

# NULLSTELLEN AND SUBDIRECT REPRESENTATION

Walter Tholen

York University, Toronto, Canada

Union College  
19-20 October 2013

# HNB THEOREM — VERSION 1

System of  $r$  polynomial equations in  $n$  variables with coefficients in a field  $k$ :

$$\begin{array}{ll} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_r(x_1, \dots, x_n) = 0 \end{array} \quad (f_i \in k[x_1, \dots, x_n]) \quad (\star)$$

## EXAMPLES

$$\begin{array}{llll} x_1 - x_2^2 = 0 & x_1^2 - 1 = 0 & x_1^2 + 1 = 0 & x_1^2 - 1 = 0 \\ & & & x_1^2 + 1 = 0 \end{array}$$

## D. HILBERT (1893)

Every system  $(\star)$  has a solution  $a = (a_1, \dots, a_n)$  in  $F^n$  for any algebraically-closed extension field  $F$  of  $k$ ,

unless there are polynomials  $g_1, \dots, g_r \in k[x]$  with

$$g_1 f_1 + \dots + g_r f_r = 1$$

Restriction is essential:

$$0 = g_1(a)f_1(a) + \dots + g_r(a)f_r(a) = 1$$

# A GALOIS CORRESPONDENCE

## Systems of equations

## Solution sets

$$P \subseteq k[x_1, \dots, x_n] \quad \longmapsto \quad S(P) = \{a \in k^n \mid \forall f \in P : f(a) = 0\}$$

$$J(X) = \{f \in k[x] \mid \forall a \in X : f(a) = 0\} \quad \longleftarrow \quad X \subseteq F^n$$

$$P \subseteq J(X) \quad \Longleftrightarrow \quad X \subseteq S(P)$$

$$P = J(S(P)) \quad \Longleftrightarrow \quad X = S(J(X))$$

Note:  $P \subseteq \sqrt{P} = \{f \in k[x] \mid \exists m \geq 1 : f^m \in P\}$

# HNB THEOREM — VERSION 2

$$\left. \begin{array}{l} P \trianglelefteq k[x_1, \dots, x_n] \text{ proper ideal} \\ F \text{ algebraically-closed} \end{array} \right\} \implies J(S(P)) = \sqrt{P}$$

That is :

$$\text{If } f \in k[x], \text{ then } (\forall a \in S(P) : f(a) = 0 \iff \exists m \geq 1 : f^m \in P)$$

$$(f_1, \dots, f_r) = P$$

$$\sqrt{P} = J(S(P)) \iff S(P)$$

$$k[x_1, \dots, x_n] \iff \emptyset$$

Note: Any  $P \trianglelefteq k[x_1, \dots, x_n]$  is finitely generated:

$$k \text{ Noetherian ring} \implies k[x] \text{ Noetherian}$$

## TRYING TO PROVE VERSION 2

Need to show  $J(S(P)) \subseteq \sqrt{P}$ :

If  $f \notin \sqrt{P}$ , must find  $a \in S(P)$  with  $f(a) \neq 0$ .

Consider  $A = k[x_1, \dots, x_n]/\sqrt{P}$

Wish: Find

$$\begin{array}{ccccc} k[x] & \xrightarrow{\pi} & A & \xrightarrow{\varphi} & F \\ f \vdash & \longrightarrow & \pi(f) = u & \vdash & \varphi(u) \neq 0 \end{array}$$

Consider  $a = (\varphi(\pi(x_1)), \dots, \varphi(\pi(x_n)))$ .

Then  $f(a) \neq 0$ , but  $\forall p \in P : p(a) = 0$ .

