CLASSIFICATION OF PROBLEMATIC SUBGROUPS OF $U(n)$

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ABSTRACT. We classify p-toral subgroups of $U(n)$ that can have non-contractible fixed points under the action of $U(n)$ on the complex $\mathcal{L}_n$ of partitions of $\mathbb{C}^n$ into mutually orthogonal subspaces.

1. INTRODUCTION

Let $\mathcal{P}_n$ be the Σ_n-space given by the nerve of the poset category of proper nontrivial partitions of the set $\{1, \ldots, n\}$. In [ADL13], the authors compute Bredon homology and cohomology groups of $\mathcal{P}_n$ with certain p-local coefficients. The computation is part of a program to give a new proof of the Whitehead Conjecture and the collapse of the homotopy spectral sequence for the Goodwillie tower of the identity, one that does not rely on the detailed knowledge of homology used in [Kuh82], [KP85], and [Beh11]. For appropriate p-local coefficients, the Bredon homology and cohomology of $\mathcal{P}_n$ turn out to be trivial when n is not a power of the prime p, and nontrivial in only one dimension when $n = p^k$ ([ADL13] Theorems 1.1 and 1.2). A key ingredient of the proof is the identification of the fixed point spaces of p-subgroups of Σ_n acting on $\mathcal{P}_n$. If a p-subgroup $H \subseteq \Sigma_n$ has noncontractible fixed points, then H gives an obstruction to triviality of Bredon homology, so it is "problematic" (see Definition 1.1 below). It turns out that only elementary abelian p-groups subgroups of Σ_n with free action on $\{1, \ldots, n\}$ can have noncontractible fixed points ([ADL13] Proposition 6.1). The proof of the main result of [ADL13] then proceeds by showing that, given appropriate conditions on the coefficients, these problematic subgroups can be "pruned" or "discarded" in most cases, resulting in sparse Bredon homology and cohomology.

In this paper, we carry out the fixed point calculation analogous to that of [ADL13] in the context of unitary groups. The calculation is part of a program to establish the conjectured bu-analogue of

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the Whitehead Conjecture and the conjectured collapse of the homotopy spectral sequence of the Weiss tower for the functor $V \mapsto BU(V)$ (see [AL07] Section 12). Let $\mathcal{L}_n$ denote the (topological) poset category of partitions of $\mathbb{C}^n$ into nonzero proper orthogonal subspaces. The category $\mathcal{L}_n$ was first introduced in [Aro02]. It is internal to the category of topological spaces: both its object set and its morphism set have a topology. (See [BJL$^+$] for a detailed discussion and examples.) The action of the unitary group $U(n)$ on $\mathbb{C}^n$ induces a natural action of $U(n)$ on $\mathcal{L}_n$, and the Bredon homology of the $U(n)$-space $\mathcal{L}_n$ plays an analogous role in the unitary context to the part played by the Bredon homology of the Σ_n-space $\mathcal{P}_n$ in the classical context.

Remark. The complex $\mathcal{L}_n$ has a similar flavor to the "stable building" or "common basis complex" studied by John Rognes in [Rog92, Rog].

As finite p-groups in the discrete setting are usually replaced in the compact Lie setting by p-toral subgroups (i.e., extensions of finite p-groups by tori), our goal in this paper is to identify p-toral subgroups of $U(n)$ with noncontractible fixed point sets on $\mathcal{L}_n$. These groups will be the obstructions to triviality of the Bredon homology and cohomology of $\mathcal{L}_n$ with suitable p-local coefficients. Hence we make the following definition.

Definition 1.1. A subgroup $H \subseteq U(n)$ is called *problematic* if the fixed point set of H acting on the nerve of $\mathcal{L}_n$ is not contractible.

Our result, loosely stated, is that there are only a few conjugacy classes of problematic p-toral subgroups of $U(n)$. To describe our results in detail, we need a little more terminology. The center of $U(n)$ is S^1, acting on $\mathbb{C}^n$ via scalar multiples of the identity matrix, and the *projective unitary group* $PU(n)$ is the quotient $U(n)/S^1$. We say that a closed subgroup $H \subseteq U(n)$ is a *projective elementary abelian* p*-group* if the image of H in $PU(n)$ is an elementary abelian p-group. Lastly, if H is a closed subgroup of $U(n)$, we write χ_H for the character of H acting on $\mathbb{C}^n$ through the standard action of $U(n)$ on $\mathbb{C}^n$.

Theorem 1.2. *Let H be a problematic p-toral subgroup of $U(n)$. Then*

(1) *H is a projective elementary abelian p-group, and*

(2) *the character of H is* $\begin{cases} \chi_H(h) = 0 & h \notin S^1 \\ \chi_H(h) = nh & h \in H \cap S^1. \end{cases}$

The first part of the theorem was the principal result of [BJL$^+$] (Theorem 1.1). For completeness, we give a streamlined proof here, in order to obtain the second part of the theorem, the subgroup's character.

The character data of Theorem 1.2 allows us to narrow down the problematic p-toral subgroups of $U(n)$ to a very small, explicitly described collection. Indeed, there are well-known p-toral subgroups of unitary groups that satisfy the conclusions of Theorem 1.2, namely the "p-stubborn" subgroups $\Gamma_k \subseteq U\big(p^k\big)$, which appear in numerous works, such as [Gri91], [JMO92], [Oli94], [Aro02], [AL07], and [AGMV08]. These subgroups are described in Section 4. For any $n = mp^k$, we can consider the diagonal embedding

$$\Gamma_k \longrightarrow \underbrace{\Gamma_k \times \dots \Gamma_k}_{m} \longrightarrow \underbrace{U\big(p^k\big) \times \dots U\big(p^k\big)}_{m} \longrightarrow U(n)\,,$$

whose image in $U(n)$ we denote $\mathrm{Diag}_m(\Gamma_k)$. Our major result is the following.

Theorem 1.3. *Let H be a problematic p-toral subgroup of $U(n)$, and suppose that $n = mp^k$, where m and p are coprime. Then H is conjugate to a subgroup of $\mathrm{Diag}_m(\Gamma_k) \subseteq U(n)$.*

The strategy of the proof is to use the first part of Theorem 1.2 and bilinear forms to classify H up to abstract group isomorphism. Then character theory, together with the second part of Theorem 1.2, allows us to pinpoint the actual conjugacy class of H in $U(n)$.

