Nullstellen and Subdirect Representation

Walter Tholen

York University, Toronto, Canada

Union College
19-20 October 2013

HNB Theorem — Version 1

System of r polynomial equations in n variables with coefficients in a field k:

$$\begin{aligned} f_1(x_1,\dots,x_n) &= 0 \\ &\vdots \qquad\qquad (f_i \in k[x_1,\dots,x_n]) \qquad (\star) \\ f_r(x_1,\dots,x_n) &= 0 \end{aligned}$$

Examples

$x_1 - x_2^2 = 0 \qquad x_1^2 - 1 = 0 \qquad x_1^2 + 1 = 0 \qquad \begin{aligned} x_1^2 - 1 &= 0 \\ x_1^2 + 1 &= 0 \end{aligned}$

HNB Theorem — Version 1

D. Hilbert (1893)

Every system $(\star)$ has a solution $a = (a_1, \dots, a_n)$ in F^n for any algebraically-closed extension field F of k,

unless there are polynomials $g_1, \dots, g_r \in k[x]$ with

$$g_1 f_1 + \dots + g_r f_r = 1$$

Restriction is essential:

$$0 = g_1(a)f_1(a) + \dots + g_r(a)f_r(a) = 1$$

A Galois Correspondence

Systems of equations Solution sets

$$P \subseteq k[x_1, \dots, x_n] \quad \longmapsto \quad S(P) = \{a \in k^n \mid \forall f \in P : f(a) = 0\}$$

$$J(X) = \{f \in k[x] \mid \forall a \in X : f(a) = 0\} \quad \longleftarrow \quad X \subseteq F^n$$

$$P \subseteq J(X) \quad \iff \quad X \subseteq S(P)$$

$$P = J(S(P)) \quad \longleftrightarrow \quad X = S(J(X))$$

Note: $P \subseteq \sqrt{P} = \{f \in k[x] \mid \exists m \geq 1 : \ f^m \in P\}$

HNB Theorem — Version 2

$$\left.\begin{array}{l} P \trianglelefteq k[x_1, \ldots, x_n] \text{ proper ideal} \\ \\ F \text{ algebraically-closed} \end{array}\right\} \implies J(S(P)) = \sqrt{P}$$

That is :

If $f \in k[x]$, then $(\forall a \in S(P) : f(a) = 0 \iff \exists m \geq 1 : f^m \in P)$

Version 2 implies Version 1

$$(f_1, \ldots, f_r) = P$$

$$\sqrt{P} = J(S(P)) \quad \longleftrightarrow \quad S(P)$$

$$k[x_1, \ldots, x_n] \quad \longleftrightarrow \quad \emptyset$$

Note: Any $P \trianglelefteq k[x_1, \ldots, x_n]$ is finitely generated:

$$k \text{ Noetherian ring} \quad \Longrightarrow \quad k[x] \text{ Noetherian}$$

Trying to prove Version 2

Need to show $J(S(P)) \subseteq \sqrt{P}$:

If $f \notin \sqrt{P}$, must find $a \in S(P)$ with $f(a) \neq 0$.

Consider $A = k[x_1, \dots, x_n]/\sqrt{P}$

Wish: Find

$$\begin{array}{ccccc} k[x] & \xrightarrow{\quad\pi\quad} & A & \xrightarrow{\quad\varphi\quad} & F \\ f & \longmapsto & \pi(f) = u & \longmapsto & \varphi(u) \neq 0 \end{array}$$

Consider $a = (\varphi(\pi(x_1)), \dots, \varphi(\pi(x_n)))$.

Then $f(a) \neq 0$, but $\forall p \in P : p(a) = 0$.

Version 3

$$\left.\begin{array}{l} A \text{ fin. gen. comm. } k\text{-algebra} \\ k \subseteq F \text{ alg. closed} \\ u \in A \text{ non-nilpotent} \end{array}\right\} \Longrightarrow \begin{array}{l} \exists \varphi : A \to F \\ \qquad u \mapsto \varphi(u) \neq 0 \end{array}$$

Version 4

$$\left.\begin{array}{l} A \text{ fin. gen. comm. } k\text{-algebra} \\ u \in A \text{ non-nilpotent} \end{array}\right\} \Longrightarrow \begin{array}{l} \exists \chi : A \to K, \quad \chi(u) \neq 0, \\ k \subseteq K \text{ field,} \\ K \text{ fin. gen. (as unital } k\text{-algebra)} \end{array}$$

Version 4 implies Version 3

$$\begin{tikzcd} k \arrow[r] \arrow[d] & F & \text{algebraically closed} \\ A \arrow[r, "\chi"] & K \arrow[u, dashed] & \text{algebraic} \end{tikzcd}$$

HNB Theorem — Version 5, Version 5 implies Version 4

$$\left.\begin{array}{l} A \text{ comm. ring} \\ \\ u \text{ non-nilpotent} \end{array}\right\} \implies \exists Q \trianglelefteq A \text{ prime}, u \notin Q$$

