Putnam Practice Problems

Those taking the Putnam Exam are generally expected to know some "basic" inequalities. The first one (AM-GM) has been used to solve several recent Putnam problems.

(1) **Arithmetic Mean - Geometric Mean Inequality** For n nonnegative values, a_i,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}$$

The tricky part in applying this inequality in a particular problem is determining the n values to which to apply this inequality.

(2) **The Cauchy-Schwarz Inequality**

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$$

Often, this is presented as

$$a_1 b_1 + \cdots + a_n b_n \leq \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}$$

Note to Math 115 students (past or present): Can you rewrite this using norms and dot products?

(3) **Jensen's Inequality** Let $f(x)$ be a *convex* function (in calculus parlance, a concave-up function) on an interval in $\mathbb{R}$. Let x_1 and x_2 be two points in the interval. Then

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

How can you remember this? Think graphically. This inequality says that for a convex function, the image of the midpoint of two input points is "lower" or less than the midpoint of the images. More generally

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

What happens for *concave* (in calculus, concave down) functions? The direction of the inequalities reverse.

The tricky part in applying this inequality in a particular problem is determining what function to use.

Now try the following:

(1) Prove $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$

(2) For $a, b, c \geq 0$, prove $(a+b)(b+c)(c+a) \geq 8abc$. (Hint: What does the AM-GM say when $n = 2$?)

(3) (Putnam 2003) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$

(4) (Putnam 2005) Find all positive integers $n, k_1, \ldots, k_n$ such that

$$k_1 + \cdots + k_n = 5n - 4$$
$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$