The organization of the paper is as follows. In Section 2 we give a normal subgroup condition from which one can deduce contractibility of a fixed point set $(\mathcal{L}_n)^H$. We follow up in Section 3 with the proof of Theorem 1.2, by finding an appropriate normal subgroup unless H is a projective elementary abelian p-subgroup of $U(n)$. Section 4 is expository and discusses the salient properties of the subgroup $\Gamma_k \subseteq U\big(p^k\big)$. The projective elementary abelian p-subgroups of $U(n)$ are classified up to abstract group isomorphism using bilinear forms in Section 5. Finally, in Section 6, we prove Theorem 1.3 and give an example.

Definitions, Notation, and Terminology

When we speak of a subgroup of a Lie group, we always mean a closed subgroup. If we speak of $S^1 \subseteq U(n)$ without any further description, we mean the center of $U(n)$.

We generally do not distinguish in notation between a category and its nerve, since the context will make clear which we mean.

We are concerned with actions of subgroups $H \subseteq U(n)$ on $\mathbb{C}^n$; we write ρ_H for the restriction of the standard representation of $U(n)$ to H, and χ_H for the corresponding character. We apply the standard terms from representation theory to H if they apply to ρ_H. For example, we say that H is *irreducible* if ρ_H is irreducible, and we say that H is *isotypic* if ρ_H is the sum of all isomorphic irreducible representations of H. We also introduce a new term: if H is not isotypic, we say that H is *polytypic*, as a succinct way to say that "the action of H on $\mathbb{C}^n$ is not isotypic."

A *partition* λ of $\mathbb{C}^n$ is an (unordered) decomposition of $\mathbb{C}^n$ into mutually orthogonal, nonzero subspaces. We say that λ is *proper* if it consists of subspaces properly contained in $\mathbb{C}^n$. It is often useful to regard λ as specified by an equivalence relation $\sim_\lambda$ on $\mathbb{C}^n \backslash \{0\}$, where $x \sim_\lambda y$ if x and y are in the same subspace. We suppress λ and write $\sim$ when it is clear what partition is under discussion. If $v_1, \ldots, v_m$ are the subspaces in λ, then the equivalence classes of $\sim_\lambda$ are given by $v_1 - \{0\}$, $\ldots$, $v_m - \{0\}$. By abuse of terminology, we regard $v_1, \ldots, v_m$ as the equivalence classes of $\sim_\lambda$, and we write $\mathrm{cl}(\lambda) := \{v_1, \ldots, v_m\}$ when we want to emphasize the set of subspaces in the decomposition λ.

The action of $U(n)$ on $\mathbb{C}^n$ induces a natural action of $U(n)$ on $\mathcal{L}_n$, and we are interested in fixed points of this action. If λ consists of the subspaces $v_1, \ldots, v_m$, we say that λ is *weakly fixed* by a subgroup $H \subseteq U(n)$ if for every $h \in H$ and every v_i, there exists a j such that $hv_i = v_j$. We write $(\mathcal{L}_n)^H$ for the full subcategory of weakly H-fixed objects of $\mathcal{L}_n$, or for the nerve of this category, depending on context.

By contrast, we say that λ is *strongly fixed* by $H \subseteq U(n)$ if for all i, we have $Hv_i = v_i$, that is, each v_i is a representation of H. We write $(\mathcal{L}_n)^H_{\mathrm{st}}$ for the full subcategory of strongly H-fixed objects of $\mathcal{L}_n$ (and for its nerve). Finally, we write $(\mathcal{L}_n)^H_{\mathrm{iso}}$ for the full subcategory of partitions in $\mathcal{L}_n$ that are not only strongly fixed, but whose classes consist of isotypical representations of H. (See [BJL$^+$].)

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2. A contractibility criterion

This section establishes that subgroups $H \subseteq U(n)$ with certain normal, polytypic subgroups have contractible fixed point sets on $\mathcal{L}_n$. The main result is Proposition 2.1 below, which first appeared as [BJL$^+$] Theorem 4.5; its proof is given at the end of the section. Our goal in Section 3 will be to find normal subgroups J satisfying the hypotheses of Proposition 2.1.

Proposition 2.1. *Let H be a subgroup of $U(n)$, and suppose H has a normal subgroup J with the following properties:*

(1) *J is polytypic.*
(2) *For every $\lambda \in (\mathcal{L}_n)^H$, the action of J on $\mathrm{cl}(\lambda)$ is not transitive.*

Then $(\mathcal{L}_n)^H$ is contractible.

To prove Proposition 2.1, we need two operations on partitions to construct natural retractions between various subcategories of $\mathcal{L}_n$.

Definition 2.2. Let $J \subseteq U(n)$ be a subgroup, and let λ be a weakly J-fixed partition of $\mathbb{C}^n$.

(1) The partition $\lambda_{\mathrm{st}(J)}$ is defined by the relation generated by $x \sim_{\lambda_{\mathrm{st}(J)}} y$ if there exists $j \in J$ such that $x \sim_\lambda jy$.
(2) If λ is strongly J-fixed, let $\lambda_{\mathrm{iso}(J)}$ be the refinement of λ obtained by breaking each equivalence class of λ into its canonical J-isotypical summands.

A routine check establishes the following properties of the operations above.

Lemma 2.3.

(1) *The partition $\lambda_{\mathrm{st}(J)}$ is strongly J-fixed, natural in λ, and minimal among coarsenings of λ that are strongly J-fixed.*
(2) *The partition $\lambda_{\mathrm{iso}(J)}$ is natural in strongly J-fixed partitions λ and is maximal among refinements of λ whose classes are isotypical representations of J.*

The following two lemmas give criteria for $\lambda_{\mathrm{st}(J)}$ and $\lambda_{\mathrm{iso}(J)}$ to be weakly fixed by a supergroup $H \subseteq U(n)$ of J. Lemma 2.4 is a straightforward check, and Lemma 2.5 involves some basic representation theory.

Lemma 2.4. *Let $J \lhd H$, and suppose that λ is weakly fixed by H. Then $\lambda_{\mathrm{st}(J)}$ is weakly fixed by H.*

Lemma 2.5. *Let $J \lhd H$, and suppose that λ is strongly fixed by J and weakly fixed by H. Then $\lambda_{\mathrm{iso}(J)}$ is weakly fixed by H.*

Proof. Let v be an equivalence class of λ. For any $h \in H$, the subspace hv is another equivalence class of λ since λ is weakly H-fixed. Also, since v is stabilized by J, the class hv is likewise stabilized by J, because $J \lhd H$. We need to check that if w is a canonical J-isotypical summand of v, then hw is a canonical J-isotypical summand of hv.