# HNB THEOREM — VERSION 3 AND 4

## Version 3

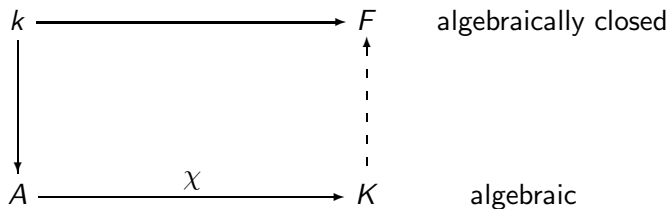
$$\left. \begin{array}{l} A \text{ fin. gen. comm. } k\text{-algebra} \\ k \subseteq F \text{ alg. closed} \\ u \in A \text{ non-nilpotent} \end{array} \right\} \implies \begin{array}{l} \exists \varphi : A \rightarrow F \\ u \mapsto \varphi(u) \neq 0 \end{array}$$

## Version 4

$$\left. \begin{array}{l} A \text{ fin. gen. comm. } k\text{-algebra} \\ u \in A \text{ non-nilpotent} \end{array} \right\} \implies \begin{array}{l} \exists \chi : A \rightarrow K, \quad \chi(u) \neq 0, \\ k \subseteq K \text{ field,} \\ K \text{ fin. gen. (as unital } k\text{-algebra)} \end{array}$$



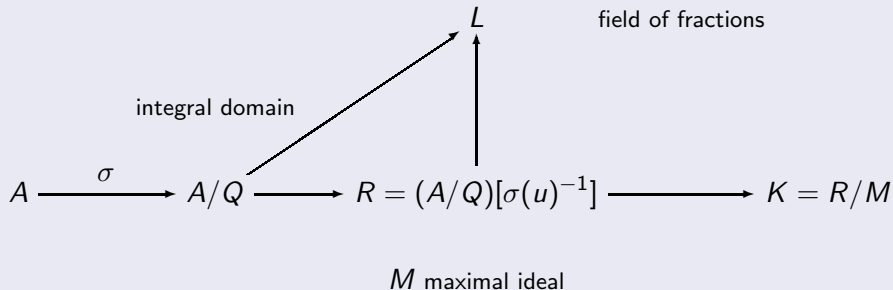
# VERSION 4 IMPLIES VERSION 3



# HNB THEOREM — VERSION 5, VERSION 5 IMPLIES VERSION 4

$$\left. \begin{array}{l} A \text{ comm. ring} \\ u \text{ non-nilpotent} \end{array} \right\} \implies \exists Q \trianglelefteq A \text{ prime, } u \notin Q$$

5  $\Rightarrow$  4



$$S := \{u^n \mid n \geq 1\} \not\subseteq 0$$

Zorn:  $\exists Q \trianglelefteq A$  maximal w.r.t.  $Q \cap S = \emptyset$

$Q$  is prime:  $a, b \notin Q$

$$\implies (a) + Q, (b) + Q \text{ meet } S$$

$$\implies \exists l, m : u^l \in (a) + Q, u^m \in (b) + Q$$

$$\implies u^{l+m} \in (ab) + Q$$

$$\implies ab \notin Q$$

QED

# DECOMPOSITION THEOREMS

$A$  comm., reduced  $\implies \bigcap \{Q \trianglelefteq A \mid Q \text{ prime}\} = (0)$

$$A \longmapsto \prod_Q A/Q$$

LASKER (1904), NOETHER (1920)

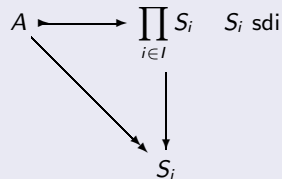
$\left. \begin{array}{l} R \text{ comm. Noetherian} \\ P \trianglelefteq R \text{ ideal} \end{array} \right\} \implies P = Q_1 \cap \dots \cap Q_n \text{ with } Q_i \trianglelefteq R \text{ irreducible}$

Alternative formulation of Version 5

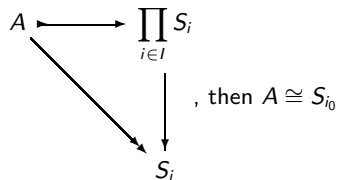
$R$  comm.,  $P \trianglelefteq R$  radical  $\iff P = \bigcap \{Q \trianglelefteq R \mid P \subseteq Q, Q \text{ prime}\}$

## BIRKHOFF'S SUBDIRECT REPRESENTATION THEOREM (1944)

A general finitary algebra  $\implies$



A subdirectly irreducible (sdi)  $:\iff$  If



This notion is categorical!

