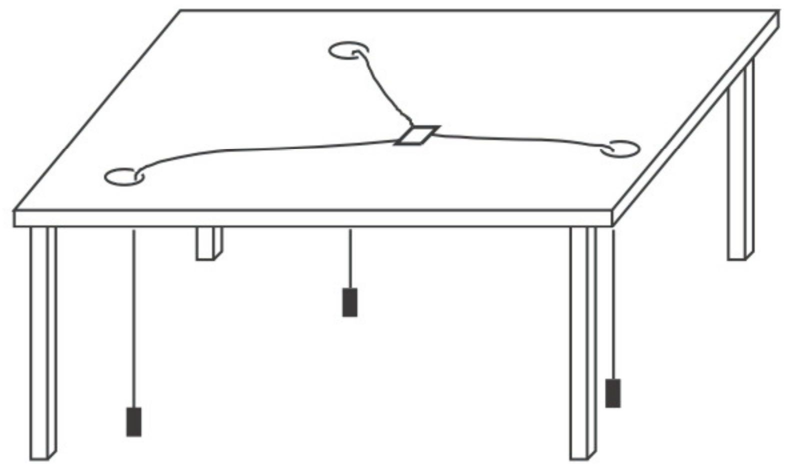

3 **BIG** Questions

1) How does the mediancentre differ from the mean?

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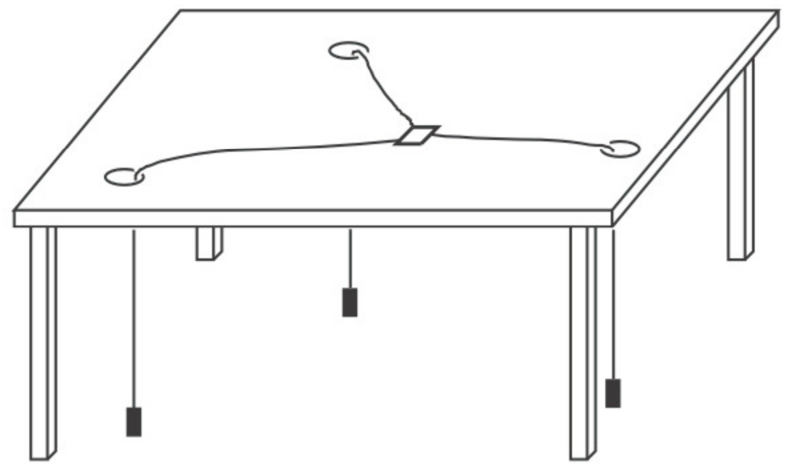
3) How are the answers to the previous two questions linked?



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1) How does the mediancentre* differ from the mean?

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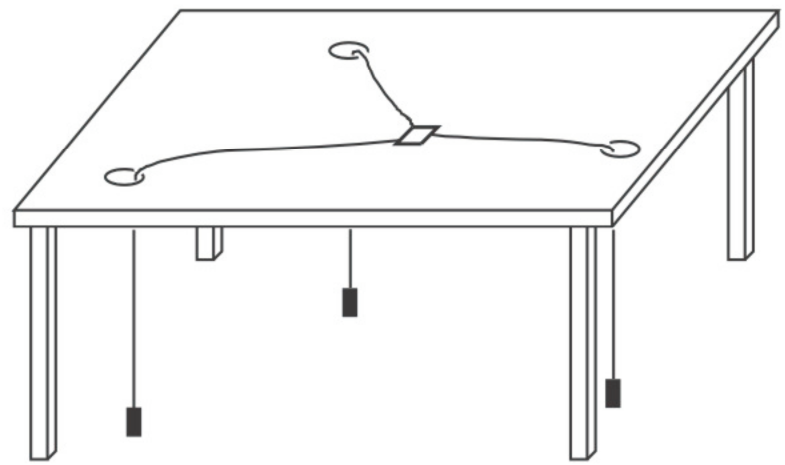
3) How are the answers to the previous two questions linked?

* *And how is the mediancentre related to the median?*



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WE'LL EXPERIMENT . . .

. . . USING DAVIDE CERVONE'S SOFTWARE