As a representation of J, the subspace hv is conjugate to the representation of J on v. Thus if $w \subseteq v$ is the isotypical summand of v for an irreducible representation ρ of J, then hw is the isotypical summand of hv for the conjugate of ρ by h. We conclude that h maps the canonical J-isotypical summands of v to the canonical J-isotypical summands of hv, so H weakly fixes $\lambda_{\mathrm{iso}(J)}$. $\square$

These constructions allow us to retract $(\mathcal{L}_n)^H$ onto subcategories that in many cases have a terminal object and hence contractible nerve.

Proposition 2.6. *Let H be a subgroup of $U(n)$, and suppose $J \lhd H$. Then the inclusion functor*

$$\iota_1\colon\ (\mathcal{L}_n)^J_{\mathrm{iso}}\ \cap\ (\mathcal{L}_n)^H \longrightarrow (\mathcal{L}_n)^J_{\mathrm{st}}\ \cap\ (\mathcal{L}_n)^H$$

induces a homotopy equivalence on nerves. If J further has the property that $\lambda_{\mathrm{st}(J)}$ is proper for every $\lambda \in (\mathcal{L}_n)^H$, then the inclusion functor

$$\iota_2\colon\ (\mathcal{L}_n)^J_{\mathrm{st}}\ \cap\ (\mathcal{L}_n)^H \longrightarrow (\mathcal{L}_n)^H$$

induces a homotopy equivalence on nerves.

Proof. By Lemma 2.5, a functorial retraction of ι_1 is given by $r_1\colon \lambda \mapsto \lambda_{\mathrm{iso}(J)}$. The coarsening morphism $\lambda_{\mathrm{iso}(J)} \to \lambda$ provides a natural transformation from $\iota_1 r_1$ to the identity, establishing the first desired equivalence.

Similarly, by Lemma 2.4 a functorial retraction of ι_2 is given by $r_2\colon \lambda \mapsto \lambda_{\mathrm{st}(J)}$, because by assumption $\lambda_{\mathrm{st}(J)}$ is always a proper partition of $\mathbb{C}^n$. The coarsening morphism $\lambda \to \lambda_{\mathrm{st}(J)}$ provides a natural transformation from the identity to $\iota_2 r_2$, which establishes the second desired equivalence. $\square$

Proof of Proposition 2.1. Let μ denote the decomposition of $\mathbb{C}^n$ into the canonical isotypical components of J. If J is polytypic, then μ has more than one component and thus is proper, and μ is terminal in $(\mathcal{L}_n)^J_{\mathrm{iso}}$.

We assert that μ is a terminal object in $(\mathcal{L}_n)^J_{\mathrm{iso}}\ \cap\ (\mathcal{L}_n)^H$, and we need only establish that μ is weakly H-fixed. But μ is the J-isotypical refinement of the indiscrete partition of $\mathbb{C}^n$, i.e., the partition consisting of just $\mathbb{C}^n$ itself. Since the indiscrete partition is certainly H-fixed, Lemma 2.5 tells us that μ is weakly H-fixed.

The result now follows from Proposition 2.6, because the assumption that the action of J on $\mathrm{cl}(\lambda)$ is always intransitive means that $\lambda_{\mathrm{st}(J)}$ is always proper. $\square$

3. Finding a normal subgroup

Throughout this section, suppose that $H \subseteq U(n)$ is a p-toral subgroup of $U(n)$ whose image in $PU(n)$ is $\overline{H}$. Note that although $PU(n)$ does not act on $\mathbb{C}^n$, it does act on $\mathcal{L}_n$, because the central $S^1 \subseteq U(n)$ stabilizes any subspace of $\mathbb{C}^n$. Hence, for example, if a partition λ is weakly H-fixed, we can speak of the action of $\overline{H}$ on $\mathrm{cl}(\lambda)$.

Our goal is to prove Theorem 1.2, which says that if H is problematic, then $\overline{H}$ is elementary abelian and χ_H is zero away from the center of $U(n)$. The plan for the proof is to apply Proposition 2.1 after locating a relevant normal polytypic subgroup of H. The following lemma gives us a starting point, and the proof of Theorem 1.2 appears at the end of the section.

Lemma 3.1. *If $J \subseteq U(n)$ is abelian, then either J is polytypic or $J \subseteq S^1$.*

Proof. Decompose $\mathbb{C}^n$ into a sum of J-irreducible representations, all of which are necessarily one-dimensional because J is abelian. If J is isotypic, then an element $j \in J$ acts on every one-dimensional summand by multiplication by the same scalar, so $j \in S^1$. $\square$

Lemma 3.1 places an immediate restriction on problematic subgroups.

Lemma 3.2. *If H is a problematic p-toral subgroup of $U(n)$, then $\overline{H}$ is discrete.*

Proof. The group $\overline{H}$ is p-toral (e.g., [BJL+], Lemma 3.3), so its identity component, denoted $\overline{H}_0$, is a torus. Let K denote the inverse image of $\overline{H}_0$ in H, and note that $K \triangleleft H$ because $\overline{H}_0 \triangleleft \overline{H}$. Let J be the identity component of K; then $J \triangleleft H$ as well, since the conjugation action of H on K must stabilize the identity component of K. Further, J is a torus, because J is a connected closed subgroup of the identity component of H, which is a torus. If $\overline{H}$ is not discrete, then J is not contained in S^1 and thus is polytypic by Lemma 3.1. Since J is connected, its action on the set of equivalence classes of any proper partition is trivial. The lemma follows from Proposition 2.1. $\square$

In terms of progress towards Theorem 1.2, we now know that if H is a problematic p-toral subgroup of $U(n)$, then $\overline{H}$ must be a finite p-group. The next part of our strategy is to show that if $\overline{H}$ is not an elementary abelian p-group, then $\overline{H}$ has a normal subgroup V satisfying the conditions of the following lemma.