$5 \Rightarrow 4$

$$\begin{tikzcd}
 & & L & \\
A \arrow[r, "\sigma"] & A/Q \arrow[ur] \arrow[r] & R = (A/Q)[\sigma(u)^{-1}] \arrow[u] \arrow[r] & K = R/M
\end{tikzcd}$$

field of fractions

integral domain

M maximal ideal

Proof of Version 5

$S := \{u^n \mid n \geq 1\} \not\ni 0$

Zorn: $\exists Q \trianglelefteq A$ maximal w.r.t. $Q \cap S = \emptyset$

Q is prime: $a, b \notin Q$

$$\Longrightarrow (a) + Q, (b) + Q \text{ meet } S$$

$$\Longrightarrow \exists l, m : \; u^l \in (a) + Q, \; u^m \in b + Q$$

$$\Longrightarrow u^{l+m} \in (ab) + Q$$

$$\Longrightarrow ab \notin Q$$

QED

Decomposition Theorems

$$A \text{ comm., reduced} \quad \Longrightarrow \quad \bigcap\{Q \trianglelefteq A \mid Q \text{ prime}\} = (0)$$

$$A \rightarrowtail \prod_Q A/Q$$

Lasker (1904), Noether (1920)

$$\left.\begin{array}{l} R \text{ comm. Noetherian} \\ P \trianglelefteq R \text{ ideal} \end{array}\right\} \Longrightarrow P = Q_1 \cap \ldots \cap Q_n \text{ with } Q_i \trianglelefteq R \text{ irreducible}$$

Alternative formulation of Version 5

$$R \text{ comm., } P \trianglelefteq R \text{ radical} \quad \Longleftrightarrow \quad P = \bigcap\{Q \trianglelefteq R \mid P \subseteq Q,\ Q \text{ prime}\}$$

Birkhoff's Subdirect Representation Theorem (1944)

A general finitary algebra $\Longrightarrow$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr] & \prod\limits_{i\in I} S_i \arrow[d] & S_i \text{ sdi} \\ & S_i & \end{tikzcd}$$

A subdirectly irreducible (sdi) $:\Longleftrightarrow$ If

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr] & \prod\limits_{i\in I} S_i \arrow[d] \\ & S_i \end{tikzcd}$$

, then $A \cong S_{i_0}$

This notion is categorical!

HNB-categories

- $\mathcal{A}$ has (strong epi, mono)-factorizations
- $\mathcal{A}$ is weakly cowellpowered
- $\mathcal{A}$ has generator consisting of finitary objects

(no existence requirements for limits or colimits)

Examples

- quasi-varieties of finitary algebras
- locally finitely presentable categories
- presheaf categories
- the category of topological spaces

HNB Theorem — Version 7

A object of an HNB-category $\Longrightarrow$ A has subdirect representation

$$A \rightarrowtail \prod_{i\in I} S_i$$

$$\begin{aligned} S \text{ sdi} \quad &\Longleftrightarrow \quad \exists a \neq b : P \to S \\ &\qquad\quad \forall f : A \to B \; (fa \neq fb \Rightarrow f \text{ monic}) \\ &\Longleftrightarrow \quad \mathrm{Con}S \setminus \{\Delta_S\} \text{ has a least element} \end{aligned}$$

<u>Set</u>: 2 $\qquad$ k-<u>Vec</u>: k $\qquad$ <u>CRng</u>: $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$

$f : A \to B$ morphism of an HNB-category with finite products

$$\begin{tikzcd} & S_i \arrow[dr, "s_i"] & \\ A \arrow[ur, "p_i"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

$((p_i)_i$ monic) each s_i sdi

$\underline{\text{Set}}/B \;\cong\; \underline{\text{Set}}^B$

$f : A \to B$ $\qquad\qquad (A_b)_{b \in B}$

$$A_b = f^{-1} b$$

$$A = \bigcup_{b \in B} A_b \,, \quad f_{|_{A_b}} = \text{const } b$$

$$f \text{ sdi} \iff \exists! b_0 \in B : \; |A_{b_0}| = 2 \text{ and } |A_b| \leq 1 \; (b \neq b_0)$$

Constructive? Functorial?

Generally: No!   Zorn's Lemma everywhere!

Set: constructive, but not functorial:

$$\begin{tikzcd}
\emptyset \arrow[r, hook] \arrow[d, hook] & 2^0 = 1 \arrow[d, equal] \\
1 \arrow[r, equal] \arrow[d, "x"'] & 2^0 = 1 \arrow[d] \\
X \arrow[r] & 2^I
\end{tikzcd}$$

$\mathcal{A}$ residually small $\iff$ $\{A \in \mathcal{A} \mid A \text{ sdi}\}/\cong$ small

$\iff$ $\mathcal{A}$ has cogenerator (of sdi objects)

(if $\mathcal{A}$ is HNB, wellpowered)

$\underline{\mathrm{Set}}$, $\underline{\mathrm{AbGrp}}$, $\underline{\mathrm{Mod}}_R$: yes

$\underline{\mathrm{Grp}}$, $\underline{\mathrm{CompAbGrp}}$: no

Residually small finitary varieties: Taylor (1972)

Equivalent Completeness properties for residually small HNB-categories

(i) $\mathcal{A}$ totally cocomplete ($\mathcal{A} \to [\mathcal{A}^{\mathsf{op}}, \underline{\mathsf{Set}}]$ has left adjoint)

(ii) $\mathcal{A}$ hypercomplete

(iii) $\mathcal{A}$ small-complete with large intersections of monos

(i)$^{\mathsf{op}}$ $\mathcal{A}$ totally complete

(ii)$^{\mathsf{op}}$ $\mathcal{A}$ hypercocomplete

(iii)$^{\mathsf{op}}$ $\mathcal{A}$ small-cocomplete with large cointersections of epis

Recent work by M. Menni: An Exercise with Sufficient Cohesion, 2011