- $\mathcal{A}$  has (strong epi, mono)-factorizations
- $\mathcal{A}$  is weakly cowellpowered
- $\mathcal{A}$  has generator consisting of finitary objects

(no existence requirements for limits or colimits)

## EXAMPLES

- quasi-varieties of finitary algebras
- locally finitely presentable categories
- presheaf categories
- the category of topological spaces

# HNB THEOREM — VERSION 7

A object of an HNB-category  $\implies A$  has subdirect representation

$$A \longrightarrow \prod_{i \in I} S_i$$

$$S \text{ sdi} \iff \exists a \neq b : P \rightarrow S$$

$$\forall f : A \rightarrow B \ (fa \neq fb \Rightarrow f \text{ monic})$$

$$\iff \text{Con}S \setminus \{\Delta_S\} \text{ has a least element}$$

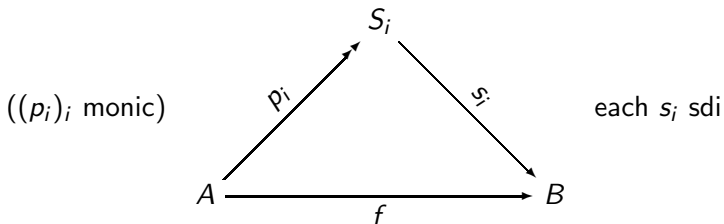
$$\underline{\text{Set}}: 2$$

$$k\text{-}\underline{\text{Vec}}: k$$

$$\underline{\text{CRng}}: \mathbb{Z}_p, \mathbb{Z}_{p^2}$$

# HNB THEOREM — VERSION 8

$f : A \rightarrow B$  morphism of an HNB-category with finite products





$$\underline{\text{SET}}/B \cong \underline{\text{SET}}^B$$

$$f : A \rightarrow B$$

$$(A_b)_{b \in B}$$

$$A_b = f^{-1}b$$

$$A = \bigcup_{b \in B} A_b, \quad f|_{A_b} = \text{const } b$$

$$f \text{ sdi} \iff \exists! b_0 \in B : |A_{b_0}| = 2 \text{ and } |A_b| \leq 1 \ (b \neq b_0)$$

# CONSTRUCTIVE? FUNCTORIAL?

Generally: No!      Zorn's Lemma everywhere!

Set: constructive, but not functorial:

$$\begin{array}{ccc} \emptyset \subset & \xrightarrow{\quad} & 2^0 = 1 \\ \downarrow \cap & & \Downarrow \\ 1 & \xlongequal{\quad} & 2^0 = 1 \\ \downarrow \times & & \downarrow \\ X & \xrightarrow{\quad} & 2^I \end{array}$$

# RESIDUALLY SMALL HNB-CATEGORIES

$\mathcal{A}$  residually small  $\iff \{A \in \mathcal{A} \mid A \text{ sdi}\} / \cong$  small

$\iff \mathcal{A}$  has cogenerator (of sdi objects)

(if  $\mathcal{A}$  is HNB, wellpowered)

Set, AbGrp, Mod<sub>R</sub>: yes

Grp, CompAbGrp : no

Residually small finitary varieties: Taylor (1972)

# EQUIVALENT COMPLETENESS PROPERTIES FOR RESIDUALLY SMALL HNB-CATEGORIES

(i)  $\mathcal{A}$  totally cocomplete  $\quad (\mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \underline{\text{Set}}]$  has left adjoint)

(ii)  $\mathcal{A}$  hypercomplete

(iii)  $\mathcal{A}$  small-complete with large intersections of monos

(i)<sup>op</sup>  $\mathcal{A}$  totally complete

(ii)<sup>op</sup>  $\mathcal{A}$  hypercocomplete

(iii)<sup>op</sup>  $\mathcal{A}$  small-cocomplete with large cointersections of epis

Recent work by M. Menni: An Exercise with Sufficient Cohesion, 2011