Lemma 3.3. *Let $H \subseteq U(n)$, and assume there exists $V \lhd \overline{H}$ such that $V \cong \mathbb{Z}/p$ and V does not act transitively on* $\mathrm{cl}(\lambda)$ *for any $\lambda \in (\mathcal{L}_n)^H$. Then $(\mathcal{L}_n)^H$ is contractible.*

Proof. Let J be the inverse image of V in H. Then $J \lhd H$, and because the action of J on $\mathcal{L}_n$ factors through V, the action of J on $\mathrm{cl}(\lambda)$ is not transitive for any $\lambda \in (\mathcal{L}_n)^H$. Further, J is abelian, because a routine splitting argument shows $J \cong V \times S^1$ (see [BJL$^+$], Lemma 3.1). Therefore J is polytypic by Lemma 3.1, and the lemma follows from Proposition 2.1. □

The following notation will be helpful in looking for subgroups satisfying the conditions of Lemma 3.3.

Definition 3.4. For a compact Lie group G, let G/p denote the maximal quotient of G that is an elementary abelian p-group.

Thus G/p is the quotient of G by the normal subgroup generated by commutators, p-th powers, and the connected component of the identity. The map $G \to G/p$ is initial among maps from G to elementary abelian p-groups. Observe that G/p is the analogue for a compact Lie group of the quotient of a finite group by its p-Frattini subgroup[Gor68].

We now have all the ingredients we require to prove Theorem 1.2, which we restate here for convenience. We note that both the statement and the proof are closely related to those of Proposition 6.1 of [ADL13].

Theorem 1.2. Let H be a problematic p-toral subgroup of $U(n)$. Then

(1) H is a projective elementary abelian p-group, and

(2) the character of H is $\begin{cases} \chi_H(h) = 0 & h \notin S^1 \\ \chi_H(h) = nh & h \in H \cap S^1. \end{cases}$

Proof. For (1), we must show that $\overline{H}$ is an elementary abelian p-group. By Lemma 3.2, we know that $\overline{H}$ is a finite p-group. If $\overline{H}$ is not elementary abelian, then the kernel of $\overline{H} \to \overline{H}/p$ is a nontrivial normal p-subgroup of $\overline{H}$, and thus has nontrivial intersection with the center of $\overline{H}$. Choose $V \cong \mathbb{Z}/p$ with

$$V \subseteq \ker\left(\overline{H} \to \overline{H}/p\right) \cap Z\left(\overline{H}\right).$$

We assert that V satisfies the conditions of Lemma 3.3. Certainly $V \lhd \overline{H}$, because V is contained in the center of $\overline{H}$. Suppose that there exists $\lambda \in (\mathcal{L}_n)^H$ such that V acts transitively on $\mathrm{cl}(\lambda)$. The set $\mathrm{cl}(\lambda)$ must then have exactly p elements. The action of $\overline{H}$ on $\mathrm{cl}(\lambda)$ induces a map $\overline{H} \to \Sigma_p$, and since $\overline{H}$ is a p-group, the image of this map must lie in a Sylow p-subgroup of Σ_p. But then the map $\overline{H} \to \Sigma_p$ factors as

$$\overline{H} \to \overline{H}/p \to \mathbb{Z}/p \hookrightarrow \Sigma_p,$$

with $V \subseteq \overline{H}$ mapping nontrivially to $\mathbb{Z}/p \subseteq \Sigma_p$. We have thus contradicted the assumption that $V \subseteq \ker\left(\overline{H} \to \overline{H}/p\right)$. We conclude that $\overline{H}$ is in fact an elementary abelian p-group, thus (1) is proved.

For (2), first note that if $h \in S^1$, then its matrix representation in $U(n)$ is hI, hence $\chi_H(h) = \mathrm{tr}(hI) = nh$. Suppose that $\overline{H}$ is an elementary abelian p-group, and let $h \in H$ such that $h \notin S^1$. The image of h in $\overline{H}$ generates a subgroup $V \cong \mathbb{Z}/p \subseteq \overline{H}$, and V is a candidate for applying Lemma 3.3. Since $(\mathcal{L}_n)^H$ is not contractible, there must be a partition $\lambda \in (\mathcal{L}_n)^H$ such that V acts transitively on $\mathrm{cl}(\lambda)$. The action of V on $\mathrm{cl}(\lambda)$ is necessarily free because $V \cong \mathbb{Z}/p$, so a basis for $\mathbb{C}^n$ can be constructed that is invariant under V and consists of bases for the subspaces in $\mathrm{cl}(\lambda)$. This represents h as a fixed-point free permutation of a basis of $\mathbb{C}^n$. Hence $\chi_H(h) = 0$. $\square$

4. The subgroups Γ_{p^k} of $U(p^k)$

Theorem 1.2 tells us that if H is a problematic p-toral subgroup of $U(n)$, then H is a projective elementary abelian p-group and the character of H is zero away from the center of $U(n)$. In fact, there are well-known subgroups of $U(n)$ that satisfy these conditions, namely the p-stubborn subgroups $\Gamma_k \subseteq U(p^k)$ that arise in, for example, [Gri91], [JMO92], [Oli94], [Aro02], [AL07], [AGMV08], and others. In this section, we review background on the groups Γ_k.

We begin with the discrete analogue of Γ_k. Let $n = p^k$ and choose an identification of the elements of $(\mathbb{Z}/p)^k$ with the set $\{1, ..., n\}$. The action of $(\mathbb{Z}/p)^k$ on itself by translation identifies $(\mathbb{Z}/p)^k$ as a transitive elementary abelian p-subgroup of Σ_{p^k}, denoted Δ_k. Up to conjugacy, Δ_k is the unique transitive elementary abelian p-subgroup of Σ_{p^k}. Note that every nonidentity element of Δ_k acts without fixed points. The embedding

$$\Delta_k \hookrightarrow \Sigma_{p^k} \hookrightarrow U\left(p^k\right)$$

given by permuting the standard basis elements is the regular representation of Δ_k, and has character $\chi_{\Delta_k} = 0$ except at the identity.

In the unitary context, the projective elementary abelian p-subgroup $\Gamma_k \subseteq U\left(p^k\right)$ is generated by the central $S^1 \subseteq U\left(p^k\right)$ and two different embeddings of Δ_k in $U\left(p^k\right)$, which we denote $\mathcal{A}_k$ and $\mathcal{B}_k$ and describe momentarily. Just as Δ_k is the unique (up to conjugacy) elementary abelian p-subgroup of Σ_{p^k} with transitive action, it turns out that Γ_k is the unique (up to conjugacy) projective elementary abelian p-subgroup of $U(p^k)$ containing the central S^1 and having irreducible action (see, for example, [Zol02]). For the explicit description of Γ_k, we follow [Oli94].

The subgroup $\mathcal{B}_k \cong \Delta_k$ of $U(p^k)$ is given by the regular representation of Δ_k. Explicitly, for any $r = 0, 1, \ldots, k-1$, let $\sigma_r \in \Sigma_{p^k}$ denote the permutation defined by

$$\sigma_r(i) = \begin{cases} i + p^r & \text{if } i \equiv 1, \ldots, (p-1)p^r \mod p^{r+1}, \\ i - (p-1)p^r & \text{if } i \equiv (p-1)p^r + 1, \ldots, p^{r+1} \mod p^{r+1}. \end{cases}$$

For each r, let $B_r \in U(p^k)$ be the corresponding permutation matrix,

$$(B_r)_{ij} = \begin{cases} 1 & \text{if } \sigma_r(i) = j \\ 0 & \text{if } \sigma_r(i) \neq j. \end{cases}$$

For later purposes, we record the following lemma.

Lemma 4.1. *The character $\chi_{\mathcal{B}_k}$ is zero except at the identity.*

Proof. Every nonidentity element of $\mathcal{B}_k$ acts by a fixed-point free permutation of the standard basis of $\mathbb{C}^{p^k}$, and so has only zeroes on the diagonal. □

Our goal is to define Γ_k with irreducible action on $\mathbb{C}^{p^k}$, but $\mathcal{B}_k$ alone clearly does not act irreducibly: being abelian, the subgroup $\mathcal{B}_k$ has only one-dimensional irreducibles. In fact, since $\mathcal{B}_k$ is acting on $\mathbb{C}^{p^k}$ by the regular representation, each of its p^k irreducible representations is present exactly once. The role of the other subgroup $\mathcal{A}_k \cong \Delta_k \subseteq \Gamma_k$ is to permute the irreducible representations of $\mathcal{B}_k$. To be specific, let $\zeta = e^{2\pi i/p}$, and consider $\mathbb{Z}/p \subseteq U(p)$ generated by the diagonal matrix with entries $1, \zeta, \zeta^2, ..., \zeta^{p-1}$. Then $\mathcal{A}_k \subseteq U(p^k)$ is the group $(\mathbb{Z}/p)^k$ acting on the k-fold tensor power $(\mathbb{C}^p)^{\otimes k}$. Explicitly, for $r = 0, \ldots, k-1$ define $A_r \in U(p^k)$ by

$$(A_r)_{ij} = \begin{cases} \zeta^{[(i-1)/p^r]} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where $[-]$ denotes the greatest integer function. The matrices $A_0,...,A_{k-1}$ commute, are of order p, and generate a rank k elementary abelian p-group $\mathcal{A}_k$.

Lemma 4.2. *The character $\chi_{\mathcal{A}_k}$ is zero except at the identity.*

Proof. The character of $\mathbb{Z}/p \subseteq U(p)$ generated by the diagonal matrix with entries $1, \zeta, \zeta^2, ..., \zeta^{p-1}$ is zero away from the identity by direct computation. The same is true for $\mathcal{A}_k$ because the character is obtained by multiplying together the characters of the individual factors. $\square$

Since the characters $\chi_{\mathcal{A}_k}$ and $\chi_{\mathcal{B}_k}$ are the same, we obtain the following corollary.

Corollary 4.3. *The subgroups $\mathcal{A}_k$ and $\mathcal{B}_k$ are conjugate in $U(p^k)$.*

Although $\mathcal{A}_k$ and $\mathcal{B}_k$ do not quite commute in $U(p^k)$, the commutator relations are simple and can be established by direct computation:

$$\begin{aligned}
&[A_r, A_s] = I = [B_r, B_s], && \text{for all } r, s \\
&[A_r, B_s] = I, && \text{for all } r \neq s \\
&[B_r, A_r] = \zeta I, && \text{for all } r.
\end{aligned} \tag{4.4}$$

Definition 4.5. The subgroup $\Gamma_k \subseteq U(p^k)$ is generated by the subgroups $\mathcal{A}_k$, $\mathcal{B}_k$, and the central $S^1 \subseteq U(p^k)$.

Lemma 4.6. *There is a short exact sequence*

$$1 \to S^1 \to \Gamma_k \to (\Delta_k \times \Delta_k) \to 1. \tag{4.7}$$

where S^1 is the center of $U(p^k)$.

Proof. The subgroups $\mathcal{A}_k$ and $\mathcal{B}_k$ can be taken as the preimages of the two copies of Δ_k in Γ_k. The commutator relations (4.4) show that $\mathcal{A}_k$ and $\mathcal{B}_k$ do not generate any noncentral elements. $\square$

Remark. When $k = 0$ we have $\Gamma_0 = S^1 \subseteq U(1)$, and Δ_0 is trivial, so Lemma 4.6 is true even for $k = 0$.

For later purposes, we record the following lemma.

Lemma 4.8. *The subgroup $\Gamma_k \subseteq U(p^k)$ contains subgroups isomorphic to $\Gamma_s \times \Delta_t$ for all nonnegative integers s and t such that $s + t \leq k$.*

Proof. The required subgroup is generated by S^1, the matrices $A_0, \ldots, A_{s+t-1}$, and the matrices $B_0, \ldots, B_{s-1}$. $\square$

A consequence of Lemma 4.6 is that $\Gamma_k \subseteq U(p^k)$ is an example of a p-toral subgroup that satisfies the first conclusion of Theorem 1.2. The next lemma says that Γ_k satisfies the second conclusion as well.

Lemma 4.9. *The character of Γ_k is nonzero only on the elements of S^1.*

Proof. The character on elements of $\mathcal{A}_k$ and $\mathcal{B}_k$ is zero by Lemmas 4.1 and 4.2. Products of nonidentity elements of $\mathcal{B}_k$ with elements of $\mathcal{A}_k$ and/or the center are obtained by multiplying matrices in $\mathcal{B}_k$, which have only zero entries on the diagonal, by diagonal matrices. The resulting products likewise have no nonzero diagonal entries and thus have zero trace. □

Remark. We note that, by inspection of the commutator relations, $S^1 \times \mathcal{A}_k$ and $S^1 \times \mathcal{B}_k$ normalize each other in Γ_k. Hence the partition μ of $\mathbb{C}^{p^k}$ by the p^k one-dimensional irreducible representations of $\mathcal{B}_k$ is weakly fixed by $\mathcal{A}_k$. The classes of μ are permuted transitively by $\mathcal{A}_k$. Likewise, $\mathcal{B}_k$ transitively permutes the p^k classes of the partition of $\mathbb{C}^{p^k}$ given by the irreducible representations of $\mathcal{A}_k$.

5. Alternating forms

From Theorem 1.2, we know that if H is a problematic p-toral subgroup of $U(n)$, then its image $\overline{H}$ in $PU(n)$ is an elementary abelian p-group. For our purposes, it is sufficient to assume that H contains S^1, because $H \subseteq HS^1$, the images of HS^1 and H in $PU(n)$ are the same, and $(\mathcal{L}_n)^H = (\mathcal{L}_n)^{HS^1}$. We would like a classification of projective elementary abelian subgroups H of $U(n)$ that contain S^1, up to abstract group isomorphism.

Before proceeding, we note that the remarkable paper [AGMV08] of Andersen-Grodal-Møller-Viruel classifies non-toral elementary abelian p-subgroups (for odd primes p) of the simple and center-free Lie groups. In particular, Theorem 8.5 of that work contains a classification of all the elementary abelian p-subgroups of $PU(n)$, building on earlier work of Griess [Gri91]. Our approach is independent of this classification, and works for all primes, using elementary methods.

We make the following definition.

Definition 5.1. A p-toral group H is an *abstract projective elementary abelian p-group* if H can be written as a central extension

$$1 \to S^1 \to H \to V \to 1,$$

where V is an elementary abelian p-group.

We use bilinear forms to classify abstract elementary abelian p-groups up to group isomorphism (Propositions 5.10 and 5.11). Once the group-theoretic classification is complete, we use representation theory in Section 6 to pin down the conjugacy classes of elementary abelian p-subgroups of $U(n)$ that can be problematic.

We begin by recalling some background on forms. Let A be a finite-dimensional $\mathbb{F}_p$-vector space, and let $\alpha\colon A \times A \to \mathbb{F}_p$ be a bilinear form. We say that α is *totally isotropic* if $\alpha(a,a) = 0$ for all $a \in A$. (A totally isotropic form is necessarily skew-symmetric, as seen by expanding $\alpha(a+b, a+b) = 0$, but the reverse is not true for $p = 2$.) If α is not only totally isotropic, but also non-degenerate, then it is called a *symplectic form.* Any vector space with a symplectic form is even-dimensional and has a (nonunique) basis $e_1, \ldots, e_s, f_1, \ldots, f_s$, called a *symplectic basis*, with the property that $\alpha(e_i, e_j) = 0 = \alpha(f_i, f_j)$ for all i, j, and $\alpha(e_i, f_j)$ is 1 if $i = j$ and zero otherwise. All symplectic vector spaces of the same dimension are isomorphic (i.e., there exists a linear isomorphism that preserves the form), and if the vector space has dimension $2s$ we use $\mathbb{H}_s$ to denote the associated isomorphism class of symplectic vector spaces. Let $\mathbb{T}_t$ denote the vector space of dimension t over $\mathbb{F}_p$ with trivial form. We have the following standard classification result.

Lemma 5.2. *Let A be a vector space over $\mathbb{F}_p$ with a totally isotropic bilinear form α. Then there exist s and t such that $A \cong \mathbb{H}_s \oplus \mathbb{T}_t$ by a form-preserving isomorphism.*

Our next task is to relate the preceding discussion to abstract projective elementary abelian p-groups. For the remainder of the section, assume that H is an abstract elementary abelian p-group

$$1 \to S^1 \to H \to V \to 1.$$

Choose an identification of $\mathbb{Z}/p$ with the elements of order p in S^1. Given $x, y \in V$, let $\tilde{x}, \tilde{y}$ be lifts of x, y to H. Define the *commutator form associated to H* as the form on V defined by

$$\alpha(x,y) = [\tilde{x}, \tilde{y}] = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}.$$

Lemma 5.3. *Let H and α be as above, and suppose that $x, y \in V$. Then*

(1) *$\alpha(x,y)$ is a well-defined element of $\mathbb{Z}/p \subseteq S^1$, and*
(2) *α is a totally isotropic bilinear form on V.*

Proof. Certainly $[\tilde{x}, \tilde{y}] \in S^1$, since x and y commute in V. If $\zeta \in S^1$ then $[\zeta\tilde{x}, \tilde{y}] = [\tilde{x}, \tilde{y}] = [\tilde{x}, \zeta\tilde{y}]$ because S^1 is central in H, which shows that α is independent of the choice of lifts $\tilde{x}$ and $\tilde{y}$, and that α is linear with respect to scalar multiplication.

To show that $[\tilde{x}, \tilde{y}]$ has order p, we note that commutators in a group satisfy the following versions of the Hall-Witt identities, as can be verified by expanding and simplifying:

$$[a, bc] = [a, b] \cdot \big[b, [a, c]\big] \cdot [a, c]$$
$$[ab, c] = \big[a, [b, c]\big] \cdot [b, c] \cdot [a, c].$$

We know that $[H, H]$ is contained in the center of H, so $[H, [H, H]]$ is the trivial group. Hence for H, the identities reduce to

$$\begin{aligned}[a, bc] &= [a, b][a, c] \\ [ab, c] &= [a, c][b, c].\end{aligned} \tag{5.4}$$

In particular, $[\tilde{x}, \tilde{y}]^p = [\tilde{x}, \tilde{y}^p] = e$, since $\tilde{y}^p \in S^1$ commutes with $\tilde{x}$.

Finally, bilinearity of α with respect to addition follows directly from (5.4). The form is totally isotropic because an element commutes with itself. □

Lemma 5.3 takes an abstract projective elementary abelian p-group H and associates to it a commutator form α. Conversely, we can start with a form α and directly construct an abstract projective elementary abelian p-group H_α. Let $\alpha\colon V \times V \to \mathbb{Z}/p$ be a totally isotropic bilinear form, which can be regarded as a function $\alpha\colon V \times V \to S^1$. Let

$$\begin{aligned} V_k &= \{v \in V \mid \alpha(v, x) = 0 \ \text{ for all } x \in V\} \\ &= \ker\left(V \to V^*\right), \end{aligned}$$

where V^* denotes the dual of V. Then α restricted to V_k is trivial. Choose a complement V' to V_k in V, which is necessarily orthogonal to V_k. Since α must be symplectic on V', we can choose a symplectic basis $e_1, \ldots, e_r, f_1, \ldots, f_r$ for V'; let V_e and V_f denote the spans of $\{e_1, \ldots, e_r\}$ and $\{f_1, \ldots, f_r\}$, respectively. By construction, we can write any $v \in V$ uniquely as a sum $v = v_k + v_e + v_f$ where $v_k \in V_k$, $v_e \in V_e$, and $v_f \in V_f$.

Let H_α be the set $S^1 \times V$. We endow H_α with the following operation:

$$(z, v) *_\alpha (z', v') = \big(zz'\,\alpha(v_f, v'_e),\ v + v'\big)\,. \tag{5.5}$$

Proposition 5.6. *For an elementary abelian p-group V and a totally isotropic bilinear form $\alpha\colon V \times V \to \mathbb{Z}/p$, the operation* (5.5) *defines an abstract projective elementary abelian p-group H_α whose associated commutator pairing is α. If α and α' are isomorphic as forms, then H_α and $H_{\alpha'}$ are isomorphic as groups.*

Proof. The element $(1, 0)$ serves as the identity in H_α. By bilinearity of α, we know $\alpha\left(v_f, -v_e\right) + \alpha\left(v_f, v_e\right) = 0$, which allows us to check that the inverse of (z, v) is $(z^{-1}\alpha\left(v_f, v_e\right), -v)$. A straightforward computation verifies associativity, showing that $*_\alpha$ defines a group law, and another shows that H_α has α for its commutator pairing. Finally, the construction of H_α guarantees that isomorphic forms α and α' give isomorphic groups H_α and $H_{\alpha'}$. $\square$

To show that totally isotropic forms over $\mathbb{Z}/p$ classify abstract elementary abelian p-groups, it remains to show that nonisomorphic groups cannot have the same commutator form. The following proposition gives us this final piece.

Proposition 5.7. *Let H be an abstract projective elementary abelian p-group with associated commutator form $\alpha\colon V \times V \to \mathbb{Z}/p$. Then H is isomorphic to H_α.*

Proof. Let V_k, V_e, V_f be the subspaces of V defined just prior to Proposition 5.6. The basis elements of V_e can be lifted (nonuniquely) to elements of H of order p, and the lifts commute since the form is trivial on V_e. Mapping basis elements of V_e to their lifts in H gives a monomorphism of groups $V_e \hookrightarrow H$ whose image we call W_e. Likewise, we can choose lifts $V_k \hookrightarrow H$ and $V_f \hookrightarrow H$, whose images are subgroups W_k and W_f of H, respectively.

Recall that as a set, $H_\alpha = S^1 \times V$, and for $v \in V$, we have $v = v_k + v_e + v_f$ as before. Let w_k, w_e, and w_f be the images of v_k, v_e, and v_f under the lifting homomorphisms of the previous paragraph. Define a function $\phi\colon H_\alpha \to H$ by

$$\phi\,(z, v_k + v_e + v_f) = z\, w_k w_e w_f.$$

(Note that we write the group operation additively in V, which is abelian, but multiplicatively in H, which may not be.)

We assert that ϕ is a group homomorphism. To see that, suppose we have two elements (z, v) and (z, v') of H_α. If we multiply first in H_α we get $\big(zz'\,\alpha(v_f, v_e'),\ v + v'\big)$, and then application of ϕ gives us

$$zz'\,\alpha(v_f, v_e')(v_k v_k')(v_e v_e')(v_f v_f'). \tag{5.8}$$

On the other hand, if we apply ϕ first and then multiply, we get $(z\, v_k v_e v_f)\,\big(z'\, v_k' v_e' v_f'\big)$, which can be rewritten as

$$zz'\ (v_k v_k')\,\big(v_e v_f v_e' v_f'\big) \tag{5.9}$$

To compare (5.8) to (5.9), we need to relate $v_e' v_f$ and $v_f v_e'$. However, the commutators in H are given exactly by α, so $v_f v_e' = \alpha(v_f, v_e') v_e' v_f$, which allows us to see that (5.8) and (5.9) are equal. We conclude that ϕ is a group homomorphism.

Finally, we need to know that ϕ is a bijection. To see that ϕ is surjective, observe that if $h \in H$ maps to $v \in V$ where $v = v_k + v_e + v_f$, then h and $w_k w_e w_f$ differ only by some element z of the central S^1. Hence every element of H can we written as $z w_k w_e w_f$ for some z, w_k,

w_e, w_f. To see that ϕ is injective, note that $zw_kw_ew_f$ is the identity only if $w_kw_ew_f \in S^1$, which implies $v = v_k + v_e + v_f = 0 \in V$. But then $v_k = v_e = v_f = 0 \in V$, and because the lifts of V_k, V_e, and V_f are homomorphisms, we find that w_k, w_e, and w_f are the identity in H. It follows that z is also, and ϕ is injective. □

Proposition 5.10. *The commutator form gives a one-to-one correspondence between isomorphism classes of abstract projective elementary abelian p-groups and isomorphism classes of totally isotropic bilinear forms over $\mathbb{Z}/p$.*

Proof. Every totally isotropic form α is realized as the commutator form of the group H_α. By Propositions 5.6 and 5.7, if H and H' have isomorphic commutator forms α and α', then

$$H \cong H_\alpha \cong H_{\alpha'} \cong H'.$$

□

Proposition 5.10 allows us to give an explicit classification of abstract projective elementary abelian p-groups. The classification result can also be found in [Gri91] Theorem 3.1, though in this section we have given an elementary and self-contained discussion and proof.

Proposition 5.11. *Suppose that H is an abstract projective elementary abelian p-group*

$$1 \to S^1 \to H \to V \to 1.$$

Let $2s$ be the maximal rank of a symplectic subspace of V under the commutator form of H, and let $t = \mathrm{rk}(V) - 2s$. Then H is isomorphic to $\Gamma_s \times \Delta_t$.

Proof. The commutator form of $\Gamma_s \times \Delta_t$ is isomorphic to that of H, so the result follows from Proposition 5.10. To interpret the proposition when $s = 0$, note that $\Gamma_0 = S^1$, so the proposition says that if $s = 0$, then $H \cong S^1 \times \Delta_t \cong S^1 \times V$. □

Remark 5.12. As pointed out by D. Benson, one could also approach this classification result via group cohomology, using the fact that equivalence classes of extensions as in Definition 5.1 correspond to elements of $H^2(V; S^1)$. An argument using the Bockstein homomorphism shows that the group $H^2(V; S^1) \cong H^3(V, \mathbb{Z})$ can be identified with the exterior square of $H^1(V, \mathbb{Z}/p)$, which is in turn isomorphic to the space of alternating forms $V \times V \to \mathbb{Z}/p$. The standard factor set approach to H^2 can be used to identify such a form with the commutator pairing of the extension (as in [Bro94] or [Wei94]). The factor set associated to an extension is similarly defined and often identically denoted as the commutator pairing, but the two pairings are *not* the same. In particular, a factor set need not be bilinear or totally isotropic.

6. Classification Theorem

Suppose that H is a problematic p-toral subgroup of $U(n)$. We know from the first part of Theorem 1.2 that H must be a projective elementary abelian p-group. If H contains $S^1 \subseteq U(n)$, we know the possible group isomorphism classes of H from Proposition 5.11. The purpose of this section is to use the character criterion of Theorem 1.2 to determine the possible conjugacy classes of H in $U(n)$.

We begin by defining certain p-toral subgroups of $U(n)$.

Definition 6.1. For an integer $m > 0$, let $\mathrm{Diag}_m(\Gamma_k)$ be the image of the following composite of inclusions:

$$\Gamma_k \longrightarrow U(p^k) \xrightarrow{\Delta} \prod_{i=1}^{m} U(p^k) \longrightarrow U\left(mp^k\right).$$

Because $\mathrm{Diag}_m(\Gamma_k)$ is represented by block diagonal matrices with blocks Γ_k, we immediately obtain the following from Lemma 4.9.

Lemma 6.2. *The character of* $\mathrm{Diag}_m(\Gamma_k)$ *is nonzero only on the elements of* S^1.

Our goal is to show that if H is a problematic subgroup of $U(n)$, then H is a subgroup of $\mathrm{Diag}_m(\Gamma_k)$ where $n = mp^k$ and m is prime to p. Although H itself may not be finite, we can use its finite subgroups to get information about n using the following basic representation theory.

Lemma 6.3. *Suppose that G is a finite subgroup of $U(n)$ and that $\chi_G(y) = 0$ unless $y = e$. Then $|G|$ divides n, and the action of G on $\mathbb{C}^n$ is by $n/|G|$ copies of the regular representation.*

Proof. The number of copies of an irreducible character χ in χ_G is given by the inner product

$$\langle \chi_G, \chi \rangle = \frac{1}{|G|} \sum_{y \in G} \chi_G(y) \overline{\chi(y)}.$$

Take χ to be the character of the one-dimensional trivial representation of G. The only nonzero term in the summation occurs when $y = e$, and since $\chi_G(e) = n$, we find $\langle \chi_G, \chi \rangle = n/|G|$. Since $\langle \chi_G, \chi \rangle$ must be an integer, we find that $|G|$ divides n. To finish, we observe that the character of $n/|G|$ copies of the regular representation is the same as χ_G, which finishes the proof. $\square$

We now have all the ingredients we require to prove the main result of this paper.

Theorem 1.3. Let H be a problematic p-toral subgroup of $U(n)$, and suppose that $n = mp^k$, where m and p are coprime. Then H is conjugate to a subgroup of $\mathrm{Diag}_m(\Gamma_k) \subseteq U(n)$.

Proof. By definition, if H is problematic, then $(\mathcal{L}_n)^H$ is not contractible. We assert that, without loss of generality, we can assume that H actually contains S^1. This is because $(\mathcal{L}_n)^H = (\mathcal{L}_n)^{HS^1}$ so H is problematic if and only if HS^1 is problematic. Likewise, H is a projective elementary abelian p-group if and only if HS^1 has the same property. Lastly, $H \subseteq \mathrm{Diag}_m(\Gamma_k)$ if and only if $HS^1 \subseteq \mathrm{Diag}_m(\Gamma_k)$, since $S^1 \subseteq \mathrm{Diag}_m(\Gamma_k)$. Hence we assume that $S^1 \subseteq H$, replacing H by HS^1 if necessary.

We know from Theorem 1.2 that H is an abstract projective elementary elementary abelian p-group. By Proposition 5.11, H is abstractly isomorphic to $\Gamma_s \times \Delta_t$ for some s and t. Then H contains a subgroup $(\mathbb{Z}/p)^{s+t}$ (say, $\mathcal{A}_s \times \Delta_t$), so p^{s+t} divides n by Lemma 6.3.

Since $n = mp^k$ with m coprime to p, we necessarily have $s + t \leq k$. Hence by Lemma 4.8, we know $\Gamma_s \times \Delta_t \subseteq \Gamma_k$. Then the subgroup $\mathrm{Diag}_m(\Gamma_k) \subseteq U(n)$ contains a subgroup $\Gamma_s \times \Delta_t$, which is abstractly isomorphic to H and has the same character by Theorem 1.2 and Lemma 6.2. We conclude that H is conjugate in $U(n)$ to this subgroup $\Gamma_s \times \Delta_t \subseteq \mathrm{Diag}_m(\Gamma_k)$, which completes the proof. □

Example 6.4. Suppose that p is an odd prime, and let $n = 2p$. Let H be a problematic subgroup of $U(2p)$ acting on $\mathcal{L}_{2p}$. According to Theorem 1.3, the subgroup H is conjugate in $U(2p)$ to a subgroup of $\mathrm{Diag}_2(\Gamma_1)$. If in addition we assume that H contains the central S^1, which does not change the fixed point set of H acting on $\mathcal{L}_{2p}$, then there are only three possibilities for H: S^1 itself, or the images under the diagonal map $\Gamma_1 \xrightarrow{\cong} \mathrm{Diag}_2(\Gamma_1)$ of Γ_1 or of $S^1 \times \Delta_1$.

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